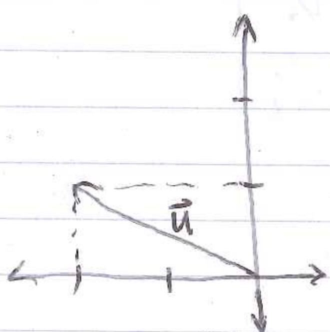
$$\vec{v}_B = \begin{bmatrix}\beta_1\\ \beta_2\\ \vdots \\ \beta_n\end{bmatrix}_B$$

<u>Ex</u>: Let $\vec{u}_s = \begin{bmatrix}-2\\ 1\end{bmatrix}_s$, relative to the standard basis, i.e;

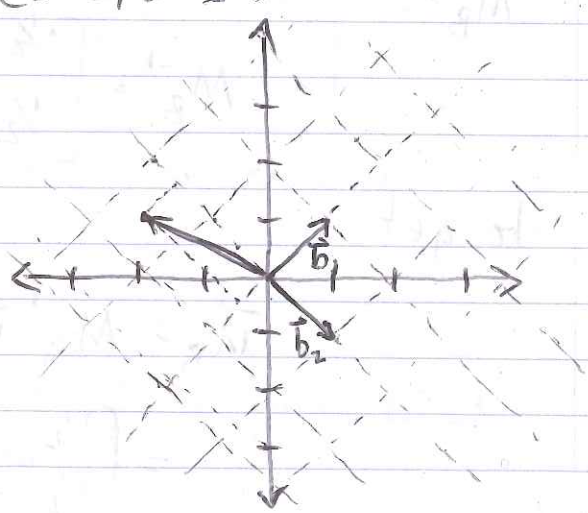
$$\vec{u}_s = -2\cdot\vec{e}_1 + 1\cdot\vec{e}_2$$

Consider the basis

$$B = \left\{\begin{bmatrix}1\\ 1\end{bmatrix}, \begin{bmatrix}1\\ -1\end{bmatrix}\right\}$$



standard basis



basis B

so in the basis B:

$$\vec{u}_B = \begin{bmatrix}-1/2\\ -3/2\end{bmatrix}_B$$

~~In other words~~ we got $\begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix}$ by solving

$$\beta_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

which is the same as

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{M_B} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\vec{u}_B} = \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{u}_S}$$

To solve this, multiply both sides by M_B^{-1}:

$$M_B^{-1} = \begin{bmatrix} +1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

to get

$$\vec{u}_B = M_B^{-1} \vec{u}_S$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix}$$

- This gives the general method for converting from one basis to another:

Let $\vec{u}$ be a vector in $\mathbb{R}^n$, and let $\vec{u}_B$ be the coordinate vector relative to $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$, and let $\vec{u}_S$ be the coordinate vector relative to $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$

- To change coordinates from B to S:

$$\boxed{\vec{u}_S = M_B \vec{u}_B}, \quad \text{where}$$

$$M_B = \left[\vec{b}_1 \middle| \vec{b}_2 \middle| \cdots \middle| \vec{b}_n \right]$$

- To change coordinates from S to B:

$$\vec{u}_B = M_B^{-1} \vec{u}_S$$

- M_B is called the change of coordinate matrix from B to S

Ex: Consider the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Express $\vec{u}_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in terms of the basis B.

We want to solve

$$\beta_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{M_B} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Inverting M_B:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

So:

$$\vec{u}_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = M_B^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -2 \\ 6 \end{bmatrix}$$

- Recall from the previous section that eigenvectors of a $n \times n$ matrix are linearly independent.
  - If there are n eigenvectors, then the set of eigen vectors form a basis

<u>Ex</u>: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

the eigenstoff is

$$\lambda_1 = 3, \ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} ; \ \lambda_2 = -1, \ \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Let's call the matrix P

$$P = \left[\vec{v}_1 \middle| \vec{v}_2 \right] = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

What is $A \cdot P$?

