5.1 Linear Transformations

- Review vector spaces!

- A $\boxed{\text{linear transformation}}$ is a function $T: V \to W$ (V and W are vector spaces) that preserves addition and scalar multiplication, i.e. for every $\vec{u}, \vec{v} \in V$ and $c \in \mathbb{R}$,

$$\boxed{\begin{aligned} T(c\vec{u}) &= cT(\vec{u}) \\ T(\vec{u}+\vec{v}) &= T(\vec{u})+T(\vec{v}) \end{aligned}} \quad (6)$$

  - V is called the $\boxed{\text{domain}}$ of T
  - W is called the $\boxed{\text{codomain}}$ of T

- You can actually replace the conditions in (6) with one equation:

$$\boxed{T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})} \quad (7)$$

- It is important to note that the codomain is <u>not</u> the same as the image: The $\boxed{\text{image}}$ of a linear transformation $T: V \to W$ is the set of vectors in W which have ~~the element~~ a vector in V which which T maps to it:

$$Im(T) = \{\vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V\}$$

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x,y,z) = (x,y,0)$$

Is T a linear transformation?

- Check equation (6a). Let $(u_1, u_2, u_3) \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} T(c(u_1,u_2,u_3)) &= T(cu_1, cu_2, cu_3) \\ &= (cu_1, cu_2, 0) \\ &= c(u_1, u_2, 0) \\ &= cT(u_1, u_2, u_3) \end{aligned}$$

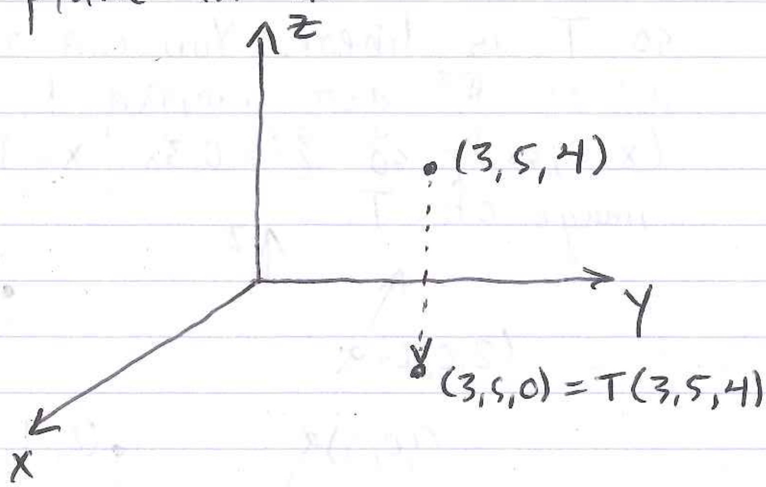
so eqn (6a) holds for T

- Check equation (6b). Let $(u_1,u_2,u_3) \in \mathbb{R}^3$, and $(v_1,v_2,v_3) \in \mathbb{R}^3$. Then

$$\begin{aligned} &T((u_1,u_2,u_3) + (v_1,v_2,v_3)) \\ &\quad = T(u_1+v_1, u_2+v_2, u_3+v_3) \\ &\quad = (u_1+v_1, u_2+v_2, 0) \\ &\quad = (u_1, u_2, 0) + (v_1, v_2, 0) \\ &\quad = T(u_1,u_2,u_3) + T(v_1,v_2,v_3) \end{aligned}$$

Therefore, T satisfies eqn (6b), so T is linear

What is the image of T? All T does is set $z=0$, so $\mathrm{Im}(T)$ is the xy-plane in $\mathbb{R}^3$



- T projects all of $\mathbb{R}^3$ onto the xy-plane

<u>Ex</u>: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x,y,z) = (x, 0, 3x)$$

Is T linear?

