- Thus, the general solution is

$$y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$$

- Thing to remember:

complex roots $r = \alpha \pm i\beta$ $\Rightarrow$ $y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$

or

$y = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$

<u>Ex</u>: Find the solution to the IVP

$$y'' + 2y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = -1$$

- First we write the characteristic eqn:

$$r^2 + 2r + 4 = 0$$

$$\Rightarrow r = \frac{-2 \pm \sqrt{4 - 4\cdot 4}}{2}$$

$$\Rightarrow r = -1 \pm i\sqrt{3} \quad (\alpha = -1,\ \beta = \sqrt{3})$$

- Then the general solution is given by

$$y(t) = c_1 e^{-t}\cos\sqrt{3}t + c_2 e^{-t}\sin\sqrt{3}t$$

$$\Rightarrow y'(t) = (-c_1 + \sqrt{3}c_2)e^{-t}\cos\sqrt{3}t + (-\sqrt{3}c_1 - c_2)e^{-t}\sin\sqrt{3}t$$

- Using the initial values, we get

$$y(0) = 0 = c_1 \qquad\qquad \Rightarrow \quad c_1 = 0,$$
$$y'(0) = -1 = -c_1 + \sqrt{3}c_2 \qquad c_2 = -1/\sqrt{3}$$

- Then the solution is

$$y(t) = -\frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t$$

__Ex:__ (Undamped Harmonic Oscillator) Find the general solution to $y'' + y = 0$.

- The characteristic roots are

$$r^2 + 1 = 0 \Rightarrow r = 0 \pm i \quad (\alpha = 0, \beta = 1)$$

- The general solution is then

$$y = c_1 e^{0t}\cos t + c_2 e^{0t}\sin t$$
$$= c_1\cos t + c_2 \sin t$$

- Exactly the same result as from section 4.1!

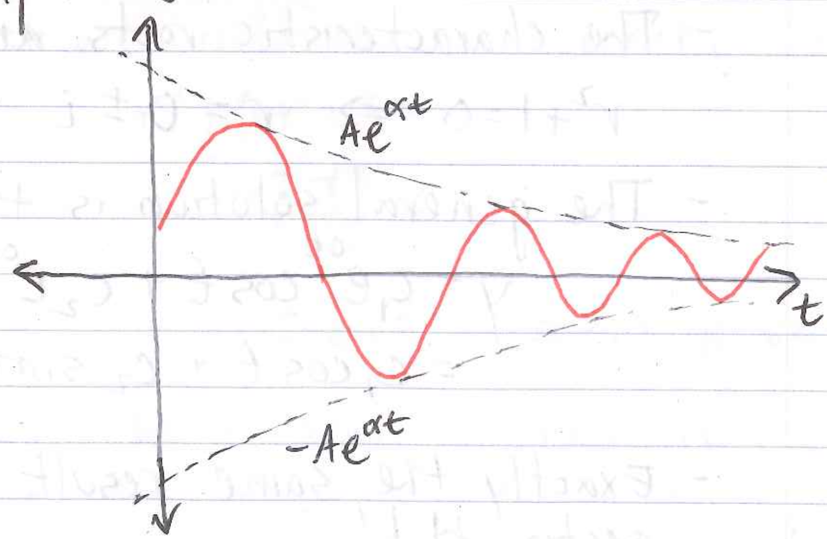
Underdamped Mass-Spring Systems

- The motion of a mass-spring system is called <u>underdamped</u> when the discriminant $\boxed{\Delta < 0}$

- In this case, we get solutions of the form

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$
$$= e^{\alpha t}(A \cos(\beta t - \delta))$$

using the same identities as in sec. 4.4.
- this is just like undamped motion, except with a time-varying amplitude:



- once again you can use the transformations

$$A = \sqrt{c_1^2 + c_2^2} \qquad \tan\delta = \frac{c_2}{c_1}$$

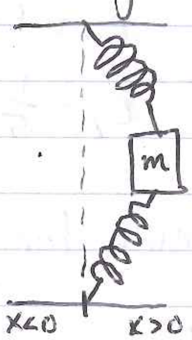
$$c_1 = A\cos\delta \qquad c_2 = A\sin\delta$$

- the natural frequency for this system is given by

$$f = \frac{\beta}{2\pi}$$

and is measured in hertz

<u>Ex</u>: Suppose we have a guitar string weighing 1g, and that we stretch the string so that the "spring constant" is 26 $\frac{g}{s^2}$, and that the air resistance produces a "damping constant" of 2 $\frac{g}{s}$. We pull the string 1 cm to the right, and let go. About how long does the note last? What note is it?



- The equation of motion for the string is

$$\ddot{x} + 2\dot{x} + 26x = 0$$

which gives as characteristic eqn:

