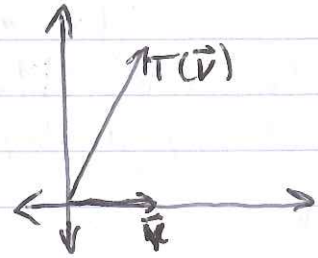
# 5.3 Eigenvalues and Eigenvectors

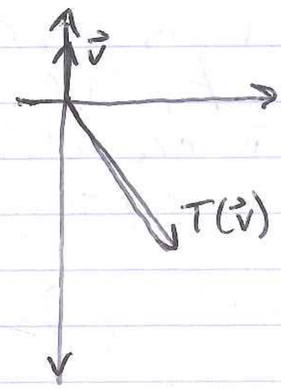
- Let's start with an example: Let $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

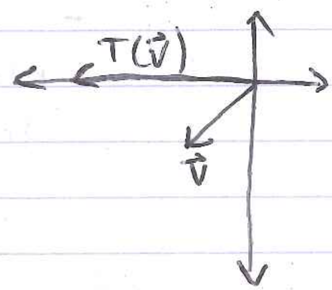
In general, $T$ maps vectors $\vec{v}$ in a different direction than $\vec{v}$:




$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

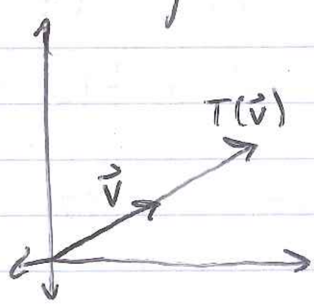



$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

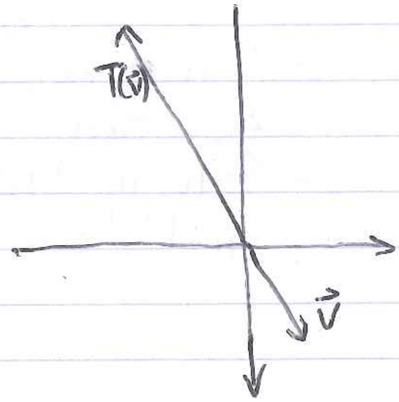



$$T\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

But, there are a few "lucky" vectors which get mapped to the same direction:




$$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$




$$T\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

- Let $T: V \to W$ be a linear transformation. A scalar $\lambda$ is an eigenvalue of $T$ if there is a nonzero vector $\vec{v} \in V$ such that

$$T(\vec{v}) = \lambda \vec{v}$$

The vector $\vec{v}$ is called an eigenvector of $T$ corresponding to $\lambda$.

  - If $T$ is represented by an $n \times n$ matrix $A$, (i.e. $T(\vec{v}) = A\vec{v}$) then $\lambda$ and $\vec{v}$ are characterized by

$$A\vec{v} = \lambda \vec{v}$$

- For matrices, $A\vec{v} = \lambda\vec{v}$ is equivalent to

$$(A - \lambda I)\vec{v} = \vec{0}$$

We want nonzero $\vec{v}$ solutions to this equation. This only happens when $A - \lambda I$ is nonsingular, or equivalently

$$\boxed{|A - \lambda I| = 0 \qquad (8)}$$

  - This equation is called the characteristic equation
  - Solving the characteristic equation will yield the eigenvalues $\lambda$.

- Steps for finding eigenstuff for a matrix $A$:

  ① Write down the characteristic equation $|A-\lambda I|=0$

  ② Solve the characteristic eqn to obtain the eigenvalues

  ③ For each eigenvalue $\lambda_i$, find the eigenvector $\vec{v}_i$ by solving the system

$$(A-\lambda_i I)\vec{v}_i = \vec{0}$$

<u>EX:</u> Find the eigenstuff for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

① The characteristic equation:

$$|A-\lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow (1-\lambda)(-2-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 + \lambda - 6 = 0$$

② Solving:

$$\lambda^2 + \lambda - 6 = 0$$
$$\Rightarrow (\lambda+3)(\lambda-2) = 0$$
$$\Rightarrow \lambda = 2, -3$$

so our eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -3$

③ for $\lambda_1 = 2$:

$$(A - \lambda_1 I)\vec{v} = \vec{0}$$
$$\Rightarrow \begin{bmatrix} 1-2 & 2 \\ 2 & -2-2 \end{bmatrix} \vec{v} = \vec{0}$$
$$\Rightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$
$$\Rightarrow v_1 - 2v_2 = 0$$

