Midterm Review

- A mass-spring system (harmonic oscillator) has the following equation of motion:

$$\boxed{m\ddot{x} + b\dot{x} + kx = f(t)}$$

  - m is the mass of the object
  - b is the damping constant
  - k is the spring constant
  - $f(t)$ is the external forces acting on the system

- This is precisely equivalent to the DE

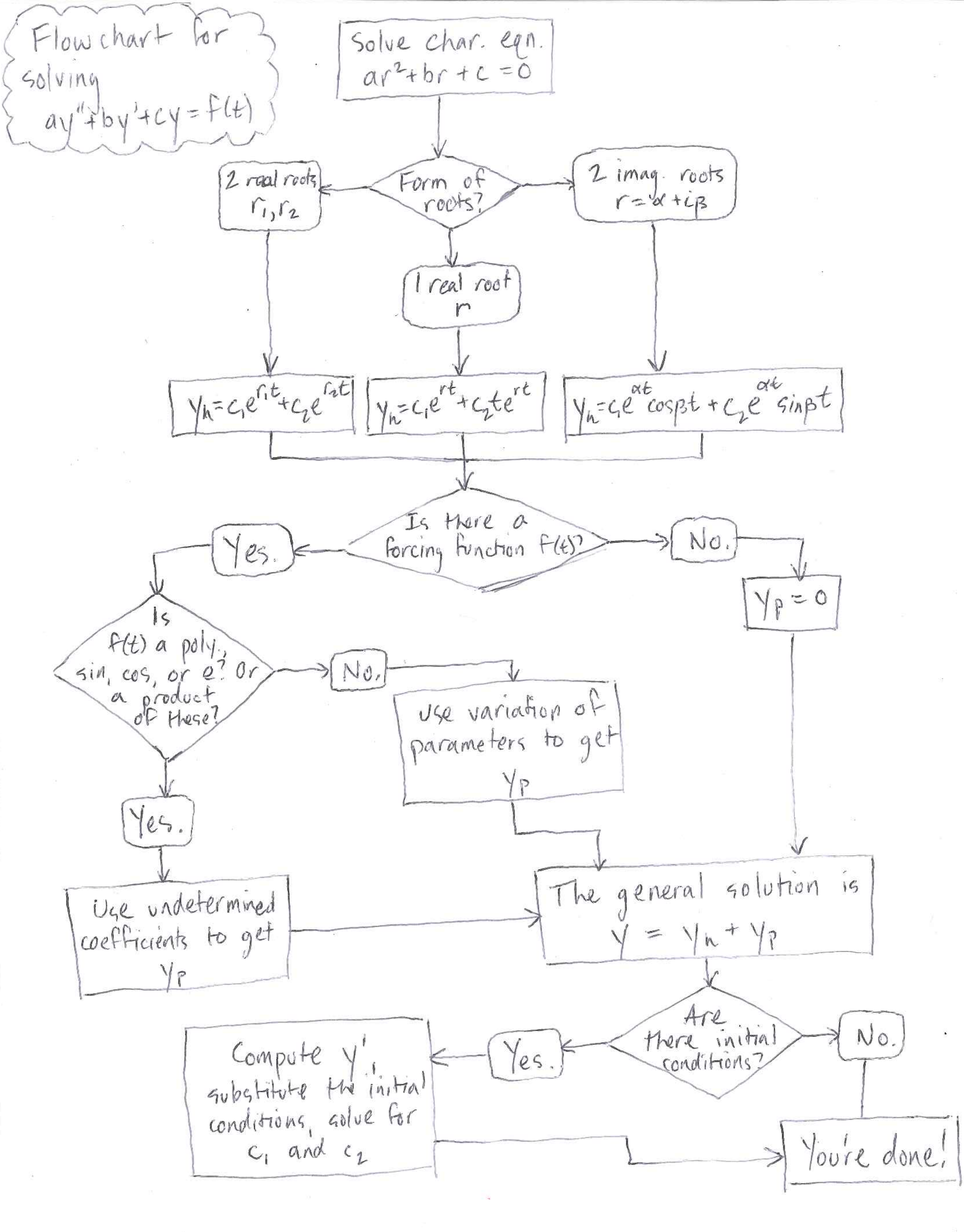
$$\boxed{ay'' + by' + cy = f(t)}$$

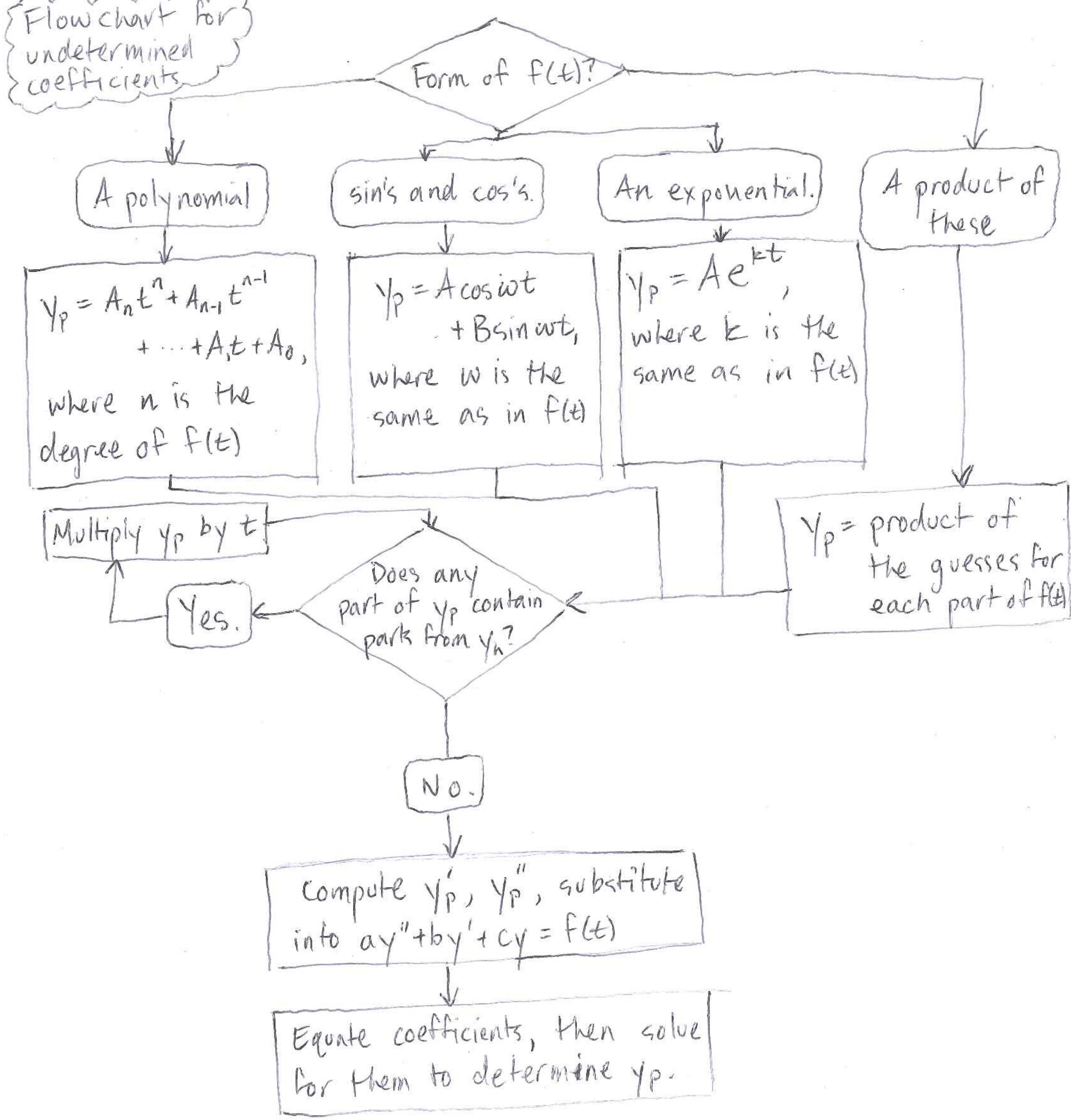
- Sometimes there are initial conditions attached to the problem:

$$\boxed{ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1}$$

practice problems:
4.2
1-22
4.3
1-16

- See flow charts for how to solve these types of problems


Flow chart for solving $ay''+by'+cy=f(t)$
Solve char. eqn. $ar^2+br+c=0$
Form of roots?
2 real roots r_1, r_2
1 real root r
2 imag. roots $r=\alpha+i\beta$
$y_h=c_1e^{r_1t}+c_2e^{r_2t}$
$y_h=c_1e^{rt}+c_2te^{rt}$
$y_h=c_1e^{\alpha t}\cos\beta t+c_2e^{\alpha t}\sin\beta t$
Is there a forcing function $f(t)$?
Yes.
No.
$y_p=0$
Is $f(t)$ a poly., sin, cos, or e? Or a product of these?
No.
Use variation of parameters to get y_p
Yes.
Use undetermined coefficients to get y_p
The general solution is $y=y_h+y_p$
Are there initial conditions?
Yes.
No.
Compute y', substitute the initial conditions, solve for c_1 and c_2
You're done!