$$A \cdot P = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$$

$$= \left[3\vec{v}_1 \middle| -1 v_2 \right]$$

Interesting... continuing:

$$A \cdot P = \left[3\vec{v}_1 \,\middle|\, -\vec{v}_2 \right]$$

$$= \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \right] \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= P \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

If we multiply on the left by P^{-1}:

$$P^{-1} A P = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- So in a sense, the matrix P turns A into a diagonal matrix. This is true in general: Let A be an $n \times n$ matrix, with n linearly independent eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$P^{-1} A P = D,$$

where

$$P = \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \cdots \,\middle|\, \vec{v}_n \right], \quad D = \begin{bmatrix} \lambda_1 & & \bigcirc \\ & \lambda_2 & \\ \bigcirc & & \ddots \quad \lambda_n \end{bmatrix}.$$

- Diagonalization Theorem: An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent e-vectors.

- Steps for diagonalizing a matrix A:

① Find the eigenstuff for A: e-values λ_i, and e-vectors $\vec{v}_i$.

② Construct D, where D is the diagonal matrix with the e-values along the main diagonal

③ Construct P, whose columns are the e-vectors, listed in the same order as the e-values in D.

- Once diagonalized, the following eqns all hold:

$$AP = PD, \quad A = PDP^{-1}, \quad P^{-1}AP = D$$

- the first eqn is usually used to check your work.

- Why do we care about diagonalizing matrices? It makes solving systems of DE's much simpler:

  - Suppose we have ~~[illegible]~~ ~~[illegible]~~ a system of n DE's:

$$\vec{x}' = A\vec{x}$$

  - ~~[illegible]~~ Diagonalize A, to get matrices P and D such that $P^{-1}AP = D$

  - Now make the substitution $\vec{x} = P\vec{u}$:

$$\vec{x}' = A\vec{x} \Rightarrow (P\vec{u})' = A(P\vec{u})$$

$$\Rightarrow P\vec{u}' = (AP)\vec{u}$$

$$\Rightarrow \vec{u}' = (P^{-1}AP)\vec{u}$$

$$\Rightarrow \vec{u}' = D\vec{u}$$

  - Recall the form of D:

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

because D has this form, $\vec{u}' = D\vec{u}$ simplifies to

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1 \\ \lambda_2 u_2 \\ \vdots \\ \lambda_n u_n \end{bmatrix}$$

- Now each equation in the system doesn't depend on any others: the system is <u>decoupled</u>.

• Moral of the story: diagonalizing a system of DE's produces a decoupled system, which is much easier to solve

Ex: Solve

$$x_1' = x_1 + x_2$$
$$x_2' = 4x_1 + x_2$$

- In matrix form, this is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The eigenstuff for A is

$$\lambda_1 = 3, \ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \ \lambda_2 = -1, \ \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

- Making the substitution $\vec{x} = P\vec{u}$:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} u_1' = 3u_1 \\ u_2' = -u_2 \end{array}$$

- These equations have solutions

$$u_1(t) = k_1 e^{3t}$$
$$u_2(t) = k_2 e^{-t}$$

- To get back to x_1 and x_2, we recall our substitution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} k_1 e^{3t} \\ k_2 e^{-t} \end{bmatrix}$$
$$= \begin{bmatrix} k_1 e^{3t} + k_2 e^{-t} \\ 2k_1 e^{3t} - 2k_2 e^{-t} \end{bmatrix}$$

$$= k_1 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + k_2 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

- To sum up, diagonalizing is good for:

  - Factoring matrices. Let's you express
  $$A = PDP^{-1}$$
  where P is a list of e-vectors, and D contains info on e-valus

  - "Natural" coordinates: P is a change of basis to a coordinate system w/ the e-vectors as the new axes

  - Decoupling: substituting $\vec{x} = P\vec{u}$ makes the system decoupled, which is much easier to solve

  - Computing powers of matrices: If A is diagonalizable, then
  $$A^k = PD^kP^{-1}$$