6.2 Linear Systems w/ Real E-values

- When we worked with 2^{nd} order DE's:

$$ay''+by'+cy=0$$

we found solutions by "guessing" they were of the form e^{rt}

- Because solutions of $\vec{x}'=A\vec{x}$ are vectors, let's "guess" that solutions are of the form $\vec{x}(t)=e^{\lambda t}\vec{v}$, where $\vec{v}$ is some vector. Substituting:

$$(e^{\lambda t}\vec{v})'=A(e^{\lambda t}\vec{v})$$
$$\Rightarrow \lambda e^{\lambda t}\vec{v}=A\,e^{\lambda t}\vec{v}$$
$$\Rightarrow A e^{\lambda t}\vec{v}-\lambda e^{\lambda t}\vec{v}=\vec{0}$$
$$\Rightarrow e^{\lambda t}(A\vec{v}-\lambda\vec{v})=\vec{0}$$
$$\Rightarrow e^{\lambda t}(A-\lambda I)\vec{v}=\vec{0}$$
$$\Rightarrow (A-\lambda I)\vec{v}=\vec{0}$$

  - What we basically already knew: In order for $e^{\lambda t}\vec{v}$ to be a solution, $\lambda, \vec{v}$ must be an e-value/e-vector pair!

- We'll be working mainly w/ 2x2 systems, so:

Let A be a 2×2 matrix, with linearly independent e-vectors $\vec{v}_1, \vec{v}_2$ and corresponding e-values λ_1, λ_2 (Note: It is possible that $\lambda_1 = \lambda_2$)
Then
$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$
is the general solution to
$$\vec{x}' = A\vec{x}$$

Ex: Consider $\vec{x}' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\vec{x}$.

- Find the e-stuff:
$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$
$$\Rightarrow \lambda_1 = 4,\ \lambda_2 = 1$$
$$\rightsquigarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

- the general solution is then
$$\vec{x}(t) = c_1 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

~~Let's look at some trajectories~~

- What happens if $\vec{x}(0) = \vec{v}_1$?

$$\vec{x}(0) = c_1 e^{4(0)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{(0)} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\Rightarrow c_1 = 1, \; c_2 = 0$$

$$\Rightarrow \vec{x}(t) = e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{4t} \vec{v}_1$$

- What if $\vec{x}(0) = \vec{v}_2$?

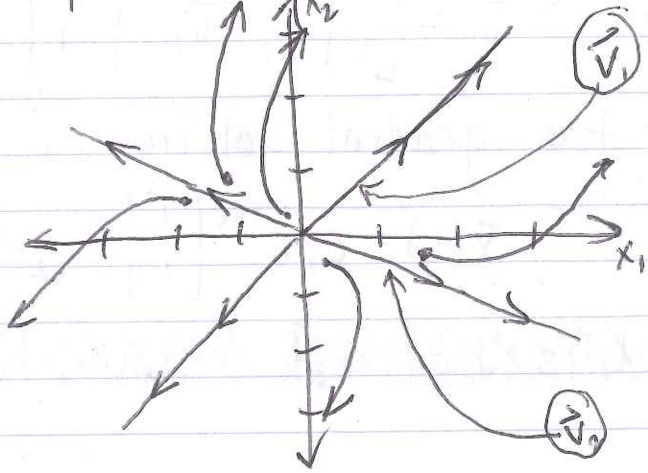
$$x(0) = c_1 e^{0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{0} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow c_1 = 0, \; c_2 = 1$$

$$\Rightarrow \vec{x}(t) = e^{t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = e^{t} \vec{v}_2$$

- If the function starts on an e-vector, it stays on an e-vector <u>forever</u>.

- phase portrait:



Ex: $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The e-stuff:

$$\lambda_1 = -1, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \lambda_2 = -3, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the general solution is

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

To determine c_1 and c_2, use the initial condition:

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

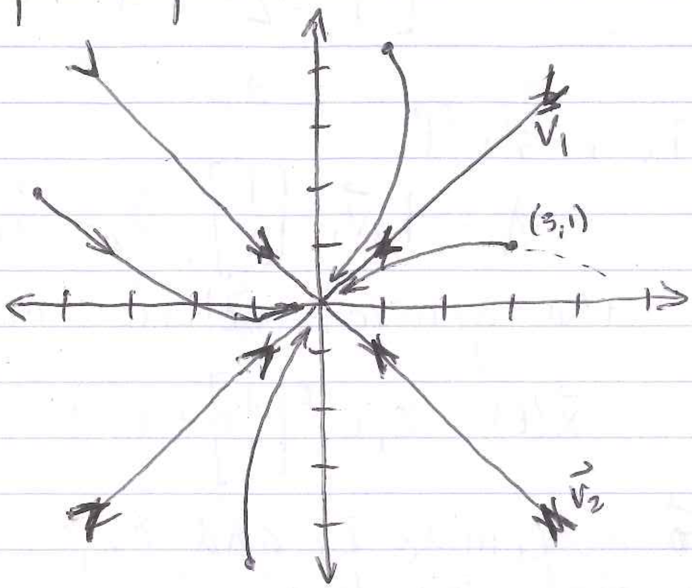
$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

so the exact solution is

$$\vec{x}(t) = 2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The phase portrait:



- Moral of the story: e-vectors tell you the basic structure of the phase portrait.

- We know what to do when e-vectors are linearly ind., but what happens when we don't have enough?

<u>Ex</u>: Consider the system

$$\vec{x}' = \underbrace{\begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}}_{A} \vec{x}$$

The e-stuff is

$$\lambda_1 = \lambda_2 = 4, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

only <u>one</u> e-vector!

We know one solution is

$$\vec{x}_1 = e^{4t}\begin{bmatrix}1\\-2\end{bmatrix}$$

but we need another linearly ind. solution! Let's try the previous method, and guess

$$\vec{x}_2 = te^{4t}\begin{bmatrix}1\\-2\end{bmatrix}$$

Is this also a solution?

LHS: $\vec{x}_2{}' = e^{4t}\begin{bmatrix}1\\-2\end{bmatrix} + 4te^{4t}\begin{bmatrix}1\\-2\end{bmatrix}$

RHS: $A\vec{x}_2 = te^{4t}\begin{bmatrix}4\\-8\end{bmatrix} = 4te^{4t}\begin{bmatrix}1\\-2\end{bmatrix}$

So, LHS $\neq$ RHS, thus $\vec{x}_2$ is <u>not</u> a solution! Oh noes!

- The problem is that $e^{4t}\vec{v}_1$ and $te^{4t}\vec{v}_1$ are <u>not</u> linearly independent.

- the solution? Lets add another vector and see what happens:

$$\vec{x}_2 = te^{4t}\vec{v}_1 + e^{4t}\vec{u}$$

- In order for $\vec{x}_2$ to be a solution:

$$\text{LHS: } \vec{x}_2' = \underline{e^{4t}\vec{v}_1} + 4te^{4t}\vec{v}_1 + \underline{4e^{4t}\vec{u}}$$

$$\text{RHS: } A\vec{x}_2 = A(te^{4t}\vec{v}_1 + \underline{e^{4t}\vec{u}})$$

- In order for LHS = RHS, we must equate the e^{4t} terms and the te^{4t} terms:

$$\cancel{e^{4t}}\vec{v}_1 + 4\cancel{e^{4t}}\vec{u} = A\cancel{e^{4t}}\vec{u}$$

and

$$4te^{4t}\vec{v}_1 = Ate^{4t}\vec{v}_1$$

$$\Rightarrow \vec{v}_1 + 4\vec{u} = A\vec{u} \quad \text{and} \quad 4\vec{v}_1 = A\vec{v}_1$$

$$\Rightarrow A\vec{u} - 4\vec{u} = \vec{v}_1 \quad \text{and} \quad A\vec{v}_1 - 4\vec{v}_1 = \vec{0}$$

$$\Rightarrow (A-4I)\vec{u} = \vec{v}_1 \quad \text{and} \quad (A-4I)\vec{v}_1 = \vec{0}$$

- This says $\vec{v}_1$ is an e-vector, which we already knew. Then we can determine $\vec{u}$ by solving

$$(A-4I)\vec{u} = \begin{bmatrix} -2 & -1 \\ 4 & +2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & -1 & 1 \\ 4 & +2 & -2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & +\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}r - \frac{1}{2} \\ r \end{bmatrix}$$

$$= r\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

- Letting $r=-1$ (so there aren't any fractions), we get one possible $\vec{u}$ to be

$$\vec{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- Thus, our second linearly ind. soln is

$$\vec{x}_2 = te^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- Then the general solution is

$$\vec{x}_h(t) = c_1 e^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\left(te^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

- $\vec{u}$ is called a <u>generalized eigenvector</u>