6.3 Linear System with Nonreal E-values

- Now we determine what happens when we get a complex-valued e-value.

- First, let's show why you actually only need to compute one e-vector in this case

  - We know that complex e-values always come in pairs:

$$\lambda = \alpha \pm i\beta$$

means switch the sign on imag. part

*i.e. $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta \Rightarrow \lambda_1^* = \lambda_2$

  - Finding the e-vector corr. to λ_1:

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$$

$$\Rightarrow \left[(A - \lambda_1 I)\vec{v}_1\right]^* = \vec{0}$$

$$\Rightarrow (A - \lambda_1^* I)\vec{v}_1^{\,*} = \vec{0}$$

$$\Rightarrow (A - \lambda_2 I)\vec{v}_1^{\,*} = \vec{0}$$

  - So the e-vector corr. to λ_2 is the conjugate of $\vec{v}_1$

Ex:

$$A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow (6-\lambda)(4-\lambda) + 5 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 29 = 0$$

$$\Rightarrow \lambda_1, \lambda_2 = 5 \pm 2i$$

- Calculating $\vec{v}_1$:

$$(A - (5+2i)I)\vec{v}_1 = \vec{0}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right]$$

$$\xrightarrow{R_1 = (1+2i)R_1} \left[\begin{array}{cc|c} 5 & -1-2i & 0 \\ 5 & -1-2i & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 5 & -1-2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (1/5 + 2/5\, i)r \\ r \end{bmatrix} = r \begin{bmatrix} 1/5 + 2/5\, i \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1+2i \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- Now we know that

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- For a matrix A, complex e-values come in pairs:

$$\lambda_1, \lambda_2 = \alpha \pm i\beta$$

and the corr. e-vectors <u>also</u> come in pairs:

$$\vec{v}_1, \vec{v}_2 = \vec{p} \pm i\vec{q}$$

- How do we solve the system $\vec{x}' = A\vec{x}$ when the e-stuff is complex valued?
  - Recall that in the 2nd order case, we split the solutions into the <u>real part</u> and the <u>imaginary part</u>:

$$\vec{x}(t) = \vec{x}_{Re}(t) + i\,\vec{x}_{Im}(t)$$

  - Just like before, we can show that if $\vec{x}(t)$ is a soln, then so are $\vec{x}_{Re}(t)$ and $\vec{x}_{Im}(t)$:

$$\vec{x}\,'(t) = \left[\vec{x}_{Re}(t) + i\,\vec{x}_{Im}(t)\right]' = \vec{x}\,'_{Re}(t) + i\,\vec{x}\,'_{Im}(t)$$

$$A\vec{x}(t) = A\left[\vec{x}_{Re}(t) + i\,\vec{x}_{Im}(t)\right] = A\vec{x}_{Re}(t) + i\,A\vec{x}_{Im}(t)$$

so if $\vec{x}'(t) = A\vec{x}(t)$, then

$$\vec{x}_{Re}'(t) + i\vec{x}_{Im}'(t) = A\vec{x}_{Re}(t) + iA\vec{x}_{Im}(t)$$

$$\Rightarrow \vec{x}_{Re}'(t) = A\vec{x}_{Re}(t)$$

<u>and</u>

$$\vec{x}_{Im}'(t) = A\vec{x}_{Im}(t)$$

so both $\vec{x}_{Re}(t)$ and $\vec{x}_{Im}(t)$ are solutions

- Now, if we have a pair of e-values

$$\lambda_1, \lambda_2 = \alpha \pm i\beta$$

and a pair of e-vectors

$$\vec{v}_1, \vec{v}_2 = \vec{p} \pm i\vec{q}$$

we know <u>one</u> solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = e^{\lambda_1 t}\vec{v}_1$$

- using Euler's identity, we get

$$e^{\lambda_1 t}\vec{v}_1 = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{p} + i\vec{q})$$

$$= e^{\alpha t}\Big[\cos(\beta t)\vec{p} + i\sin(\beta t)\vec{p} + i\cos(\beta t)\vec{q} - \sin(\beta t)\vec{q}\Big]$$

$$\cancel{= e^{\alpha t}\left(\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q}\right) + ie^{\alpha t}\left(\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q}\right)}$$

$$= \underbrace{e^{\alpha t}\left(\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q}\right)}_{\vec{X}_{Re}(t)}$$

$$+ i\underbrace{e^{\alpha t}\left(\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q}\right)}_{\vec{X}_{Im}(t)}$$