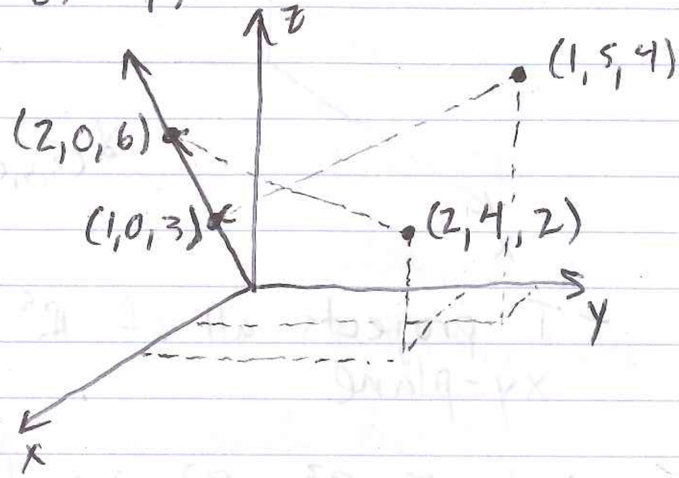
- Let's check equation (7) this time: Let $(u_1, u_2, u_3), (v_1, v_2, v_3) \in \mathbb{R}^3$ and $c, d \in \mathbb{R}$. Then

$$\begin{aligned} &T(c(u_1,u_2,u_3) + d(v_1,v_2,v_3)) \\ &\quad = T(cu_1+dv_1, cu_2+dv_2, cu_3+dv_3) \\ &\quad = (cu_1+dv_1, 0, 3(cu_1+dv_1)) \\ &\quad = (cu_1, 0, 3cu_1) + (dv_1, 0, 3dv_1) \end{aligned}$$

$= c(u_1, 0, 3u_1) + d(v_1, 0, 3v_1)$

$= cT(u_1, u_2, u_3) + dT(v_1, v_2, v_3)$

so T is linear. You can see that all of $\mathbb{R}^3$ gets mapped to the line $(x, 0, 3x)$, so $\{(x,0,3x) \mid x \in \mathbb{R}\}$ is the image of T.



Ex: Let $D: C'(a,b) \to C(a,b)$ be defined by

$$D(f) = f'$$

(here, $C(a,b)$ are the continuous functions on the interval (a,b), and $C'(a,b)$ are the functions with at least 1 continuous derivative on (a,b))

- Check equation (6a): Let $f \in C'(a,b)$, and $c \in \mathbb{R}$. Then

- Is D linear?

$$D(cf(x)) = D((cf)(x))$$
$$= \frac{d}{dx}[cf(x)]$$
$$= c\frac{d}{dx}f(x)$$
$$= cD(f(x))$$

so D satisfies (6a).

- Check equation (6b). Let $f, g \in C'(a,b)$. Then

$$D(f(x)+g(x)) = \frac{d}{dx}[f(x)+g(x)]$$
$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$
$$= D(f(x)) + D(g(x))$$

so D satisfies (6b). Therefore, D is linear.

<u>Ex</u>: Define $I : C(a,b) \to \mathbb{R}$ be defined by

$$I(f(x)) = \int_a^b f(x)\,dx$$

Is I linear?

- Check equation (7). Let $f, g \in C(a,b)$, and let $c, d \in \mathbb{R}$. Then

$$I(cf(x)+dg(x))$$
$$= \int_a^b cf(x)+dg(x)\,dx$$
$$= \int_a^b cf(x)\,dx + \int_a^b dg(x)\,dx$$
$$= c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$$
$$= cI(f(x)) + dI(g(x))$$

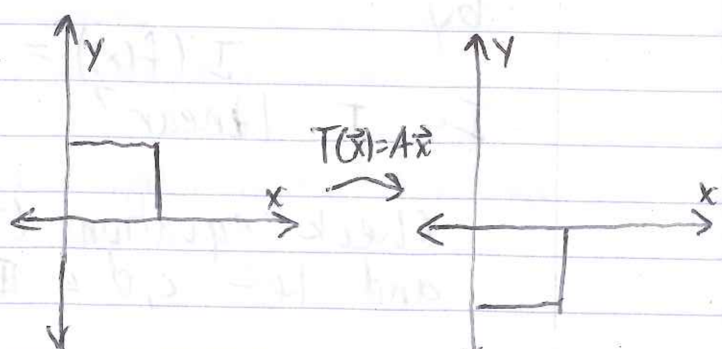
so I is a linear transformation.

