- $\boxed{\text{Case ② (one real root)}}$ happens when $\Delta = 0$. Let's call our root
$$r = \frac{-b}{2a}$$

  - We know $y = e^{rt}$ is one solution, but we need a second independent solution
  - Using a method similar to variation of parameters, we find that $y = te^{rt}$ is also a solution
  - By the superposition principle, the general solution to (2) becomes
$$y = c_1 e^{rt} + c_2 te^{rt},$$
  where c_1 & c_2 are constants determined by initial conditions

- Once again, it turns out that $\{e^{rt}, te^{rt}\}$ forms a <u>basis</u> for the solution space, which says the solution space is 2-dimensional

- Thing to remember:
$$\boxed{\text{one real root} \Rightarrow y = c_1 e^{rt} + c_2 te^{rt}}$$

<u>Ex</u>: Find the general solution to

$$y'' - 4y' + 4y = 0$$

- First, write down the characteristic equation:

$$r^2 - 4r + 4 = 0$$

$$\Leftrightarrow \quad (r-2)(r-2) = 0$$

$$\Leftrightarrow \quad r = 2.$$

- Only one root, so the general solution is

$$y = c_1 e^{2t} + c_2 t e^{2t}$$

- Suppose we're given initial conditions

$$y(0) = 1, \quad y'(0) = 1$$

Substituting 0 for t in our solution and its derivative gives

$$\begin{aligned} y(0) &= c_1 = 1 \\ y'(0) &= 2c_1 + c_2 = 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

so the exact solution is

$$y(t) = e^{2t} - te^{2t}$$

Overdamped and Critically Damped Mass-Spring Systems

- Let's go back to harmonic oscillators/mass-spring systems. The equation of motion is

$$m\ddot{x} + b\dot{x} + kx = 0$$

where m is the mass, b the damping constant, and k the spring constant

- The behavior of the system depends entirely on the discriminant $\Delta = b^2 - 4mk$

- When $\boxed{\Delta > 0}$ (2 real roots), the system is called overdamped. When $\boxed{\Delta = 0}$ (one real root), the system is called critically damped.

- Physically, a critically damped system goes to 0 as fast as possible without oscillating, while an overdamped system takes longer

Ex: Suppose we're designing a shock-absorber for a car, which consists of a spring and a hydraulic dashpot. The mass of the wheel is 20 kg, and the

spring constant of the spring is $20 \frac{N}{m}$.
We ~~can use~~ must choose between 2 different hydraulic fluids: one creates a damping constant of $40 \frac{Ns}{m}$ in the dashpot, and the other. $50 \frac{Ns}{m}$

- using the first fluid, the equation of motion is

$$20\ddot{x} + 40\dot{x} + 20x = 0$$

The characteristic equation:

$$20r^2 + 40r + 20 = 0$$

with discriminant

$$\Delta = 40^2 - 4(20)(20)$$
$$= 1600 - 1600 = 0$$

This gives the general solution

$$x_1(t) = c_1 e^{-t} + c_2 t e^{-t}$$

- using the second fluid, the eqn of motion is

$$20\ddot{x} + 50\dot{x} + 20x = 0$$

with characteristic equation

$$20r^2 + 50r + 20 = 0$$

and discriminant

$$\Delta = 50^2 - 4(20)(20)$$

$$= 2500 - 1600 = 900$$

This gives the general solution

$$x_2(t) = c_1 e^{-2t} + c_2 e^{-\frac{1}{2}t}$$

- If we start both system with the same initial conditions:

