6.1 Theory of Linear DE Systems

- Basic themes:
  - the solution space of an n^{th}-order linear differential equation has dimension n
  - Nonhomogeneous / superposition principles let us find solutions from simpler parts
  - n^{th}-order linear DE's can be converted to a first-order system of n eqns
  - Diagonalization allows for decoupling

- Here's what we'll be working with:

$$\boxed{\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t) \qquad (9)}$$

  - A is an $n \times n$ matrix
  - $\vec{f}(t)$ is a vector $(n \times 1)$ of continuous fns
  - $\vec{x}(t)$ is a solution vector $(n \times 1)$ of differentiable functions
  - If $\vec{f}(t) = \vec{0}$, the system is homogeneous.

- IVP's for systems: A combination of (9) and an initial value vector:

$$x'(t) = A\vec{x}(t) + \vec{f}(t), \quad \vec{x}(0) = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

where $\alpha_1, \ldots, \alpha_n$ are constants.

- Superposition Principle for systems: If $\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_k(t)$ are solutions to the homogeneous equation $\vec{x}' = A\vec{x}$, then

$$\vec{x}(t) = c_1\vec{x}_1(t) + \cdots + c_k\vec{x}_k(t)$$

is also a solution for any constants $c_1, \ldots, c_n$

Ex: Consider

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{bmatrix} \vec{x}(t)$$

It's easy to check that

$$\vec{x}_1(t) = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_2(t) = \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_3(t) = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}$$

are all solutions. Then letting

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t),$$

we have that

$$\begin{aligned}
\vec{x}'(t) &= \left(c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)\right)' \\
&= c_1\vec{x}_1'(t) + c_2\vec{x}_2'(t) + c_3\vec{x}_3'(t) \\
&= c_1 A\vec{x}_1(t) + c_2 A\vec{x}_2(t) + c_3 A\vec{x}_3(t) \\
&= A\left(c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)\right) = A\vec{x}(t)
\end{aligned}$$

- Solution space theorem: The set of solutions to the homogeneous DE

$$\vec{x}'(t) = A\vec{x}(t)$$

is a vector space of dimension n.

- Solution theorem for homogeneous linear DE systems: If $\vec{x}_1, \dots, \vec{x}_n$ are linearly independent solutions to $\vec{x}' = A\vec{x}$, then the general solution is

$$\vec{x}_h = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n$$

Ex: In the last example,

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} \quad \vec{x}_3 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}$$

were all solutions. Are they linearly independent?

$$\begin{array}{c} \begin{vmatrix} 0 & 2e^{2t} & e^t \\ 0 & e^{2t} & e^t \\ e^{3t} & e^{2t} & 0 \end{vmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 \end{array} \end{array} = e^{3t}\left[2e^{2t}\cdot e^{t} - e^{2t}\cdot e^{t}\right]$$

$$= e^{3t}\left[e^{3t}\right] = e^{6t} \neq 0,$$

so they are linearly independent. Thus,

$$\vec{x}_h = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

is the general solution

- We can write the general solution more succinctly:

$$\vec{x}_h = c_1 \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 2e^{2t} & e^{t} \\ 0 & e^{2t} & e^{t} \\ e^{3t} & e^{2t} & 0 \end{bmatrix}}_{X(t)} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- $X(t)$ is important: it's called the fundamental matrix. For the eqn

$$\vec{x}' = A\vec{x},$$

the fundamental matrix is formed by putting the n linearly independent solns as the columns.

- Properties of $X(t)$:

① $|X(t)| \neq 0$

② $X'(t) = AX(t)$

- When we find a solution, we can produce different kinds of graphs

- We can graph each component of

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

separately, to give the <u>component graphs</u>

- Or, we can plot $\vec{x}(t)$ parametrically, to produce a <u>phase portrait</u>

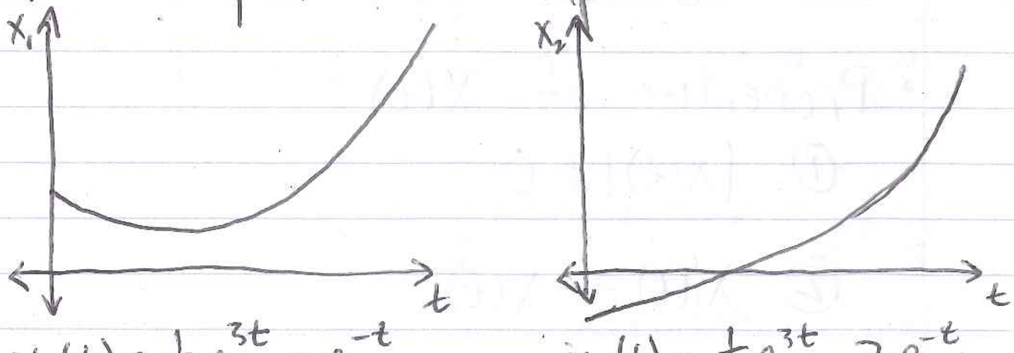
<u>Ex</u>: Consider

$$\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

We've computed the general solution before:

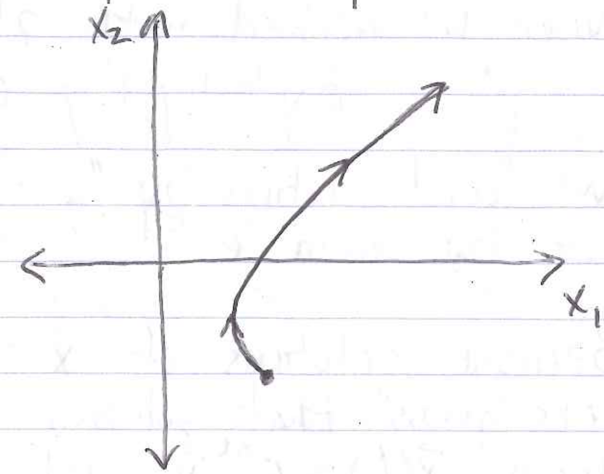
$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We can plot the components:



$x_1(t) = \frac{1}{10} e^{3t} + e^{-t}$ $\qquad$ $x_2(t) = \frac{1}{5} e^{3t} - 2e^{-t}$

Or we can plot $\vec{x}(t)$ parametrically.



- often, its useful to combine multiple solutions with different initial conditions into one phase portrait: