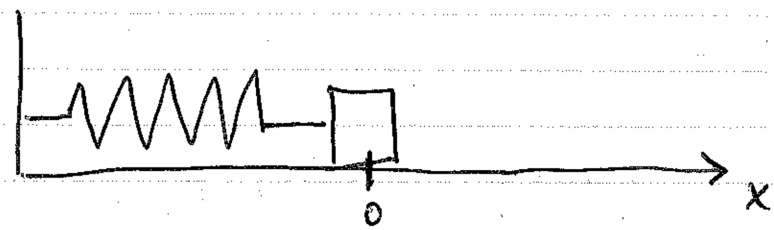
Section 4.1 - Harmonic Oscillators

- Consider an object of mass m sitting on a surface, connected to the wall via a spring:



- Let x be the horizontal displacement of the object from its rest position (at $x=0$)

- Let's develop a model for the motion of the object, using Newton's Second Law of Motion:

$$F = m\ddot{x}$$

(here $\ddot{x} = \frac{d^2x}{dt^2}$ = acceleration)

- Forces acting on the object:
  - the spring: if we pull the object far to the right, the spring pulls to the left, and vice-versa. By Hooke's Law, the force becomes

$$F_{spring} = -kx$$

k is called the spring constant.

- friction: if the object is moving to the right, friction ~~exher~~ exerts a force to the left. Let's say the force is proportional to the velocity:

b is called the **damping constant**

$$F_{friction} = -b\dot{x} = -b\frac{dx}{dt}$$

- external forces: things like wind, magnetic fields, physically pushing the object, etc. Let's just clump it all into one function:

$$F_{external} = f(t)$$

- Using Newtons 2nd Law:

$$m\ddot{x} = F_{spring} + F_{friction} + F_{external}$$

$$= -kx - b\dot{x} + f(t)$$

- This gives the equation for a <u>simple harmonic oscillator</u>:

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

- this is a second order linear differential equation with constant coefficients
- if $b=0$, the motion is called <u>undamped</u>, otherwise it is <u>damped</u>.
- if $f(t) \equiv 0$, the equation is <u>homogeneous</u> and the motion is called <u>unforced</u>, <u>undriven</u> or <u>free</u>

Ex: Let's say the mass of the object is 1 kg ($m=1$), and we determine that it takes 1 newton of force to push the object 0.25 meters. Then

$$k = \frac{1\,N}{0.25\,m} = 4\,\frac{N}{m}$$

Furthermore, we discover that it takes 0.5 newtons to keep the object moving at 0.25 meters per second:

$$b = \frac{0.5\,N}{0.25\,m/s} = 2\,\frac{Ns}{m}$$

This gives the equation

$$\ddot{x} + 2\dot{x} + 4x = 0$$

- Suppose we start the experiment by pulling the object to the right 0.5 meters, and releasing it without any initial velocity. Then we get the IVP (initial value problem):

$$\ddot{x} + 2\dot{x} + 4x = 0,$$
$$x(0) = 0.5, \quad \dot{x}(0) = 0$$

- Note: any second-order IVP requires <u>two</u> initial conditions.

Solutions to the Undamped Unforced Oscillator

- Let's try to solve the equation

$$m\ddot{x} + kx = 0$$

or equivalently,

$$\boxed{\ddot{x} = -\frac{k}{m}x \qquad (1)}$$

- one example of a function which, when differentiated, gives you the same function with a "−" is sine:

$$\frac{d^2}{dt^2}\sin(t) = -\sin(t)$$

- If we wanted a constant in front of the sin, we'd use

$$\frac{d^2}{dt^2}\sin(\omega_0 t) = -\omega_0^2 \sin(\omega_0 t)$$

$$\Updownarrow$$

$$\ddot{x} = -\omega_0^2 x$$

- letting $\omega_0 = \sqrt{\frac{k}{m}}$, we get that

$$x(t) = \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

is a solution to (1)

- Similarly, you can show that
$$x(t) = \cos\left(\sqrt{\tfrac{k}{m}}\, t\right)$$
is <u>also</u> a solution to (1).

- By the superposition principle (see section 2.1, be sure to review it!), we find the general solution:
$$x(t) = c_1 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)$$
  - c_1 + c_2 are determined by initial conditions

<u>Ex</u>: Solve the IVP
$$\ddot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 1$$

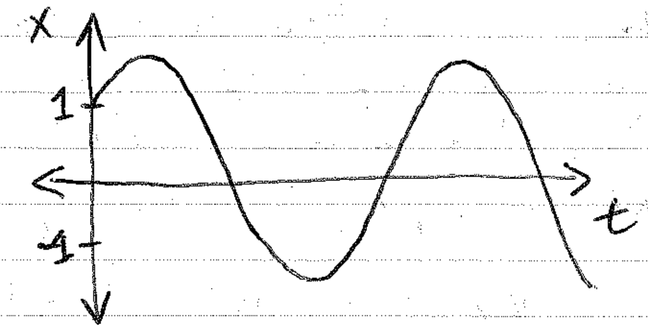
- seeing that $k = 1$ + $m = 1$, we know that the general solution is
$$x(t) = c_1 \cos(t) + c_2 \sin(t)$$
$$\Rightarrow \dot{x}(t) = -c_1 \sin(t) + c_2 \cos(t)$$

- using the initial conditions, we find that
$$x(0) = c_1 = 1$$
$$\dot{x}(0) = c_2 = 1$$

so the solution to the IVP is

$$x(t) = \cos t + \sin t$$

- the graph of this function looks like this:



which looks suspiciously like a normal cosine curve, but stretched and shifted. It is, in fact:

$$\cos t + \sin t = \sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$$

- In fact, you can convert anything of the form

$$c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

into the form

$$A \cos(\omega_0 t - \delta)$$

and back again using the transformations

$$A = \sqrt{c_1^2 + c_2^2}, \quad \tan\delta = \frac{c_1}{c_2}$$

$$c_1 = A\cos\delta, \quad c_2 = A\sin\delta$$

- A is called the <u>amplitude</u>, and δ is called the <u>phase angle</u>

~~Ex: Express $2\cos(\frac{1}{2}t) + \sin(\frac{1}{2}t)$ in phase-amplitude form.~~
~~- Since $c_1 = 2$ and $c_2 = 1$, we get that~~

$$\cancel{A = \sqrt{2^2 + 1^2} = \sqrt{5}}$$

$$\cancel{\tan\delta = \frac{2}{1} \Rightarrow \delta =}$$

<u>Ex</u>: Express $\sqrt{3}\cos(\frac{1}{2}t) + \sin(\frac{1}{2}t)$ in phase-amplitude form.
- Since $c_1 = \sqrt{3}$ and $c_2 = 1$, we get that

$$A = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$$

$$\tan\delta = \frac{\sqrt{3}}{1} \Rightarrow \delta = \frac{\pi}{6}$$

so in phase-amplitude form we get

$$2\cos\left(\frac{1}{2}t - \frac{\pi}{6}\right)$$

<u>Ex</u>: Express $\sqrt{3}\cos(5t - \frac{2\pi}{3})$ is component form.
- Since $A = \sqrt{3}$ and $\delta = \frac{2\pi}{3}$, we get that

$$c_1 = \sqrt{3}\cos\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$c_2 = \sqrt{3}\sin\frac{2\pi}{3} = \frac{3}{2}$$

so in component form, we get

$$-\frac{\sqrt{3}}{2}\cos(5t) + \frac{3}{2}\sin(5t)$$

- Note: sometimes calculating δ can be tricky: for example if

$$\tan\delta = -1,$$

then both $\delta = \frac{3\pi}{4}$ and $\delta = -\frac{\pi}{4}$ are solutions! Usually, we restrict δ to be between $-\pi$ and π to fix this issue.

~~Phase Plane Description~~