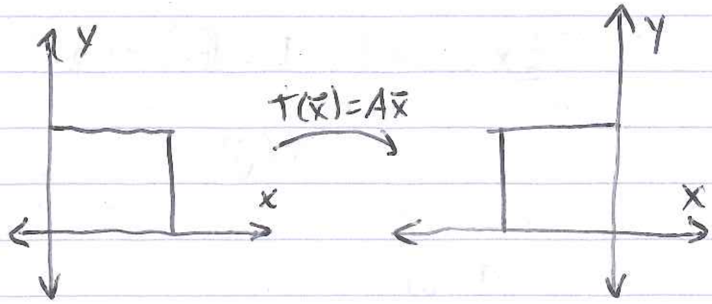
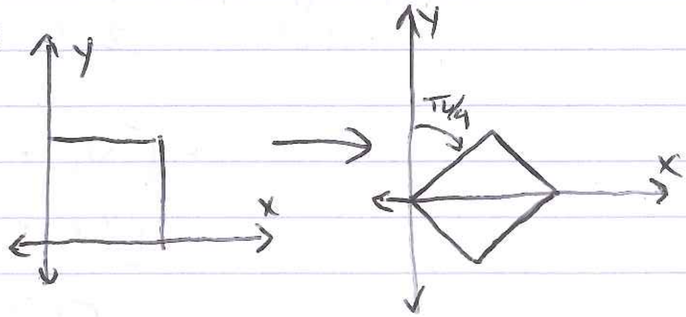
Reflection about the x-axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



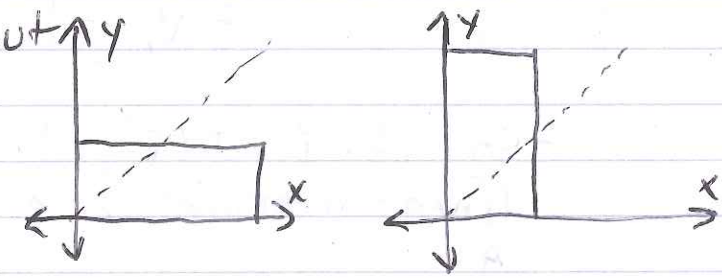
Clockwise rotation of $\pi/4$:

$$A = \begin{bmatrix} \cos \pi/4 & \sin \pi/4 \\ -\sin \pi/4 & \cos \pi/4 \end{bmatrix}$$



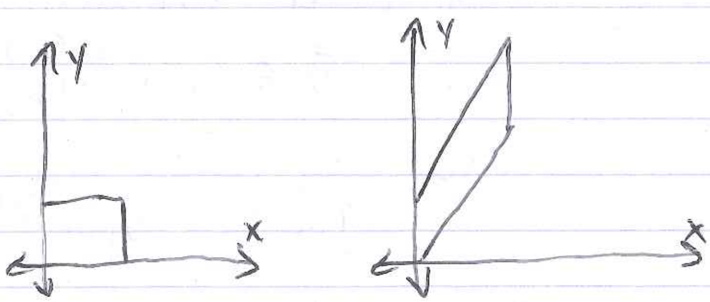
Reflection about the line $y = x$:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Shear of 2 in the y-direction

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$



- How do you find the image of a matrix multiplication?

<u>Ex</u>: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(\vec{v}) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \vec{v}$$

Then

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 + v_2 + 2v_3 \\ 2v_1 + 3v_2 + 5v_3 \end{bmatrix}$$

$$= v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + v_3 \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

- So, if $\vec{u} = T(\vec{v})$, then $\vec{u}$ is a linear combination of the columns of A
- in other words,

$$\vec{u} \in \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\},$$

so

$$\text{Im}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$$

- This is true for any matrix multiplication: If $T(\vec{v}) = A\vec{v}$, then

$$\text{Im}(T) = \text{span}\{\text{columns of } A\}$$
$$= \text{Col}(A) \quad (\text{column space of } A)$$

- We already know that matrix multiplications are linear, but it turns out that <u>any</u> linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ determines a unique matrix multiplication:

- Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The $\boxed{\text{standard matrix}}$ associated with T is defined by

$$A = \left[T(\vec{e}_1) \,\middle|\, T(\vec{e}_2) \,\middle|\, \cdots \,\middle|\, T(\vec{e}_n) \right]$$

where the columns $T(\vec{e}_i)$ are the images of the standard basis $\vec{e}_1, \vec{e}_2, \ldots$

<u>Ex</u>: Let $T(x,y) = (x-y, x+y, 2x)$. Find the standard matrix for T.

- Since $T: \mathbb{R}^2 \to \mathbb{R}^3$, we know the matrix A must be 3×2. (2 columns)

$$T(\vec{e}_1) = T(1,0) = (1,1,2) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$T(\vec{e}_2) = T(0,1) = (-1,1,0) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

so the standard matrix is:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

- Sometimes you can skip calculating $T(\vec{e}_i)$:

$$T(x,y) = (x-y,\ x+y,\ 2x)$$

$$\iff T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \\ 2x \end{bmatrix} = \begin{bmatrix} 1x - 1y \\ 1x + 1y \\ 2x + 0y \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

<u>Ex</u>: Consider $\mathbb{P}_3$, the vector space of all polynomials of degree 3 or less. An arbitrary "vector" in $\mathbb{P}_3$ is:

$$a_3x^3 + a_2x^2 + a_1x + a_0 = \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