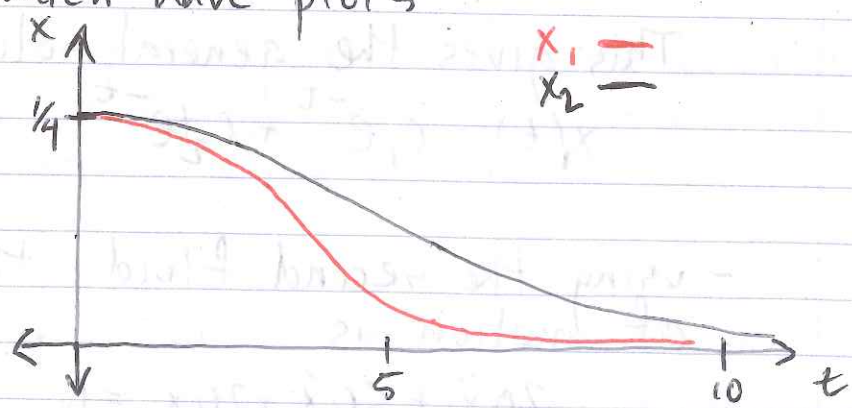
$$x(0) = 0.25, \quad \dot{x}(0) = 0$$

we get the exact solutions

$$x_1(t) = \frac{1}{4}e^{-t} + \frac{1}{4}te^{-t}$$

$$x_2(t) = -\frac{1}{12}e^{-2t} + \frac{1}{3}e^{-\frac{1}{2}t}$$

which have plots



- since x_1 returns to 0 more quickly than x_2 (because its corresponding system is <u>critically damped</u>), we should choose the first fluid.

- One last thing:
  <u>Solution Space Theorem</u>: The solution space for <u>any</u> second-order homogeneous differential equation has dimension 2.

4.3 Complex Characteristic Roots

- We've determined how to solve

$$\boxed{ay'' + by' + cy = 0 \quad (2)}$$

when the characteristic roots are real valued, so let's figure out what happens when they're complex valued

- Since we're dealing with complex #'s now, let's talk about complex valued solutions. Suppose $u(t) + iv(t)$ is a solution to (2). Then

$$a(u+iv)'' + b(u+iv)' + c(u+iv) = 0$$

$$\Leftrightarrow (au'' + bu' + cu) + i(av'' + bv' + cv) = 0$$

$$\Leftrightarrow au'' + bu' + cu = 0 \text{ and } av'' + bv' + cv = 0$$

so both u and v are solutions

- We'll use this fact later:

$$\boxed{\begin{array}{c} \text{if } u+iv \text{ is} \\ \text{a solution to (2)} \end{array} \Leftrightarrow \begin{array}{c} \text{both } u \text{ and } v \\ \text{are solutions to (2)} \end{array}}$$

- Ok, on to complex characteristic roots: Case ③ (complex roots) happen when $\Delta < 0$. Let's call the roots

$$r_1 = \frac{-b}{2a} + i\frac{\sqrt{-\Delta}}{2a} = \alpha + i\beta$$

$$r_2 = \underbrace{\frac{-b}{2a}}_{\alpha} - i\underbrace{\frac{\sqrt{-\Delta}}{2a}}_{\beta} = \alpha - i\beta$$

- This tells us that $y = e^{(\alpha+i\beta)t}$ is a solution to (2), but it is complex valued...

- Euler's formula:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta \quad (4)}$$

- Using (4), we get another way to write y:

$$\begin{aligned} y &= e^{(\alpha+i\beta)t} \\ &= e^{\alpha t}\cdot e^{i\beta t} \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t) \\ &= \underbrace{e^{\alpha t}\cos\beta t}_{u} + i\underbrace{e^{\alpha t}\sin\beta t}_{v} \end{aligned}$$

- Now we know that $y = e^{\alpha t}\cos\beta t$ and $y = e^{\alpha t}\sin\beta t$ are both solutions. Since they are linearly independent, and the solution space is 2-dimensional, then $\{e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t\}$ is a basis for the solution space

(2)

• Thus, the general solution is

$$y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

• Thing to remember:

$$\boxed{\text{complex roots } r = \alpha \pm i\beta \Rightarrow y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t}$$

or

$$\boxed{y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)}$$

<u>Ex</u>: Find the solution to the IVP

$$y'' + 2y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = -1$$

- First we write the characteristic eqn:

$$r^2 + 2r + 4 = 0$$

$$\Rightarrow r = \frac{-2 \pm \sqrt{4 - 4 \cdot 4}}{2}$$

$$\Rightarrow r = -1 \pm i\sqrt{3} \quad (\alpha = -1, \beta = \sqrt{3})$$

- Then the general solution is given by

$$y(t) = c_1 e^{-t} \cos \sqrt{3}t + c_2 e^{-t} \sin \sqrt{3}t$$