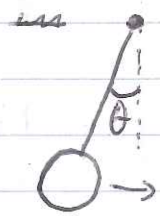
# 6.4 Stability and Linear Classification

- We'll be looking at the stability of equilibrium solutions:

  - A constant solution $\vec{x}(t) = \vec{c}$ of the autonomous system $\vec{x}' = \vec{f}(\vec{x})$ is called an <u>equilibrium solution</u>. The trajectory corresponding to this solution is just a point, called a <u>fixed point</u>.

<u>Ex</u>: Consider a pendulum: It turns out that you can model how the angle the pendulum is at with a differential equation. One equilibrium solution is $\theta(t) = 0$:




and another equilibrium solution is $\theta(t) = \pi$:

- Even though both solutions are equilibrium solutions, "nearby" solutions behave very differently

- An equilibrium solution $\vec{x}(t) = \vec{c}$ is <u>stable</u> if solutions that start sufficiently near to $\vec{c}$ remain bounded.

  - If nearby solutions move towards $\vec{c}$ as $t \to \infty$, the equilibrium soln is <u>asymptotically stable</u>

  - If nearby solutions move neither towards nor away from $\vec{c}$, the equilibrium solution is <u>neutrally stable</u>.

- An equilibrium solution that is not stable is called <u>unstable</u>

- For 2x2 systems,

$$\vec{x}\,' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

the characteristic equation is

$$|A - \lambda I| = \lambda^2 - \underbrace{(a+d)}_{\mathrm{Tr}A} \lambda + \underbrace{(ad - bc)}_{|A|} = 0$$

- the e-values are then

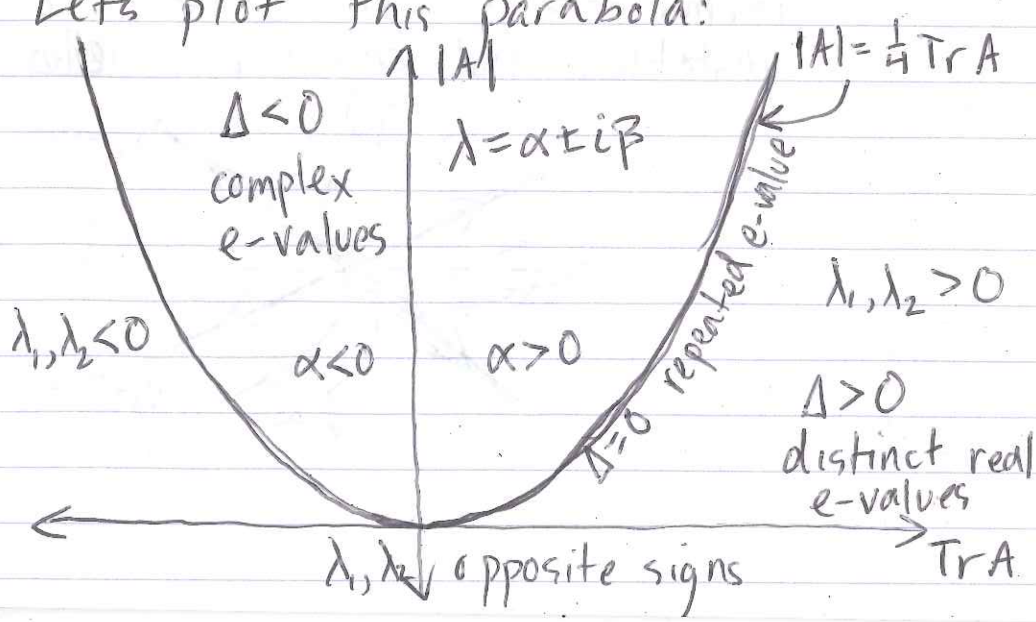
$$\lambda = \frac{TrA \pm \sqrt{(TrA)^2 - 4|A|}}{2}$$

- Now we can determine the different cases for e-values based solely on TrA and $|A|$

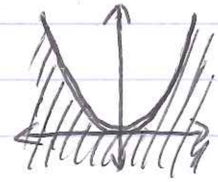
- The sign of $\Delta = (TrA)^2 - 4|A|$ determines whether we have distinct real roots, one repeated roots, or two imaginary roots

- The condition $(TrA)^2 - 4|A| = 0$ is equivalent to $|A| = \frac{1}{4}(TrA)^2$, which is a parabola with respect to TrA and $|A|$