$$r^2 + 2r + 26 = 0$$

which has roots

$$r = -1 \pm 5i \quad (\alpha = -1, \beta = 5)$$

- Thus, the general solution is

$$y = c_1 e^{-t} \cos 5t + c_2 e^{-t} \sin 5t$$

$$\Rightarrow y' = (-c_1 + 5c_2) e^{-t} \cos 5t + (-5c_1 - c_2) e^{-t} \sin 5t$$

- Using the initial conditions

$$y(0) = 1, \quad y'(0) = 0$$

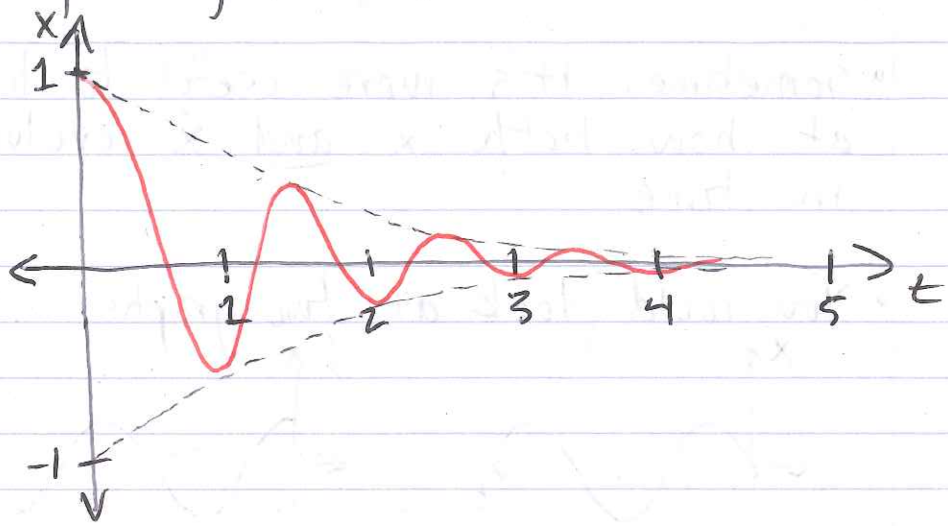
we get

$$y(0) = 1 = c_1$$
$$y'(0) = 0 = -c_1 + 5c_2 \quad \Rightarrow \quad c_1 = 1, \; c_2 = \frac{1}{5}$$

- Thus, the ~~solution~~ formula for the motion of the string is

$$x(t) = e^{-t} \cos 5t + \frac{1}{5} e^{-t} \sin 5t$$
$$= e^{-t} \left(\cos 5t + \frac{1}{5} \sin 5t \right)$$

- plotting the solution:



- we see that after 4 seconds, the vibration has almost stopped, so the note lasts for about 4 s!

- the natural frequency of the note is

$$f = \frac{5}{2\pi} \approx 0.796$$

which corresponds to approximately a G#.

- In summary:

	roots	general solution
Case 1: $\Delta > 0$	$r_1, r_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$	$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
Case 2: $\Delta = 0$	$r = \frac{-b}{2a}$	$y = c_1 e^{rt} + c_2 t e^{rt}$
Case 3: $\Delta < 0$	$r = \alpha \pm i\beta$	$y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$

4.4 Undetermined Coefficients

- We're pretty confident we can solve 2^{nd} order homogeneous DEs:

$$ay'' + by' + cy = 0$$

- Now let's try to solve the nonhomogeneous version, but first, let's generalize our notation.

- Instead of writing out $ay'' + by' + cy$, simplify by just using $L(y)$:

  - ~~[illegible]~~ L is the thing that takes a function y and outputs the corresponding combination of derivatives of y

  - L, in general, can denote <u>any</u> linear differential operator:

$$L(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

- <u>Superposition Principle for Nonhomogeneous Linear DEs</u>: If y_1 ~~[illegible]~~ and y_2 satisfy

$$L(y_1) = f_1(t) \text{ and } L(y_2) = f_2(t)$$

then

$$L(y_1 + y_2) = f_1(t) + f_2(t)$$

- Nonhomogeneous Principle for Linear DEs: The general solution of the nonhomogeneous linear DE $L(y)=f$ is

$$y = y_h + y_p$$

where y_h is the general solution to $L(y)=0$, and y_p is one solution to $L(y)=f$

- So, in order to solve

$$\boxed{ay'' + by' + cy = f(t) \qquad (4)}$$

we first solve $ay''+by'+cy=0$ to get y_h (which we're pros at), and then find <u>one</u> solution to (4), to get y_p

- Suppose y is a polynomial of degree n. Then $L(y)$ is <u>also</u> a polynomial of degree n.

- This suggests that if $f(t)$ is a poly., then $L(y)=f$ should have a polynomial as a solution, too.
  - Write down y with undetermined coefficients, and then determine them by substituting into $L(y)=f$

$\underline{\text{Ex}}$: Suppose we're given

$$y'' - y' - 2y = 3t^2 - 1$$

The RHS is a poly. of degree 2, we guess that

$$y_p = At^2 + Bt + C$$

$$\Rightarrow y_p' = 2At + B$$

$$\Rightarrow y_p'' = 2A$$

Substituting into the DE:

$$(2A) - (2At + B) - 2(At^2 + Bt + C) = 3t^2 - 1$$

$$\Rightarrow \underline{(-2A)}t^2 + \underline{(-2A-2B)}t + \underline{\underline{(2A - B - 2C)}} = \underline{3}t^2 \underline{\underline{-1}} + \underline{0}t$$

$$\Rightarrow \begin{array}{r} -2A = 3 \\ -2A - 2B = 0 \\ 2A - B - 2C = -1 \end{array} \Rightarrow \begin{array}{l} A = -3/2 \\ B = 3/2 \\ C = -7/4 \end{array}$$

and so our particular solution is

$$y_p = -\frac{3}{2}t^2 + \frac{3}{2}t - \frac{7}{4}$$

- The same procedure works for forcing functions of the form $f = Ae^{kt}$ or $f = k_1 \cos(\omega t) + k_2 \sin(\omega t)$:

$\underline{\text{Ex}}$: Given $y'' - y' - 2y = 2e^{-3t}$, we guess that $y_p = Ae^{-3t}$. Then