$v_2$ is free $\rightarrow v_2 = r$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r \begin{bmatrix} +2 \\ 1 \end{bmatrix}$$
$$\Rightarrow \vec{v}_1 = \begin{bmatrix} +2 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -3$:

$$(A - \lambda_2 I)\vec{v} = \vec{0}$$
$$\Rightarrow \begin{bmatrix} 1+3 & 2 \\ 2 & -2+3 \end{bmatrix} \vec{v} = \vec{0}$$

$$\Rightarrow \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow v_1 + \frac{1}{2} v_2 = 0$$

$v_2$ is free $\Rightarrow v_2 = r$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- From the first example, we already found that the eigenvectors were

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

but we got

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

What gives? Eigenvectors are <u>not unique</u>! Any scalar multiple of an eigenvector is <u>still</u> an eigenvector. I.e. if

$$A\vec{v} = \lambda \vec{v} \Rightarrow cA\vec{v} = \lambda \vec{v}$$

$$\Rightarrow A(c\vec{v}) = \lambda (c\vec{v})$$

so our $\vec{v}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ is the same as the first e-vector we found:

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

Ex: Find the eigenstuff for

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Step ①:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(-1-\lambda) - 1\right]$$
$$+ 1\left[(-1-\lambda) + 2\right]$$
$$+ 0\left[1 + 2(2-\lambda)\right] = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda-2)(\lambda-1)(\lambda+1) = 0$$

Step ②: the eigenvalues are
$\lambda = 2, 1, -1$

Step ③: for $\boxed{\lambda = 2}$:

$$(A-2I)\vec{v}=\vec{0} \Rightarrow \begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3$ free $\Rightarrow v_3 = r$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = r\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \boxed{\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}}$$

for $\boxed{\lambda = 1}$:

$$(A-I)\vec{v}=\vec{0} \Rightarrow \begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3$ free $\rightarrow v_3 = r$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = r\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \boxed{\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$

for $\lambda = -1$:

$$(A+I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3$ free $\rightarrow v_3 = r$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- There are a few shortcuts for finding eigenvalues
  - If the matrix is upper or lower triangular, then the eigenvalues are on the main diagonal:

<u>Ex:</u>

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & -4 \end{bmatrix} \rightarrow \lambda = 1, 2, -4$$

  - If the matrix is $2 \times 2$,

$$|A - \lambda I| = \lambda^2 - \underbrace{(\mathrm{Tr} A)}_{\text{trace of } A \text{ = sum of diagonal}} \lambda + |A|$$

Ex: If $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$$\Rightarrow |A - \lambda I| = \lambda^2 - (-2-2)\lambda + (4-1)$$
$$= \lambda^2 + 4\lambda + 3$$

- In all the previous examples, we saw that all vectors of the form $r\vec{v}$, where $\vec{v}$ is an eigenvector corresponding to $\lambda$, solve the eqn
$$A(r\vec{v}) = \lambda(r\vec{v})$$
- Actually, this says that anything in span$\{\vec{v}\}$ satisfies the eqn. Spans always form subspaces, so we can define what's called an eigen space
- For each eigen value $\lambda$ of the linear transformation $T: V \to W$, the $\boxed{\text{eigenspace}}$
$$E_\lambda = \{\vec{v} \in V \mid T(\vec{v}) = \lambda\vec{v}\}$$
is a subspace of $V$.

Ex: For
$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$
we found the eigenvectors

$\lambda = 2 \to \vec{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ $\lambda = 1 \to \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\lambda = -1 \to \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

so the eigenspaces are

$$\mathbb{E}_{\lambda=2} = \{ \vec{v} \in \mathbb{R}^3 \mid A\vec{v} = 2\vec{v} \}$$
$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

$$\mathbb{E}_{\lambda=1} = \{ \vec{v} \in \mathbb{R}^3 \mid A\vec{v} = 1 \cdot \vec{v} \}$$
$$= \text{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$E_{\lambda=-1} = \{ \vec{v} \in \mathbb{R}^3 \mid A\vec{v} = -1 \cdot \vec{v} \}$$
$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- In each of the previous cases, $\mathbb{E}_\lambda$ was a one dimensional subspace (i.e. only 1 basis element). This isn't always the case:

Ex: Find the eigen stuff for

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

① $|A - \lambda I| = 0 \rightarrow \lambda(\lambda+3)^2 = 0$

② $\lambda = 0, -3$ ($-3$ is a double root)

③ For $\lambda = 0$:

$$(A - 0 \cdot I)\vec{v} = \vec{0} \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \vec{v} = r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow E_{\lambda=0} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

For $\lambda = -3$:

$$(A + 3I)\vec{v} = \vec{0} \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \vec{v} = r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow E_{\lambda=-3} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

- the eigenspace for $\lambda = -3$ is 2-dimensional