Flow chart for undetermined coefficients
Form of f(t)?
A polynomial
sin's and cos's.
An exponential.
A product of these
$y_p = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$, where n is the degree of f(t)
$y_p = A\cos\omega t + B\sin\omega t$, where ω is the same as in f(t)
$y_p = Ae^{kt}$, where k is the same as in f(t)
y_p = product of the guesses for each part of f(t)
Multiply y_p by t
Does any part of y_p contain part from y_h?
Yes.
No.
Compute y_p', y_p'', substitute into $ay''+by'+cy = f(t)$
Equate coefficients, then solve for them to determine y_p.

practice problems:
4.4
17 - 52

4.5
1 – 12

- How to use variation of parameters to get y_p:

① Find $y_h = c_1 y_1 + c_2 y_2$

② Compute the Wronskian $W(y_1, y_2)$:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

and then compute

$$v_1' = \frac{-y_2 \cdot \frac{f}{a}}{W(y_1, y_2)}, \quad v_2' = \frac{y_1 \cdot \frac{f}{a}}{W(y_1, y_2)}$$

③ Integrate v_1' and v_2' to get v_1 and v_2.

④ $y_p = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$

Practice problems:
4.6
1-6
20-22

- Steady-state solutions: these are the parts of the general solution that remain as $t \to \infty$.
  - if y_h contains e terms w/ negative exponents, then the steady-state soln simplifies to y_p

- Converting to phase-amplitude form:

Given a function of the form

$$c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

you can convert to

$$A \cos(\omega t - \delta)$$

using the transformations

$$\boxed{A = \sqrt{c_1^2 + c_2^2} \qquad \tan \delta = \frac{c_2}{c_1}}$$

practice problems:
1-16
33-40
(5.1)

- Linear Transformations:

  - To determine if $T: V \to W$ is a linear transformation:

    ① Check the vector addition property: pick two general vectors in V (Ex: if $V = \mathbb{R}^2$, pick $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} \in \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$)

    - Compute $T(\vec{u} + \vec{v})$ and $T(\vec{u}) + T(\vec{v})$
    - If they're equal, T preserves vector addition

    ② Check the scalar multiplication property: pick a general vector $\vec{v}$ in V, and a scalar $c \in \mathbb{R}$

- Compute $T(c\vec{v})$ and $cT(\vec{v})$
- If they're equal, T preserves scalar multiplication.

③ If T preserves <u>both</u> vector addition and scalar multiplication, the T is a linear transformation

practice problems: ~~\#~~ 5.2 1-4 21-40

- <u>Images, Kernels, Rank, and Nullity</u>

- The image of a linear transformation $T: V \to W$ is

$$\boxed{Im(T) = \{\vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V\}}$$

- For matrix multiplications $(T(\vec{v}) = A\vec{v})$, it's simpler:

$$\boxed{Im(T) = Col(A) = span\{\text{columns of } A\}}$$

- To find a basis for the image of a matrix multiplication $T(\vec{v}) = A\vec{v}$:

① Row reduce A.
② Identify the pivot columns of A
③ The basis for $Im(T)$ is the columns of A which correspond to the pivot columns.

- The rank of $T: V \to W$ is defined by

$$\boxed{\text{rank}(T) = \dim(\text{Im}(T))}$$

- For matrix multiplications $T(\vec{v}) = A\vec{v}$,

$$\boxed{\begin{aligned}\text{rank}(T) &= \dim(\text{Col}(A)) \\ &= \#\text{ of pivot columns}\end{aligned}}$$

- the kernel of $T: V \to W$ is defined by

$$\boxed{\text{Ker}(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}}$$

- for matrix multiplications $T(\vec{v}) = A\vec{v}$, to find a basis for Ker(T):

① Solve the equation $A\vec{v} = \vec{0}$, using row reduction techniques

② The basis is the set of vectors corresponding to each free variable in the solution to ①.

- The nullity of $T: V \to W$ is defined by

$$\boxed{\text{nullity}(T) = \dim(\text{Ker}(T))}$$

- For matrix multiplications $T(\vec{v}) = A\vec{v}$,

$$\boxed{\begin{aligned}\text{nullity}(T) &= \#\text{ of free variable in solution of } A\vec{v} = 0 \\ &= \#\ \underline{\text{non}}\text{ pivot columns}\end{aligned}}$$