- What is the general solution to the <u>scalar</u> DE

$$y' = ay?$$

  - You can solve this using various methods, but it's easy to see that the gen. soln. is

$$y(t) = ce^{at}$$

- The general solution of

$$\vec{x}' = A\vec{x},$$

where A is an $n \times n$ matrix, is

$$\vec{x}(t) = e^{At}\vec{c},$$

where $\vec{c}$ is a vector of arbitrary constants.

  - If there is an initial condition $\vec{x}(0) = \vec{x}_0$, then the exact solution is

$$\vec{x}(t) = e^{At}\vec{x}_0$$

- But wait! Back in 6.1, we said that the general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = X(t)\vec{c},$$

where $X(t)$ is ~~the~~ a <u>fundamental matrix</u>

  - This says that $e^{At} = X(t)$!

Ex: Find the general solution to

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A^5 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$$

$$\Rightarrow e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t \cancel{A} - \frac{t^2}{2!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\cancel{A} - \frac{t^3}{3!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{t^4}{4!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots & t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$\Rightarrow$ the general solution is

$$\vec{x}(t) = e^{At}\vec{c} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

- We know e^{At} is <u>one</u> fundamental matrix, but is there a way to construct e^{At} from <u>any</u> fundamental matrix? Yes:

  - Given a fundamental matrix $X(t)$, we know the general solution is
  $$\vec{x}(t) = X(t)\vec{c}$$

  - Given the initial condition $\vec{x}(0) = \vec{x}_0$, the following holds:
  $$\vec{x}(0) = X(0)\vec{c} = \vec{x}_0$$
  $$\Rightarrow \vec{c} = X(0)^{-1}\vec{x}_0$$
  $$\Rightarrow \vec{x}(t) = X(t)X(0)^{-1}\vec{x}_0$$

  - Since the exact (unique) solution to
  $$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \vec{x}_0$$
  can be written as $\vec{x}(t) = e^{At}\vec{x}_0$, we see that
  $$\boxed{e^{At} = X(t)X(0)^{-1}}$$

- Now we're actually equipped to solve non homogeneous equations:
$$\vec{x}' = A\vec{x} + \vec{f}(t)$$

- Similar to the integrating factor method:

$$\Rightarrow \vec{x}' - A\vec{x} = \vec{f}(t)$$

$$\Rightarrow e^{-At}(\vec{x}' - A\vec{x}) = e^{-At}\vec{f}(t)$$

$$\cancel{\Rightarrow \frac{d}{dt}\left[\vec{x}\, e^{-At}\right] = e}$$

$$\Rightarrow \frac{d}{dt}\left[e^{-At}\vec{x}\right] = e^{-At}\vec{f}(t)$$

$$\Rightarrow e^{-At}\vec{x} = \int_0^t e^{-As}\vec{f}(s)\,ds + \vec{c}$$

$$\Rightarrow \vec{x}(t) = e^{At}\vec{c} + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds$$

- So, given the system $\vec{x}' = A\vec{x} + \vec{f}(t)$, the general solution is

$$\boxed{\vec{x}(t) = e^{At}\vec{c} + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds}$$

If initial conditions are supplied: $\vec{x}(0) = \vec{x}_0$, then the exact solution is

$$\boxed{\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds}$$

<u>Ex</u>: Find the general solution to

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Recall that

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$\Rightarrow e^{-At} = (e^{At})^{-1} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

The integral in the general solution is

$$\int_0^t \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix} \begin{bmatrix} s \\ 0 \end{bmatrix} ds = \int_0^t \begin{bmatrix} s\cos(s) \\ s\sin(s) \end{bmatrix} ds$$

$$= \begin{bmatrix} t\sin t + \cos t - 1 \\ -t\cos t + \sin t \end{bmatrix}$$

Then the full general solution is

$$\vec{x}(t) = e^{At}\vec{c} + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$+ \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} t\sin t + \cos t - 1 \\ -t\cos t + \sin t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\begin{bmatrix} 1-\cos t \\ \sin t - t \end{bmatrix}}_{\vec{x}_p}$$

- There is even another way to calculate e^{At}
  - Recall that with the homogeneous problem $\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$, the change of variable $\vec{x} = P\vec{u}$ gives a decoupled system

$$\vec{u}\,' = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \vec{u}$$

with solution

$$\vec{u}(t) = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & 0 \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \vec{u}_0$$

To get back in terms of $\vec{x}$, we calculated

$$\vec{x}(t) = P\vec{u}(t) = P \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & \ddots & e^{\lambda_n t} \end{bmatrix} P^{-1} \vec{x}_0$$

- We already know that $\vec{x}(t) = e^{At}\vec{x}_0$ is the exact solution, so...

$$\boxed{e^{At} = P e^{Dt} P^{-1}}$$

- Only if A is diagonalizable!

6.7 Nonhomogeneous Linear Systems

- You can extend the methods of undetermined coefficients and variation of parameters to systems of DE's

Ex: Find the general solution to

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

we've calculated the solution to the homogeneous equation $\vec{x}' = A\vec{x}$ before:

$$\vec{x}_h(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Since the forcing function $\vec{f}(t) = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ is constant, we guess that the particular solution is also:

$$\vec{x}_p(t) = \begin{bmatrix} A \\ B \end{bmatrix}$$

Substituting into the DE:

$$\vec{x}_p' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}_p + \begin{bmatrix} 3 \\ 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

So, the general solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

- In general, undetermined coefficients for systems only works for $\vec{f}(t)$ with components in the same families as from sec. 4.4, i.e.
  - polynomials
  - exponentials
  - sin and cos
  - sums and products of these

Ex: Find the general solution to

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} t-2 \\ 4t-1 \end{bmatrix}$$

We know what $\vec{x}_h$ is, so let's find $\vec{x}_p$. Since the components of $\vec{f}(t)$ are polynomials, we guess

$$\vec{x}_p(t) = \begin{bmatrix} at+b \\ ct+d \end{bmatrix} \Rightarrow \vec{x}_p' = \begin{bmatrix} a \\ c \end{bmatrix}$$