• You might think that repeated eigenvalues ~~have~~ always have multiple eigenvectors, but this isn't the case:

Ex: Find the eigenstuff for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

① + ② We know the eigenvalue is $\lambda = 1$, since A is upper triangular.

③

$$(A - I)\vec{v} = \vec{0} \xrightarrow{RREF} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

v_1 free $\Rightarrow v_1 = r$

$$\rightarrow \vec{v} = r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow E_{\lambda=1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- even though $\lambda = 1$ is a repeated eigenvalue, its eigenspace is only 1-dimensional

• One thing that's always true of eigenstuff: If $\lambda_1, \lambda_2, \dots, \lambda_p$ are distinct eigenvalues w/ corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a set of linearly independent vectors.

- $|A-\lambda I| = 0$ is always a polynomial equation, so it is possible to have <u>imaginary</u> eigenvalues:

<u>Ex</u>: Find the eigenstuff for

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

① $|A-\lambda I| = 0 \to \lambda^2 + 1 = 0$

② $\lambda = \pm i$

③ For $\lambda = i$:

$$(A - iI)\vec{v} = 0 \Rightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\to \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r\begin{bmatrix} -i \\ 1 \end{bmatrix} \to \vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

For $\lambda = -i$:

$$(A + iI)\vec{v} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r \begin{bmatrix} i \\ 1 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

- Here's the coolest part yet though: we've actually been ~~doing~~ finding eigen values before this section.

<u>Ex</u>: Consider the DE

$$y'' - y' - 2y = 0$$

Characteristic equation:

$$r^2 - r - 2 = 0$$

with characteristic roots: $r = 2, -1$

to give general solution $y(t) = c_1 e^{2t} + c_2 e^{-t}$.

We can also convert the ~~system~~ equation to a 2x2 system: Let $x_1 = y$, $x_2 = y'$. Then

$$x_1' = y' = x_2$$

$$x_2' = y'' = y' + 2y = x_2 + 2x_1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Finding the e-stuff for A:

① $|A-\lambda I| = -\lambda(1-\lambda)-2 = \lambda^2-\lambda-2=0$

② $(\lambda-2)(\lambda+1)=0 \rightarrow \lambda=2,-1$

☆ The e-values are exactly the same as the characteristic roots!

But wait, there's more! We know $y=e^{2t}$ is a component of the general soln. This corresponds to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

Now,

$$A\begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix} = 2\begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

so $\begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$ is actually an <u>eigenvector</u>!

similarly, corresponding to $y=e^{-t}$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

is the other eigenvector.

5.4 Coordinates and Diagonalization

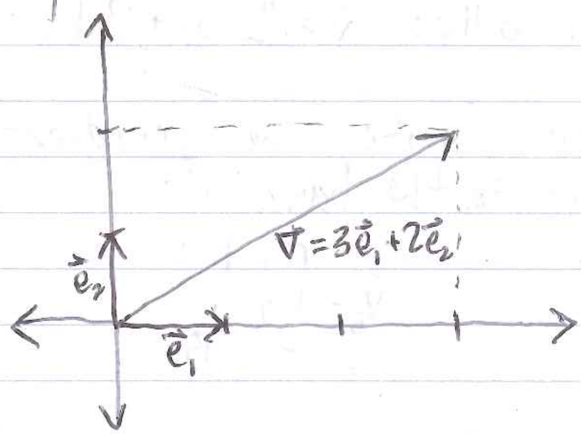
- Normally, when we write a vector, e.g.

$$\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

we assume we're using the standard basis:

$$S = \left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$$

graphically:



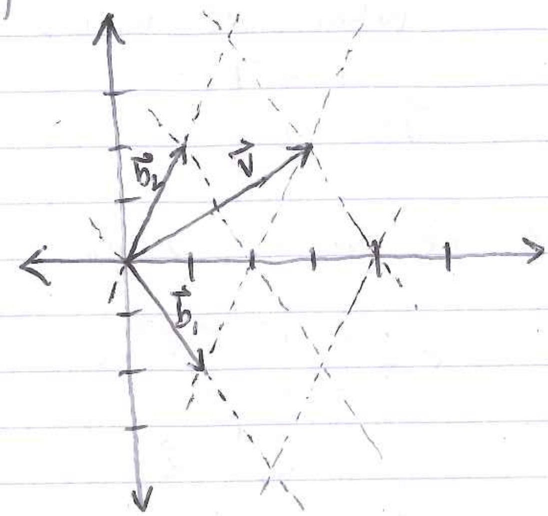
algebraically: $\vec{v} = 3\vec{e}_1 + 2\vec{e}_2$

- Really though, we can do this for <u>any</u> basis, eg:

$$\cancel{B = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}}$$

$$B = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} +1 \\ -2 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$$

graphically:



algebraically: $\vec{v} = 1 \cdot \vec{b}_1 + 2\vec{b}_2$

- We can write the coordinate vector of $\vec{v}$ relative to the basis B:

$$\vec{v}_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_B$$

- More generally: Let $\vec{v}$ be a vector in a vector space V, with basis $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$. The $\boxed{\text{coordinates}}$ of $\vec{v}$ relative to B are the constants $\beta_1, \beta_2, \ldots, \beta_n$ such that

$$\vec{v} = \beta_1 \vec{b}_1 + \beta_2 \vec{b}_2 + \ldots + \beta_n \vec{b}_n$$

The $\boxed{\text{coordinate vector}}$ of $\vec{v}$ relative to B is

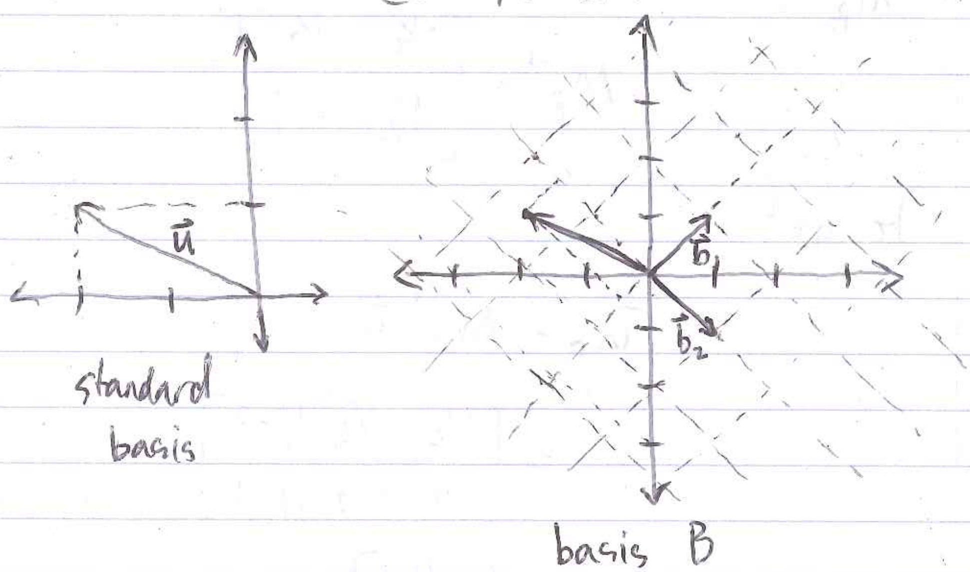
$$\vec{v}_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}_B$$

<u>Ex</u>: Let $\vec{u}_s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_s$, relative to the standard basis, i.e;

$$\vec{u}_s = -2\cdot\vec{e}_1 + 1\cdot\vec{e}_2$$

Consider the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$



so in the basis B:

$$\vec{u}_B = \begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix}_B$$