- Similarly, an arbitrary "vector" in $\mathbb{P}_1$ is

$$a_1 x + a_0 = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

Let $D: \mathbb{P}_3 \to \mathbb{P}_1$ be the second derivative operator:

$$D(a_3x^3 + a_2x^2 + a_1x + a_0) = 6a_3x + 2a_2$$

What is the standard matrix for D?

i.e. find A such that

$$A\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 6a_3 \\ 2a_2 \end{bmatrix}$$

$$D\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad D\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$D\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

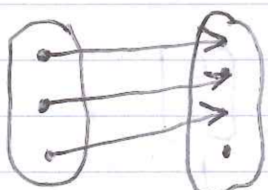
Therefore,

$$A = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

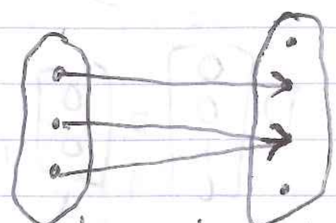
5.2 Properties of Linear Transformations

- A function $f: X \to Y$ is $\boxed{\text{one-to-one}}$ or $\boxed{\text{injective}}$ if $f(u) = f(v)$ implies $u = v$. That is, different inputs always give different outputs.

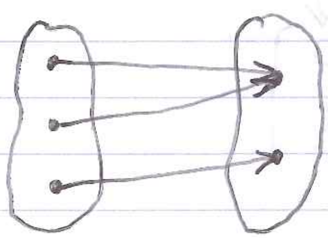
- A function $f: X \to Y$ is $\boxed{\text{onto}}$ or $\boxed{\text{surjective}}$ if the image of f is *all* of Y.



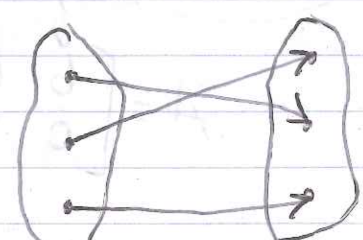
one-to-one, not onto



not one-to-one, not onto



not one-to-one, onto



one-to-one, onto

- Linear transformations have some nice properties:

  - Let $T: V \to W$ be a linear transformation. Then $\mathrm{Im}(T)$ is a *subspace* of W, i.e. $\mathrm{Im}(T)$ is *also* a vector space

<u>Ex</u>: Define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

The image of T is all vectors of the form

$$\vec{u} = A\vec{v} = v_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + v_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

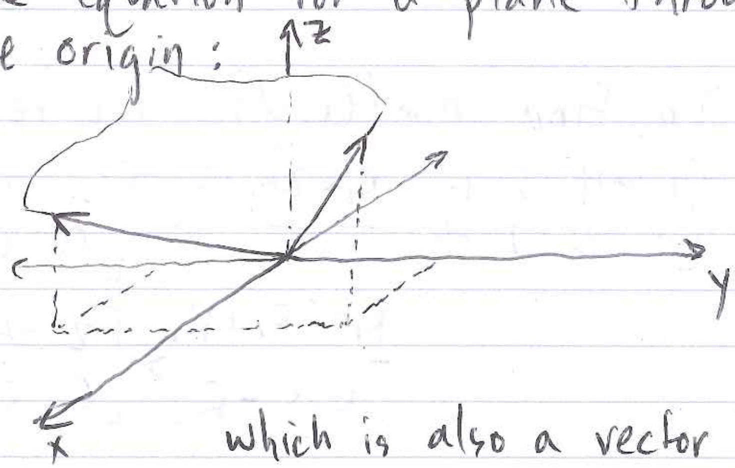
which is the column space of A, which we know is a vector space. Also, we can get three equations:

$$u_1 = v_1 + v_2 \quad u_2 = v_1 - v_2 \quad u_3 = 2v_1 + v_2$$

Eliminating v_1 and v_2, we get

$$3\underset{x}{u_1} + \underset{y}{u_2} - 2\underset{z}{u_3} = 0$$

the equation for a plane through the origin:



which is also a vector space!

- Since $Im(T)$ is a vector space, we can talk about its dimension: the dimension of $Im(T)$ is called its $\boxed{\text{rank}}$

$$\text{rank}(T) = \dim(Im(T))$$

- In the case of matrix multiplications, (i.e. $T(\vec{v}) = A\vec{v}$), we can calculate the rank pretty easily. Since

$$Im(T) = Col(A),$$

we know that

$$\begin{aligned} \text{rank}(T) &= \dim(Im(T)) \\ &= \dim(Col(A)) \\ &= \#\text{ of pivot columns in } A \end{aligned}$$

Ex: Let $T: \mathbb{R}^4 \to \mathbb{R}^2$ be

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

To find $\dim(Col(A))$, row reduce A:

$$\begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} \xrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{bmatrix} 2 & -4 & 3 & 6 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow 3R_2 + R_1} \begin{bmatrix} 2 & -4 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow \frac{1}{2} R_1} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

pivot columns
(leading 1's)

So, A has 2 pivot columns, thus
rank(T) = 2

- Another <u>very</u> <u>important</u> aspect of linear transformations: The $\boxed{\text{kernel}}$ or $\boxed{\text{nullspace}}$ of a linear transformation $T: V \to W$, denoted $\ker(T)$, is the set of vectors in V which get mapped to $\vec{0}$ in W.

$$\ker(T) = \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \}$$

<u>Ex</u>: Let $T(x,y,z) = (x,y,0)$. What is the kernel of T?

$$T(x,y,z) = (0,0,0)$$

$$\Rightarrow (x,y,0) = (0,0,0)$$

$$\Rightarrow x=0,\ y=0,\ z \text{ free}$$

$$\Rightarrow \ker(T) = \{(0,0,z) \mid z \in \mathbb{R}\}$$