- this is a 2x2 linear system of equations:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f/a \end{bmatrix}$$

- you can solve this however you want to, but you can always use Cramer's Rule:

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f/a & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f/a \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad (*)$$

- once you determine what v_1' and v_2' are, just integrate to solve for v_1 and v_2.

<u>Note</u>: The denominators in $(*)$ are the Wronskians of $\{y_1, y_2\}$. We know they're not 0 since y_1 and y_2 are linearly independent. Phew!

<u>Ex</u>: Find the general solution to

$$y'' + y = \sec(t)$$

- First, we find the homogeneous solution:

$$r^2+1=0 \Rightarrow r=\pm i$$

$$\Rightarrow y_h(t) = c_1 \underbrace{\cos(t)}_{y_1} + c_2 \underbrace{\sin(t)}_{y_2}$$

- Now, the particular solution is of the form

$$y_p(t) = v_1(t)\,y_1(t) + v_2(t)\,y_2(t)$$

- Using equation (*):

$$v_1' = \frac{\begin{vmatrix} 0 & \sin(t) \\ \sec(t) & \cos(t) \end{vmatrix}}{\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}} = \frac{-\frac{\sin(t)}{\cos(t)}}{\cos^2 t + \sin^2 t}$$

$$= -\tan(t)$$

$$v_2' = \frac{\begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \sec(t) \end{vmatrix}}{\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}} = \frac{\frac{\cos(t)}{\cos(t)}}{\cos^2 t + \sin^2 t}$$

$$= 1$$

$$\Rightarrow v_1 = \int -\tan(t)\,dt = -\ln|\sec(t)|$$

$$\Rightarrow v_2 = \int 1\,dt = t$$

Thus:

$$y_p(t) = -\ln|\sec(t)|\cos(t) + t\sin(t)$$

- The steps:

① Determine the homogeneous solution:

$$y_h = c_1 y_1 + c_2 y_2$$

② Solve for v_1' and v_2' in the system

$$y_1 v_1' + y_2 v_2' = 0 \quad (5a)$$
$$y_1' v_1' + y_2' v_2' = f/a \quad (5b)$$

or use Cramer's Rule:

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f/a & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 f/a}{W(y_1,y_2)}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f/a \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f/a}{W(y_1,y_2)}$$

where $W(y_1,y_2) = y_1 y_2' - y_1' y_2$ is the Wronskian

③ Integrate the results of step ②

④ Compute $y_p = v_1 y_1 + v_2 y_2$

- Advantages: variation of parameters allows for a wider range of forcing functions.
- Disadvantages: require integration, ugh.

4.6 Forced Oscillations

- Let's look at mass-spring system with a very particular kind of external forcing:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_f t)$$

- Let's see what happens in the undamped case (i.e. when $b=0$). We know that

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \quad \omega_0 = \sqrt{k/m}$$

  - using either undetermined coefficients, or variation of parameters, we can determine x_p

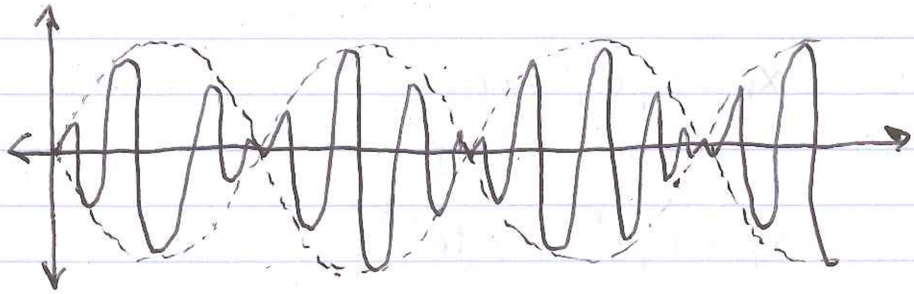
- If $\omega_f \neq \omega_0$, then

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega_f^2)} \cos \omega_f t$$

The general solution is then of the form

$$x(t) = A\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega_f^2)} \cos(\omega_f t)$$