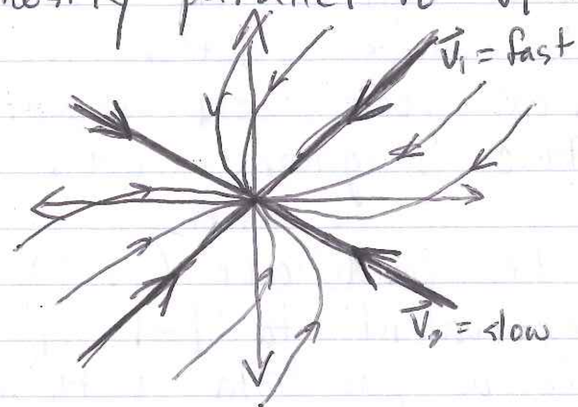
- Let's plot this parabola:



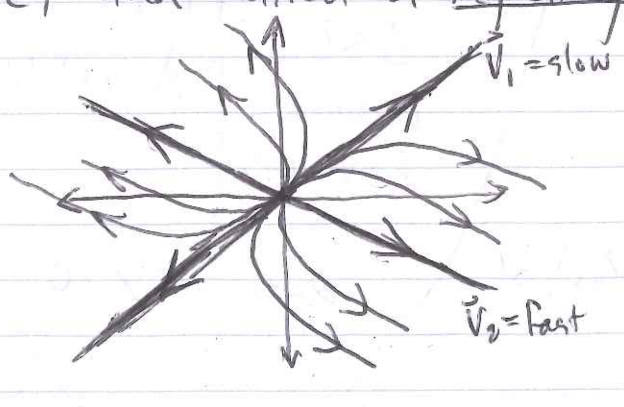
• Let's look at the case where $\Delta > 0$ (i.e. distinct real e-values)



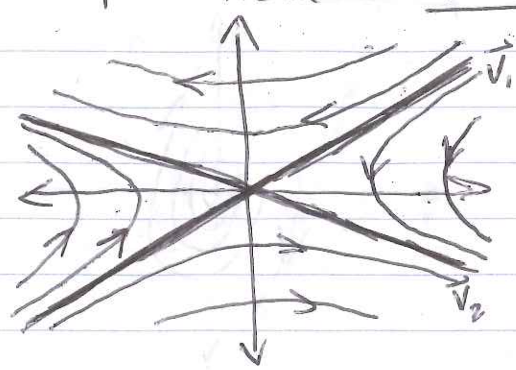
- Attracting node ($\lambda_1 < \lambda_2 < 0$) In this case, both $e^{\lambda t}$ terms $\to 0$ as $t \to \infty$. Asymptotically stable, and it's called an <u>attracting node</u>. Since $e^{\lambda_1 t} \to 0$ faster than $e^{\lambda_2 t} \to 0$, trajectories stay mostly parallel to $\vec{v}_1$:




- Repelling Node ($0 < \lambda_1 < \lambda_2$) Both $e^{\lambda t}$ terms $\to \infty$ as $t \to \infty$, so it's unstable, and called a <u>repelling node</u>

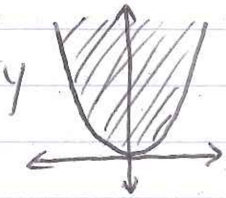


- Saddle Point ($\lambda_1 < 0 < \lambda_2$): ~~Since~~ Here $e^{\lambda_1 t} \to 0$, but $e^{\lambda_2 t} \to \infty$, Unstable, called a <u>saddle point</u>

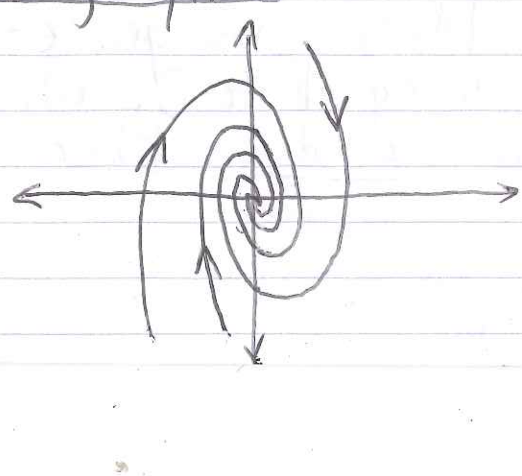



- Weird stuff happens when one or both of $\lambda_1, \lambda_2$ equal 0, so let's defer that case for now

