- A is called the <u>amplitude</u>, and δ is called the <u>phase angle</u>

~~Ex: Express $2\cos(\frac{1}{2}t) + \sin(\frac{1}{2}t)$ in phase-amplitude form.~~
~~- Since $c_1 = 2$ and $c_2 = 1$, we get that~~

$$\cancel{A = \sqrt{2^2 + 1^2} = \sqrt{5}}$$

$$\cancel{\tan\delta = \frac{2}{1} \Rightarrow \delta =}$$

<u>Ex</u>: Express $\sqrt{3}\cos(\frac{1}{2}t) + \sin(\frac{1}{2}t)$ in phase-amplitude form.
- Since $c_1 = \sqrt{3}$ and $c_2 = 1$, we get that

$$A = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$$

$$\tan\delta = \frac{\sqrt{3}}{1} \Rightarrow \delta = \frac{\pi}{6}$$

so in phase-amplitude form we get

$$2\cos\left(\frac{1}{2}t - \frac{\pi}{6}\right)$$

<u>Ex</u>: Express $\sqrt{3}\cos(5t - \frac{2\pi}{3})$ in component form.
- Since $A = \sqrt{3}$ and $\delta = \frac{2\pi}{3}$, we get that

$$c_1 = \sqrt{3}\cos\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$c_2 = \sqrt{3}\sin\frac{2\pi}{3} = \frac{3}{2}$$

so in component form, we get

$$-\frac{\sqrt{3}}{2}\cos(5t) + \frac{3}{2}\sin(5t)$$

- Note: sometimes calculating δ can be tricky: for example if

$$\tan\delta = -1,$$

then both $\delta = \frac{3\pi}{4}$ and $\delta = -\frac{\pi}{4}$ are solutions! Usually, we restrict δ to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ to fix this issue.

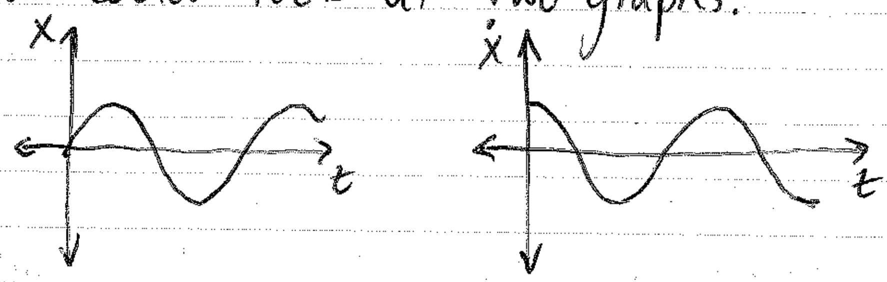
~~Phase Plane Description~~

~~Phase Plane Description~~

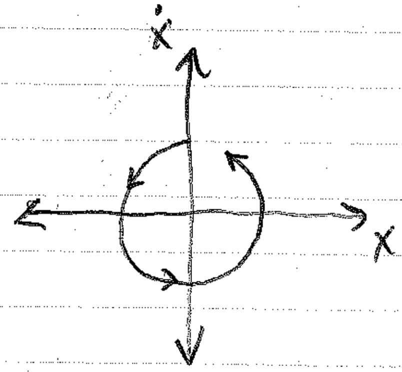
Phase Plane Description

- Sometimes it's more useful to look at how both x and $\dot{x}$ evolve in time

- You could look at two graphs:



but it's useful to ~~examine~~ treat $x(t)$ and $\dot{x}(t)$ as parametric eqns that make a curve in the $x\dot{x}$ phase plane:



"phase portrait"

- An advantage of phase portraits is that you can qualitatively sketch them without solving the DE

- Similar to direction fields
- First convert the DE into a system of 1^{st} order equations: given

$$m\ddot{x} + b\dot{x} + kx = 0,$$

let $y = \dot{x}$. Then $\dot{y} = \ddot{x}$, so

$$\dot{x} = y$$
$$\dot{y} = \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x$$
$$= -\frac{b}{m}y - \frac{k}{m}x$$

- The system of 1^{st} order eqns:

$$\dot{x} = y$$
$$\dot{y} = -\frac{b}{m}y - \frac{k}{m}x$$

- The right-hand side defines the <u>vector field</u> which trajectories in the phase plane follow