  - this is the sum of two different sinusoidal functions with different frequencies
  - the graph looks something like this:

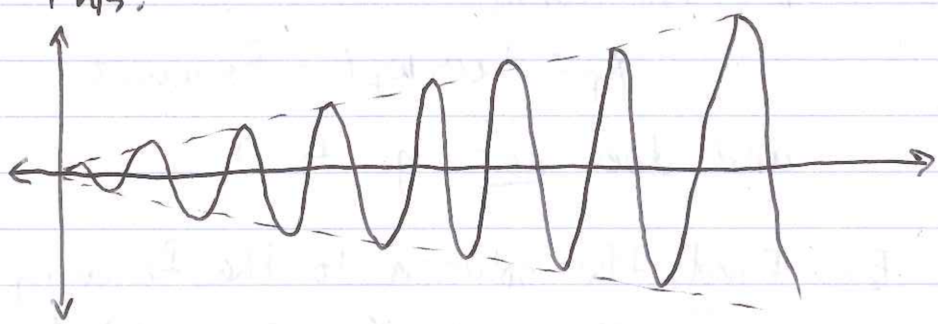


- this results in what is called "beats"

- If $\omega_f = \omega_0$, then

$$x_p(t) = \frac{F_0}{2m\omega_0} \boxed{t} \sin(\omega_0 t)$$

  - when this occurs, it is called "pure resonance," and the graph looks like this:



  - resonance occurs all the time in practice; rattling of the bus at certain speeds, feedback in a mic system, even breaking wine glasses

- What about when the system is damped? When $b > 0$, there is an "e" term in the homogeneous solution:

$$x_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

<u>or</u>

$$x_h = c_1 e^{rt} + c_2 t e^{rt}$$

<u>or</u>

$$x_h = c_1 e^{\alpha t} \cos\beta t + c_2 e^{\alpha t} \sin\beta t$$

- in all these cases, $x_h(t) \to 0$ as $t \to \infty$, because the exponential terms go to zero

- meanwhile, the particular solution is of the form

$$x_p = A\cos\omega_f t + B\sin\omega_f t$$

and does <u>not</u> go to 0.

<u>Ex</u>: Find the solution to the following IVP:

$$\ddot{x} + 4\dot{x} + 5x = 10\cos 3t, \quad x(0) = \dot{x}(0) = 0.$$

- Finding x_h:

$$r^2 + 4r + 5 = 0 \Rightarrow r = -2 \pm i$$

$$\Rightarrow x_h(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$$

- To find x_p, we can use undetermined coefficients:

$$x_p = A\cos 3t + B\sin 3t$$

$$\Rightarrow \dot{x}_p = -3A\sin 3t + 3B\cos 3t$$

$$\Rightarrow \ddot{x}_p = -9A\cos 3t - 9B\sin 3t$$

Substituting into the DE gives.

$$(12B - 4A)\cos 3t + (-12A - 4B)\sin 3t = 10\cos 3t$$

$$\Rightarrow \begin{array}{l} 12B - 4A = 10 \\ -12A - 4B = 0 \end{array} \Rightarrow \begin{array}{l} A = -\frac{1}{4} \\ B = \frac{3}{4} \end{array}$$

So the particular solution is

$$x_p(t) = -\frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t$$

- So, the general solution is then

$$x(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t - \frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t$$

- Using the initial conditions and $\dot{x}(t)$, we can find that

$$c_1 = \frac{1}{4}, \quad c_2 = -\frac{7}{4}$$

- the exact solution is then

$$x(t) = \underbrace{\tfrac{1}{4}e^{-2t}\cos t - \tfrac{7}{4}e^{-2t}\sin t}_{x_h} \underbrace{- \tfrac{1}{4}\cos 3t + \tfrac{3}{4}\sin 3t}_{x_p},$$

- We see that $x_h \to 0$ as $t \to \infty$, so we call x_h the $\boxed{\text{transient}}$ solution.
- x_p does not decay, and the exact solution "approaches" x_p as $t \to \infty$, so x_p is called the $\boxed{\text{steady-state}}$ solution.