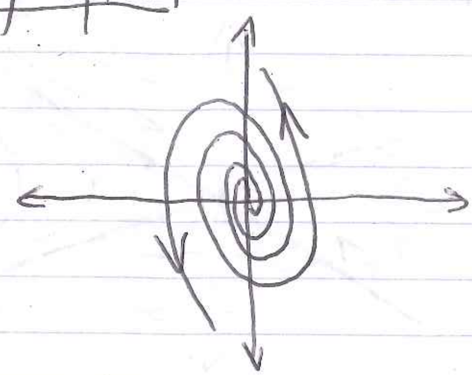
- When $\Delta < 0$, we get imaginary e-values $\lambda_1, \lambda_2 = \alpha \pm i\beta$, and sin and cos terms in the general solution $\to$ produces spirals



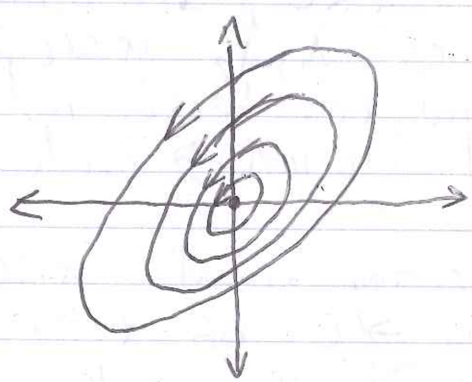
  - Attracting spiral ($\alpha < 0$) The $e^{\alpha t}$ terms $\to 0$ as $t \to \infty$, so it's asymptotically stable, and called an <u>attracting spiral</u>



- Repelling spiral ($\alpha > 0$) $e^{\alpha t} \to \infty$ as $t \to \infty$, so it's unstable, called a <u>repelling spiral</u>.

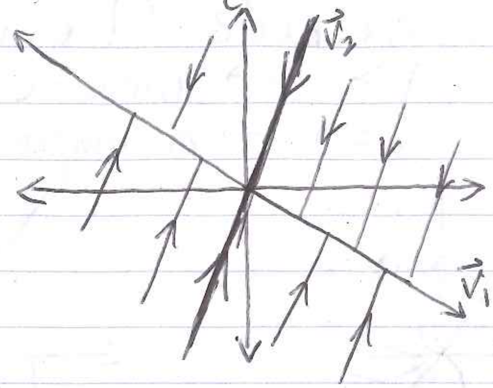


- Center ($\alpha = 0$) In this case, we only get sin and cos terms, so solutions neither move toward or away $\to$ neutrally stable, called a <u>center</u>



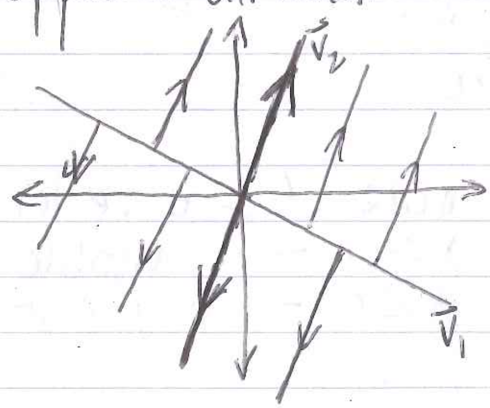
- When $|A| = 0$, we get either 1 or 2 e-values equal to 0, which produces ~~a~~ <u>non-isolated equilibria</u>

- When $\lambda_1 = 0$ and $\lambda_2 < 0$, we get a line of equilibrium solutions, with trajectories moving toward the equilibria:



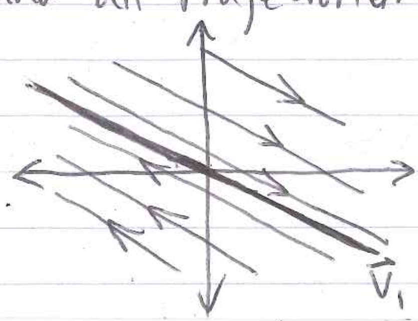

asymptotically stable

- When $\lambda_1 = 0$, $\lambda_2 > 0$, similar behavior but opposite direction:



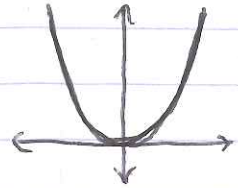

unstable

- When $\lambda_1 = \lambda_2 = 0$, there's only one e-vector and all trajectories are parallel to it:

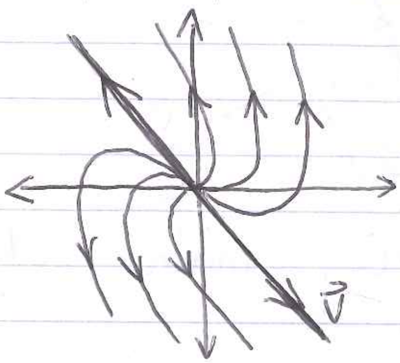



neutrally stable

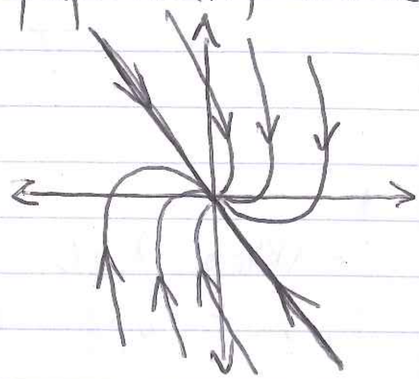
- And finally, we look at the case where $\Delta \stackrel{!}{=} 0$. We one e-vector, or two



- Degenerate node (1 e-vector)
  If $\lambda > 0 \rightarrow$ unstable
  If $\lambda < 0 \rightarrow$ asymptotically stable

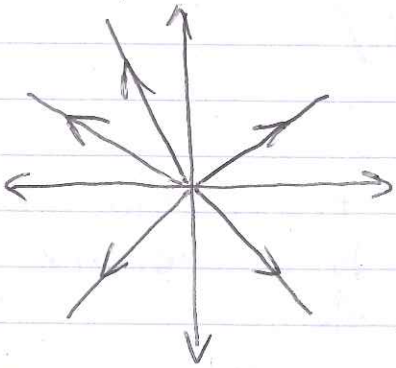



$\lambda > 0$

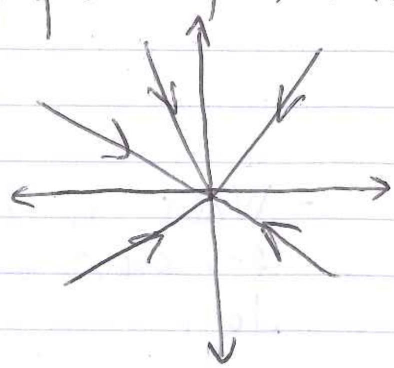


$\lambda < 0$

- Star node (2 e-vectors)
  If $\lambda > 0 \rightarrow$ unstable
  If $\lambda < 0 \rightarrow$ asymptotically stable

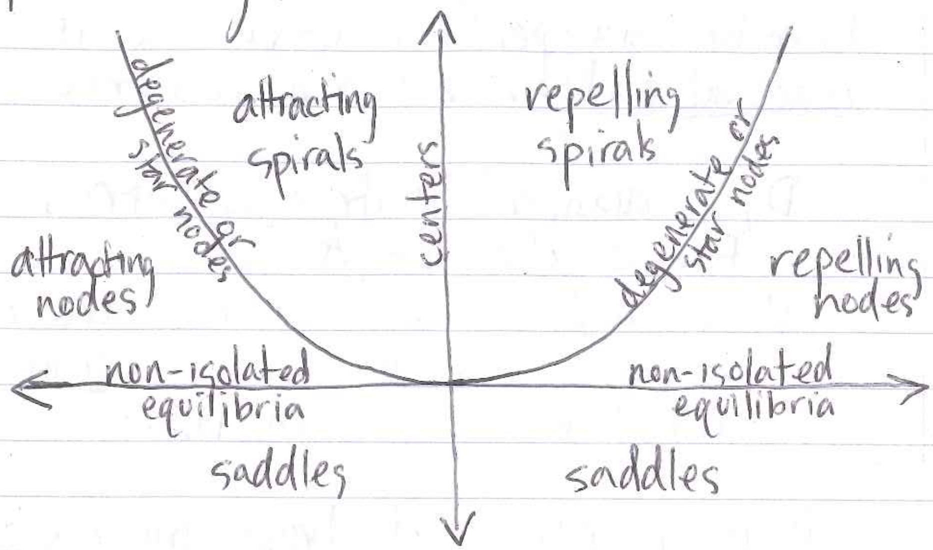


$\lambda > 0$



$\lambda < 0$

- Let's take a look at the $\mathrm{Tr}\,A / |A|$ plane again:




and in terms of stability: