Ex: Suppose we're given

$$y'' - y' - 2y = 3t^2 - 1$$

The RHS is a poly. of degree 2, we guess that

$$y_p = At^2 + Bt + C$$

$$\Rightarrow y_p' = 2At + B$$

$$\Rightarrow y_p'' = 2A$$

Substituting into the DE:

$$(2A) - (2At + B) - 2(At^2 + Bt + C) = 3t^2 - 1$$

$$\Rightarrow \underline{(-2A)}t^2 + \underline{(-2A-2B)}t + \underline{(2A - B - 2C)} = \underline{3}t^2 \underline{-1} + \underline{0}t$$

$$\Rightarrow \begin{array}{c} -2A = 3 \\ -2A - 2B = 0 \\ 2A - B - 2C = -1 \end{array} \Rightarrow \begin{array}{l} A = -3/2 \\ B = 3/2 \\ C = -7/4 \end{array}$$

and so our particular solution is

$$y_p = -\frac{3}{2}t^2 + \frac{3}{2}t - \frac{7}{4}$$

- The same procedure works for forcing functions of the form $f = Ae^{kt}$ or $f = k_1 \cos(\omega t) + k_2 \sin(\omega t)$:

Ex: Given $y'' - y' - 2y = 2e^{-3t}$, we guess that $y_p = Ae^{-3t}$. Then

$$y_p' = -3Ae^{-3t}, \quad y_p'' = 9Ae^{-3t}$$

$$\Rightarrow (9Ae^{-3t}) - (-3Ae^{-3t}) - 2(Ae^{-3t}) = 2e^{-3t}$$

$$\Rightarrow (10A)e^{-3t} = 2e^{-3t}$$

$$\Rightarrow 10A = 2 \quad \Rightarrow A = 1/5$$

so our particular solution is

$$y_p = 1/5\, e^{-3t}$$

Ex: Given $y'' - y' - 2y = 2\cos 3t$, we guess

$$y_p = A\cos 3t + B\sin 3t$$

(we need the sin term, since derivs of cos involve sin). Then

$$y_p' = -3A\sin 3t + 3B\cos 3t$$

$$y_p'' = -9A\cos 3t - 9B\sin 3t$$

$$\Rightarrow (-9A\cos 3t - 9B\sin 3t)$$
$$-(-3A\sin 3t + 3B\cos 3t)$$
$$-2(A\cos 3t + B\sin 3t) = 2\cos 3t$$

$$\Rightarrow \underline{(-11A - 3B)}\cos 3t + \underline{(3A - 11B)}\sin 3t$$
$$= \underline{2}\cos 3t + \underline{0}\sin 3t$$

$$\Rightarrow \begin{array}{l} -11A - 3B = 2 \\ 3A - 11B = 0 \end{array} \Rightarrow \begin{array}{l} A = -11/65 \\ B = -3/65 \end{array}$$

so our particular solution is

$$y_p = -\frac{11}{65}\cos 3t - \frac{3}{65}\sin 3t$$

- Furthermore, we can produce guesses for y_p when f is a product of the 3 types of functions we've seen so far:

$$f(t) = 3t^2e^t \Rightarrow y_p = (At^2+Bt+C)e^t$$

$$f(t) = \sin 2t \cdot e^t$$
$$\Rightarrow y_p = (A\cos 2t + B\sin 2t)e^t$$

$$f(t) = t\sin 3t$$
$$\Rightarrow y_p = (At+B)\cos 3t + (Ct+D)\sin 3t$$

- There's one small hitch though. Let's look at

$$y'' - y' - 2y = 5e^{2t}.$$

we guess that $y_p = Ae^{2t}$, so

$$y_p' = 2Ae^{2t}, \quad y_p'' = 4Ae^{2t}$$

$$\Rightarrow (4Ae^{2t}) - (2Ae^{2t}) - 2(Ae^{2t}) = 5e^{2t}$$

$$\Rightarrow \quad 0 = 5e^{2t} \leftarrow \underline{\text{not true}}$$

- The problem is solved via a similar method for repeated characteristic roots: multiply by t. So if

$$y_p = Ate^{2t} \Rightarrow y_p' = (2At+A)e^{2t}$$
$$\Rightarrow y_p'' = (4At+4A)e^{2t}$$

substituting:

$$\left[(4At+4A)e^{2t}\right] - \left[(2At+A)e^{2t}\right] - 2(Ate^{2t})$$
$$= 5e^{2t}$$

$$\Rightarrow 3Ae^{2t} = 5e^{2t} \Rightarrow A = 5/3$$

so our particular solution is

$$y_p(t) = \frac{5}{3}te^{2t}$$

- Why did this happen? Let's look at the homogeneous solution:

$$y'' - y' - 2y = 0$$
$$\Rightarrow r^2 - r - 2 = 0$$
$$\Rightarrow (r-2)(r+1) = 0$$
$$\Rightarrow r = -1, 2$$

so the homogeneous solution is

$$y_h(t) = c_1e^{-t} + c_2e^{2t}$$

- Now, if we guessed that $y_p = Ae^{2t}$, we see that if $c_1 = 0$ and $c_2 = A$, $y_p = y_h$, so of course $y_p'' - y_p' - 2y_p = 0$.
  - Basically, we need y_p to be <u>linearly independent</u> from the terms in y_h, so we had to multiply by t.

<u>Ex:</u>

$$y'' - 2y' + y = 3e^t$$

The homogeneous solution:

$$r^2 - 2r + r = 0$$

$$\Rightarrow (r-1)^2 = 0$$

$$\Rightarrow r = 1$$

$$\Rightarrow y_h(t) = c_1 e^t + c_2 t e^t$$

- We would first guess that $y_p = Ae^t$, but that's identical to one of the terms in y_h.
- Then we guess $y_p = Ate^t$, but that is <u>still</u> identical to one of the terms in y_h.
- Multiply by t again:

$$y_p = At^2 e^t$$

$$\Rightarrow y_p' = 2Ate^t + At^2 e^t$$

$$\Rightarrow y_p'' = 2Ae^t + 4Ate^t + At^2 e^t$$

After substituting into the DE:

$$2Ae^t = 3e^t \Rightarrow A = \frac{3}{2}$$

so the particular solution is:

$$y_p(t) = \frac{3}{2}t^2e^t$$

- The moral of the story:

$\boxed{\text{Always compute } y_h \text{ first!}}$

If y_h contains terms identical to $f(t)$, multiply y_p by t until it is linearly independent.

- Table 4.4.1 on page 252 gives the rundown of how to determine the form of y_p, given the form of $f(t)$.

4.5 Variation of Parameters

- Undetermined coefficients is nice, but what if $f(t)$ is not ~~[illegible]~~ a polynomial, an exponential, or sines and cosines?

- First, find the homogeneous solution:
$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$
Then, we find a particular solution by varying the parameters c_1 + c_2:
$$y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$$
$$\Rightarrow y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$$
  - before going on, we choose an auxiliary condition: v_1 + v_2 satisfy
$$\boxed{v_1' y_1 + v_2' y_2 = 0 \qquad (5a)}$$
  - using (5a), y_p' simplifies to
$$y_p' = v_1 y_1' + v_2 y_2'$$
$$\Rightarrow y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

- Now, substituting y_p into
$$a y_p'' + b y_p' + c y_p = f$$

we get

$$a(v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2')$$
$$+ b(v_1y_1' + v_2y_2') + c(v_1y_1 + v_2y_2) = f$$

- rearranging a bit:

$$v_1(ay_1'' + by_1' + cy_1)$$
$$+ v_2(ay_2'' + by_2' + cy_2)$$
$$+ a(v_1'y_1' + v_2'y_2') = f$$

- Since y_1 and y_2 are the components of the homogeneous solution, we know that

$$ay_1'' + by_1' + cy_1 = 0 \quad \text{and} \quad ay_2'' + by_2' + cy_2 = 0$$

so at the end of the day we're left with

$$\boxed{v_1'y_1' + v_2'y_2' = \frac{f}{a} \qquad (5b)}$$

- So, we have 2 unknowns (v_1 + v_2) and 2 equations:

$$v_1'y_1 + v_2'y_2 = 0$$
$$v_1'y_1' + v_2'y_2' = f/a$$

- this is a 2x2 linear system of equations:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f/a \end{bmatrix}$$

- you can solve this however you want to, but you can always use Cramer's Rule:

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f/a & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f/a \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad (*)$$

- once you determine what v_1' and v_2' are, just integrate to solve for v_1 and v_2.

<u>Note</u>: The denominators in $(*)$ are the Wronskians of $\{y_1, y_2\}$. We know they're not 0 since y_1 and y_2 are linearly independent. Phew!

<u>Ex</u>: Find the general solution to

$$y'' + y = \sec(t)$$