6.5 Decoupling a Linear DE System

- Recall: If an $n \times n$ matrix A has n linearly independent e-vectors it is diagonalizable. We can construct

  D, a diagonal matrix w/ entries as the e-values of A

  P, a matrix whose columns are the e-vectors of A, in the same order as the e-values in D.

  - P is a change of basis matrix with
  $$A = PDP^{-1} \text{ and } D = P^{-1}AP$$

  P diagonalizes A

- Recall: We can decouple the system
$$\vec{x}' = A\vec{x}$$
by the substitution $\vec{x} = P\vec{u}$. This transforms the system into
$$\vec{u}' = D\vec{u}$$

Ex:
$$\vec{x}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{x}$$

$$\Rightarrow \lambda_1 = -2,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ \lambda_2 = -4,\ \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

this transforms $\vec{x}' = A\vec{x}$ into

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} u_1' &= -2u_1 \\ u_2' &= -4u_2 \end{aligned}$$

$$\Rightarrow \begin{aligned} u_1(t) &= c_1 e^{-2t} \\ u_2(t) &= c_2 e^{-4t} \end{aligned}$$

$$\Rightarrow \vec{u}(t) = \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{-4t} \end{bmatrix}$$

to get the general solution, we calculate $\vec{x} = P\vec{u}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{-4t} \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-4t} \\ c_1 e^{-2t} - c_2 e^{-4t} \end{bmatrix}$$

$$\Rightarrow \quad \vec{x}(t) = c_1 \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-4t} \\ -e^{-4t} \end{bmatrix}$$

- Decoupling also allows you to solve <u>nonhomogeneous</u> systems:

To solve $\vec{x}' = A\vec{x} + \vec{f}(t)$, where A is diagonalizable:

① Calculculate the e-stuff

② Construct P, D, and calculate P^{-1}

③ Let $\vec{x} = P\vec{u}$, and solve the decoupled system

$$\vec{u}' = D\vec{u} + P^{-1}\vec{f}(t)$$

using the integrating factor method.

- Review: integrating factor method. Suppose we want to solve

$$u'(t) = \lambda u(t) + f(t)$$

Then

$$u'(t) - \lambda u(t) = f(t)$$

$$\Rightarrow u'(t)e^{-\lambda t} - \lambda u(t)e^{-\lambda t} = f(t)e^{-\lambda t}$$

$$\Rightarrow \left[u(t)e^{-\lambda t}\right]' = f(t)e^{-\lambda t}$$

$\cancel{\Rightarrow u(t)e^{-\lambda t} = \int f(t)e^{-\lambda t}}$

$$\Rightarrow u(t)e^{-\lambda t} = \int f(s)e^{-\lambda s}\,ds$$

$$\Rightarrow u(t) = e^{\lambda t}\int f(s)e^{-\lambda s}\,ds$$

<u>Ex</u>: Solve

$$\vec{x}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

①: $\lambda_1 = -2,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = -4,\ \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

②: $D = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$\Rightarrow P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

③: $\vec{u}' = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}\vec{u} + \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$

$$\Rightarrow \begin{aligned} u_1 &= -2u_1 + \tfrac{1}{2}e^{-t} \\ u_2 &= -4u_2 - \tfrac{1}{2}e^{-t} \end{aligned}$$

$$\Rightarrow \begin{aligned} &(u_1 + 2u_1 = \tfrac{1}{2}e^{-t})\cdot e^{2t} \\ &(u_2 + 4u_2 = -\tfrac{1}{2}e^{-t})\cdot e^{4t} \end{aligned}$$

$$\Rightarrow \quad \left[u_1 e^{2t}\right]' = \frac{1}{2} e^{t}$$
$$\left[u_2 e^{4t}\right]' = -\frac{1}{2} e^{3t}$$

$$\Rightarrow \quad u_1 e^{2t} = \int \frac{1}{2} e^{s}\, ds$$
$$u_2 e^{4t} = -\int \frac{1}{2} e^{3s}\, ds$$

$$\Rightarrow \quad u_1 e^{2t} = \frac{1}{2} e^{t} + c_1$$
$$u_2 e^{4t} = -\frac{1}{6} e^{3t} + c_2$$

$$\Rightarrow \quad u_1 = \frac{1}{2} e^{-t} + c_1 e^{-2t}$$
$$u_2 = -\frac{1}{6} e^{-t} + c_2 e^{-4t}$$

$$\Rightarrow \quad \vec{u} = \begin{bmatrix} \tfrac{1}{2} e^{-t} + c_1 e^{-2t} \\ -\tfrac{1}{6} e^{-t} + c_2 e^{-4t} \end{bmatrix}$$

now, to get the general solution, we compute $\vec{x} = P\vec{u}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tfrac{1}{2} e^{-t} + c_1 e^{-2t} \\ -\tfrac{1}{6} e^{-t} + c_2 e^{-4t} \end{bmatrix}$$

$$\Rightarrow \quad \vec{x}(t) = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-4t} + \tfrac{1}{3} e^{-t} \\ c_1 e^{-2t} - c_2 e^{-4t} + \tfrac{2}{3} e^{-t} \end{bmatrix}$$

6.6 Matrix Exponential

- It turns out that you can define e^A for some matrix A!

  Given an $n \times n$ matrix A,
  $$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

- Since this is an infinite series, it's usually difficult to actually compute e^A, however:

  - If A is diagonal, then
  $$e^A = \begin{bmatrix} e^{a_{11}} & & 0 \\ & e^{a_{22}} & \\ 0 & & \ddots \quad e^{a_{nn}} \end{bmatrix}$$

  - A matrix A is <u>nilpotent</u> if $A^n = 0$ for some n! In this case, the series for e^A terminates.

Ex:
$$A = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^n = 0 \text{ for all } n \geq 3$$

This means that

$$e^A = I + A + \frac{A^2}{2}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- It turns out that the matrix exponential satisfies most of the properties of scalar exponential:

  *(i) $e^0 = I$
  (ii) $(e^A)^{-1} = e^{-A}$
  (iii) If $AB = BA$ then $e^{A+B} = e^A e^B$

- If we multiply A by t, we can define the <u>matrix exponential function</u>:

$$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

<u>Ex:</u>

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{t^2}{2}$$

$$= \begin{bmatrix} 1 & -t & 2t - t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

- what is $\frac{d}{dt} e^{At}$ in this case?

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \begin{bmatrix} 1 & -t & 2t - t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 2 - t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -t & 2t - t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= A e^{At}$$

- In fact, this is always the case: for any $n \times n$ matrix A,

$$\boxed{\frac{d}{dt} e^{At} = A e^{At}}$$

- this is pretty powerful, as we'll soon see...

- What is the general solution to the <u>scalar</u> DE
$$y' = ay?$$
  - You can solve this using various methods, but it's easy to see that the gen. soln. is
$$y(t) = ce^{at}$$

- The general solution of
$$\vec{x}' = A\vec{x},$$
where A is an $n \times n$ matrix, is
$$\vec{x}(t) = e^{At}\vec{c},$$
where $\vec{c}$ is a vector of arbitrary constants.
  - If there is an initial condition $\vec{x}(0) = \vec{x}_0$, then the exact solution is
$$\vec{x}(t) = e^{At}\vec{x}_0$$

- But wait! Back in 6.1, we said that the general solution to $\vec{x}' = A\vec{x}$ is
$$\vec{x}(t) = X(t)\vec{c},$$
where $X(t)$ is ~~the~~ a <u>fundamental matrix</u>
  - This says that $e^{At} = X(t)$!