$$\xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

pivot columns
(leading 1's)

So, A has 2 pivot columns, thus
$\text{rank}(T) = 2$

- Another <u>very</u> <u>important</u> aspect of linear transformations: The $\boxed{\text{kernel}}$ or $\boxed{\text{nullspace}}$ of a linear transformation $T: V \to W$, denoted $\text{Ker}(T)$, is the set of vectors in V which get mapped to $\vec{0}$ in W.

$$\text{Ker}(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$$

<u>Ex</u>: Let $T(x,y,z) = (x,y,0)$. What is the kernel of T?

$$T(x,y,z) = (0,0,0)$$
$$\Rightarrow (x,y,0) = (0,0,0)$$
$$\Rightarrow x=0,\ y=0,\ z \text{ free}$$
$$\Rightarrow \text{Ker}(T) = \{(0,0,z) \mid z \in \mathbb{R}\}$$

<u>Ex</u>: Let

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

What does T map to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$? I.e. what values of v_1, v_2, v_3 satisfy

$$\begin{aligned} v_1 + v_2 + 2v_3 &= 0 \\ 2v_1 + 3v_2 + 5v_3 &= 0 \end{aligned}$$

This is equivalent to the augmented system

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 3 & 5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{aligned} v_1 \quad + v_3 &= 0 \\ v_2 + v_3 &= 0 \end{aligned}$$

v_3 is free $\to$ $v_3 = r$

$$\Rightarrow \begin{aligned} v_1 &= -r \\ v_2 &= -r \\ v_3 &= r \end{aligned} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

so,

$$\ker(T) = \left\{ r\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \middle| \ r \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

<u>Ex</u>: Let T_A, T_B, T_C be defined by

$$T_A(\vec{v}) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{v}, \quad T_B(\vec{v}) = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\vec{v}, \quad T_C(\vec{v}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\vec{v}$$

(for A): Solving the augmented system

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

so $v_1 = 0$, $v_2 = 0$, thus $\ker\{T_A\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

(for B): Solving the augmented system

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

we see that $v_1 + \frac{1}{2}v_2 = 0$, v_2 is free.
Setting $v_2 = r$, we get

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

so $\ker(T_B) = \left\{ r\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \middle| \ r \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$

(for C): No matter what vector $\vec{v}$ we supply, $T_c(\vec{v}) = \vec{0}$, so

$$\ker(T_c) = \mathbb{R}^2$$

- It turns out that $\ker(T)$ is <u>also</u> a subspace. (so it is a vector space)
- Also, a linear transformation T is one-to-one if and only if $\ker(T) = \{\vec{0}\}$
- Since $\ker(T)$ is a vector space, we can talk about its dimension: the dimension of $\ker(T)$ is called the $\boxed{\text{nullity}}$ of T.

<u>Ex</u>: What is the nullity of $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} ?$$

First we find the kernel:

$$\left[\begin{array}{cccc|c} 2 & -4 & 3 & 6 & 0 \\ -1 & 2 & -2 & -3 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

we see that

$$\begin{aligned} v_1 - 2v_2 \quad + 3v_4 &= 0 \\ v_3 \quad &= 0 \end{aligned}$$

v_2, v_4 are free.

Setting $v_2 = r$, $v_4 = s$, we get

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 2r - 3s \\ 1r + 0s \\ 0r + 0s \\ 0r + 1s \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So

$$\mathrm{Ker}(T) = \left\{ r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; r, s \in \mathbb{R} \right\}$$

$$= \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thus, $\dim(\mathrm{Ker}(T)) = 2$. In other words, the number of free variables in the row-reduced system.

- This leads to the $\boxed{\text{dimension theorem:}}$

$$\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = \dim(V)$$

for every linear transformation $T: V \to W$

- Why do we care so much about kernels, images, and their dimensions? We've actually been finding kernels the whole class!

Ex: Let $T: C^2(a,b) \to C(a,b)$ be defined by

$$T(y) = y'' - 2y' + y$$

What is the kernel of T?

$$T(y) = 0$$
$$\Rightarrow y'' + 2y' + y = 0$$
$\Rightarrow$ characteristic roots:
$$r^2 + 2r + 1 = 0$$
$$\rightarrow (r+1)^2 = 0$$
$$\rightarrow r = -1$$
$$\Rightarrow y = c_1 e^{-t} + c_2 t e^{-t}$$

And so:

$$\text{Ker}(T) = \{c_1 e^{-t} + c_2 t e^{-t} \mid c_1, c_2 \in \mathbb{R}\}$$
$$= \text{span}\{e^{-t}, te^{-t}\}$$

★ Finding the kernel of a linear differential operator is <u>exactly the same</u> as finding the homogeneous solution!

• As it turns out, you can extend the nonhomogeneous principle for DE's to general linear transformations:

Nonhomogeneous Principle for Linear Transformations:

Let $T: V \to W$ be a linear transformation, and suppose that $\vec{v}_p$ is one particular solution to the nonhomogeneous problem

$$T(\vec{v}) = \vec{b}$$

Then the set S of all solutions is given by

$$S = \{\vec{v}_p + \vec{v}_h \mid \vec{v}_h \in Ker(T)\}$$

Ex: Given

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 4 \\ x_1 + 2x_2 + 5x_3 &= 6 \end{aligned}$$

we can describe this as $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

To solve this system, we write the augmented system

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 6 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

so we have

$$\begin{aligned} x_1 + \phantom{x_2 +{}} x_3 &= 2 \\ x_2 + 2x_3 &= 2 \end{aligned}$$

x_3 is free

Letting $x_3 = r$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - r \\ 2 - 2r \\ 0 + 0r \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}}_{\vec{v}_p} + r \underbrace{\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}}_{\vec{v}_n}$$