- Perhaps the most interesting linear transformations are matrix multiplications: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is defined by
$$T(\vec{x}) = A\vec{x},$$
where A is an $m \times n$ matrix, then T is a linear transformation.

- It's easiest to see how matrix multiplications behave in the 2×2 case:

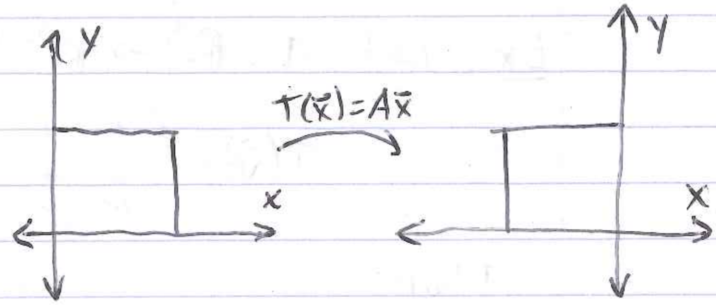
Ex:
Reflection about the y-axis:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



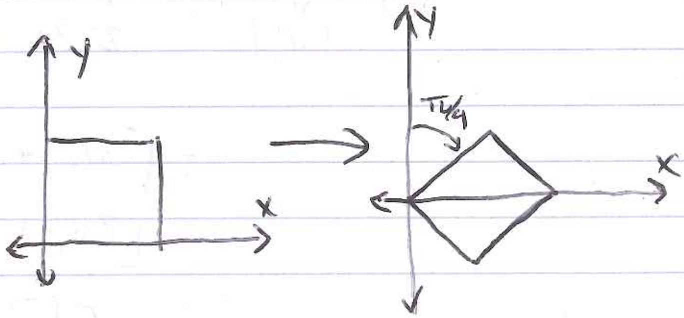
Reflection about the x-axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



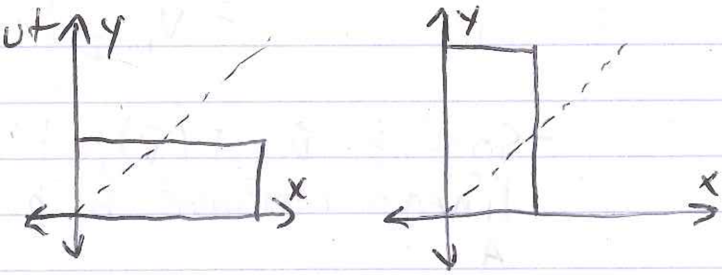
Clockwise rotation of $\pi/4$:

$$A = \begin{bmatrix} \cos\pi/4 & \sin\pi/4 \\ -\sin\pi/4 & \cos\pi/4 \end{bmatrix}$$



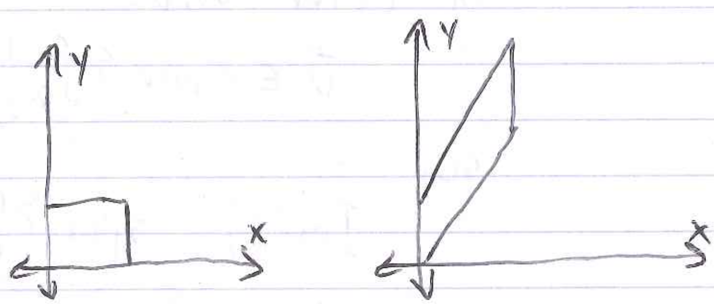
Reflection about the line $y=x$:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Shear of 2 in the y-direction

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$



- How do you find the image of a matrix multiplication?