- It's not too hard to show that $\vec{x}_{Re}(t)$ and $\vec{x}_{Im}(t)$ are linearly independent, so they form a basis for the solution space:

$$\vec{X}_h(t) = c_1 \vec{x}_{Re}(t) + c_2 \vec{x}_{Im}(t)$$

- General solutions to 2x2 systems w/ complex-valued eigen stuff:
  1. For one e-value $\lambda_1 = \alpha + i\beta_1$ find the corresponding e-vector $\vec{v}_1 = \vec{p} + i\vec{q}$
  2. The general solution is then

$$\vec{x}_h(t) = c_1 e^{\alpha t}\left(\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q}\right) + c_2 e^{\alpha t}\left(\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q}\right)$$

Ex: Find the general solution to

$$\vec{x}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}\vec{x}$$

we already calculated that

$$\lambda_1, \lambda_2 = \overset{\alpha}{5} \pm \overset{\beta}{2} i \ ; \quad \vec{v}_1, \vec{v}_2 = \overset{\vec{p}}{\begin{bmatrix} 1 \\ 5 \end{bmatrix}} \pm i \overset{\vec{q}}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

then the general solution is

$$\vec{x}_h(t) = c_1 e^{5t}\left(\cos 2t \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin 2t \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$$

$$+ c_2 e^{5t}\left(\sin 2t \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos 2t \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$$

- what is the exact solution if $\vec{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$?

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

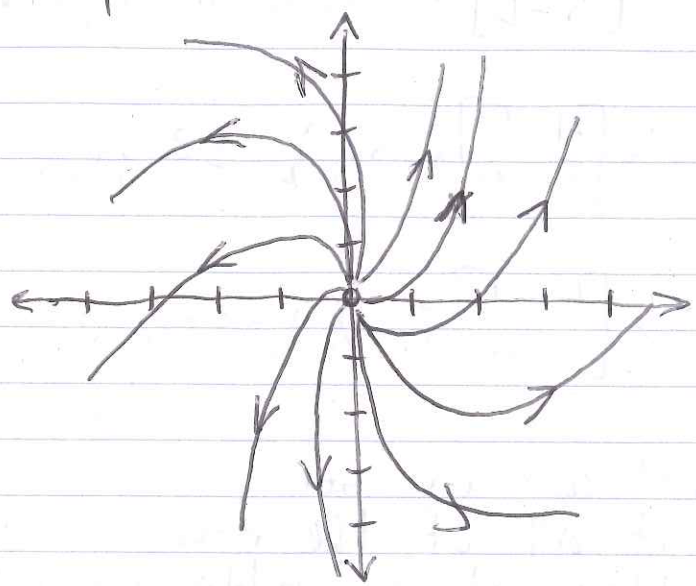
$$\Rightarrow c_1 = 1, \ c_2 = 1$$

$$\Rightarrow \vec{x}(t) = e^{5t}\left(\cos 2t \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin 2t \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$$

$$+ e^{5t}\left(\sin 2t \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos 2t \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$$

- is the stationary solution $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ stable, or unstable? Since $e^{5t} \to \infty$ as $t \to \infty$, solutions which start near $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ move away $\Rightarrow$ unstable

- phase portrait:



- Use the applet posted on the class website to examine the following systems:
  - $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x} \rightarrow \lambda_1, \lambda_2 = -1, 3 \; ; \; \vec{v}_1, \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
  - $\vec{x}' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x} \rightarrow \lambda_1, \lambda_2 = 4, 1 \; ; \; \vec{v}_1, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
  - $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x} \rightarrow \lambda_1, \lambda_2 = -1, -3 \; ; \; \vec{v}_1, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
  - $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x} \rightarrow \lambda_1, \lambda_2 = 4, \; \vec{v}_1, \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

- $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \vec{x} \rightarrow \lambda_1, \lambda_2 = -1 \pm 2i$; $\vec{v}_1, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \pm \begin{bmatrix} 0 \\ 2 \end{bmatrix} i$

- $\vec{x}' = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{x} \rightarrow \lambda_1, \lambda_2 = \pm 3i$; $\vec{v}_1, \vec{v}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \pm \begin{bmatrix} 0 \\ -3 \end{bmatrix} i$

- $\vec{x}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \vec{x} \rightarrow \lambda_1, \lambda_2 = 5 \pm 2i$, $\vec{v}_1, \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \pm \begin{bmatrix} 2 \\ 0 \end{bmatrix} i$

- What can we see?
  - if all of the e-values have $\alpha < 0$, then it is <u>stable</u> (actually, <u>asyptotically stable</u>)
  - if ~~all~~ <u>one</u> of the e-values has $\alpha > 0$, then it is <u>unstable</u>
  - if all of the e-values have $\alpha = 0$, then it is <u>stable</u>