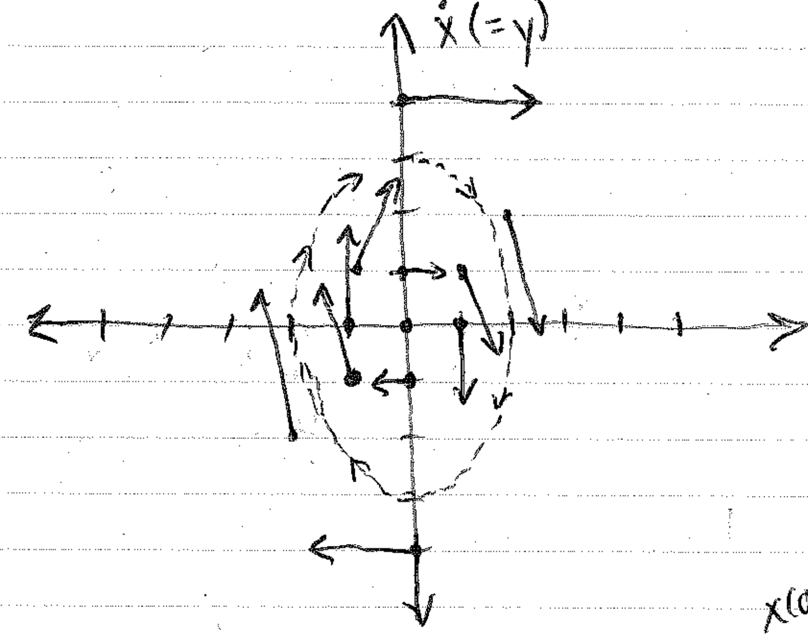
<u>Ex:</u> Suppose we're given the following DE:

$$\ddot{x} + 4x = 0$$

converting to a system:

$$\dot{x} = y$$
$$\dot{y} = -4x$$

- Now, for each point in the xy-plane (i.e. the $x\dot{x}$-plane), the RHS defines the direction vector:



$x(0)$ $\dot{x}(0)$

- the trajectory starting at $(0,3)$ follows the vectors in the vector field to trace the dotted line

4.2 Real Characteristic Roots

- In this section we'll be solving diff. eqn's of the form

$$\boxed{ay'' + by' + cy = 0 \quad (2)}$$

$$\Updownarrow$$

$$(m\ddot{x} + b\dot{x} + kx = 0)$$

- section 2.3 suggests 'guessing' that $y = e^{rt}$ is a solution. Substituting

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}$$

into (2) gives

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$\Leftrightarrow e^{rt}(ar^2 + br + c) = 0$$

$$\Leftrightarrow \boxed{ar^2 + br + c = 0 \quad (3)}$$

- (3) is called the characteristic equation. It's very important, because it says that $y = e^{rt}$ is a solution to (2) only when r satisfies (3).

- Since (3) is a quadratic eqn, we know

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- there are 3 possible outcomes for r:
  1. two distinct real-valued #'s
  2. one real-valued #
  3. two complex #'s (complex ⇒ imaginary part ≠ 0)
- we'll focus on cases 1 + 2 for now

~~Let's call~~

- the values of r which satisfy (3) are called the <u>characteristic roots</u>
- Let's call $\Delta = b^2 - 4ac$ the <u>discriminant</u> of (3) (the thing under the $\sqrt{\ }$).
- $\boxed{\text{Case 1 (two distinct roots)}}$ happens precisely when $\Delta > 0$. Let's call our two distinct real roots

$$r_1 = \frac{-b+\sqrt{\Delta}}{2a}, \quad r_2 = \frac{-b-\sqrt{\Delta}}{2a}$$

- We know that $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are <u>both</u> solutions to (2).
- By the superposition principle, the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where c_1 + c_2 are constants determined by initial conditions.

- Thing to remember:

$$\boxed{\text{2 distinct real roots} \implies y = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

<u>Ex</u>: Find the general solution to

$$y'' + 5y' + 6y = 0$$

- First, we write down the characteristic equation:

$$r^2 + 5r + 6 = 0$$
$$\Leftrightarrow (r+2)(r+3) = 0$$
$$\Leftrightarrow r = -2, -3$$

- 2 distinct real roots, so the general solution is

$$y = c_1 e^{-2t} + c_2 e^{-3t}$$

- It turns out that $e^{r_1 t}$ and $e^{r_2 t}$ are <u>linearly independent</u>, and that $\{e^{r_1 t}, e^{r_2 t}\}$ forms a <u>basis</u> for the solution space of (2).

  - This tells us that the solution space is 2-dimensional