Final Review

- 5.3 – Eigenvalues and Eigenvectors
  - know how to find them for a given matrix A:

    ① Solve the characteristic equation:

    $$\boxed{|A - \lambda I| = 0}$$

    to get the e-values

    ② For each eigenvalue λ_i, solve the system

    $$\boxed{(A - \lambda_i I)\vec{v}_i = \vec{0}}$$

    to get the corresponding eigenvector(s)

    Note: For a repeated e-value, it is possible to get 1 or more corresponding e-vectors!

    Note: E-vectors are not unique. Any nonzero multiple of an e-vector is also an e-vector, so if you can multiply to get rid of fractions, you should.

  - know the shortcuts: If A is upper or lower triangular, the e-values appear along the main diagonal.

    For 2x2 systems, the characteristic equation simplifies to

    $$|A - \lambda I| = \lambda^2 - (\mathrm{Tr}\, A)\lambda + |A| = 0$$

  - know how e-stuff applies to DE's of the form $\vec{x}\,' = A\vec{x}$

Ex: Find the e-stuff for

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

1

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda+3)^2 = 0 \Rightarrow \lambda = 0, -3$$

2 For $\lambda_1 = 0$:

$$(A - 0I)\vec{v}_1 = 0 \Rightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} v_1 - v_3 = 0 \\ v_2 - v_3 = 0 \\ v_3 \text{ is free} \end{array}$$

$$\Rightarrow \vec{v} = r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -3$:

$$(A + 3I)\vec{v}_2 = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} v_1 + v_2 + v_3 = 0 \\ v_2 \text{ is free} \\ v_3 \text{ is free} \end{array}$$

$$\Rightarrow \vec{v} = r\begin{bmatrix}-1\\1\\0\end{bmatrix} + s\begin{bmatrix}-1\\0\\1\end{bmatrix} \Rightarrow \vec{V}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Practice Problems: #1–16, 19–34

- 5.4 – Coordinates and Diagonalization

- know what "coordinates" means: given a basis $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$, the coordinates of a vector $\vec{v}$ relative to B are the constants $\beta_1, \beta_2, \ldots, \beta_n$ in

$$\vec{v} = \beta_1\vec{b}_1 + \beta_2\vec{b}_2 + \cdots + \beta_n\vec{b}_n .$$

The coordinate vector of $\vec{v}$ relative to B is

$$\vec{v} = \begin{bmatrix}\beta_1\\\beta_2\\\vdots\\\beta_n\end{bmatrix}_B$$

- know how to perform a change of basis: given a basis $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$, the change of coordinate matrix is

$$M_B = \left[\vec{b}_1 \middle| \vec{b}_2 \middle| \cdots \middle| \vec{b}_n\right]$$

To express the vector $\vec{v}$ in the basis B:

$$\boxed{\vec{v}_B = M_B^{-1}\vec{v}}$$

- Know the steps for diagonalizing a matrix A:

  ① Find the e-values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, and the corresponding e-vectors $\vec{v}_1, \ldots, \vec{v}_n$.

  ② Construct D:

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

  ③ Construct P:

$$P = \Bigg[\underbrace{\vec{v}_1 \,\Big|\, \vec{v}_2 \,\Big|\, \cdots \,\Big|\, \vec{v}_n}_{\text{in the same order as the e-values!}} \Bigg]$$

  To check your work: verify that

$$\boxed{AP = PD}$$

  Important note! An $n \times n$ matrix A is only diagonalizable if A has n linearly independent e-vectors!

Practice Problems: #1-12, 25-48

• 6.1 - Theory of Linear DE Systems

  - the $n \times n$ system $\vec{x}' = A\vec{x}$ must have n linearly independent solutions $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$

- the fundamental matrix is formed by putting $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ as the columns:

$$X(t) = \left[\vec{x}_1(t) \,\middle|\, \vec{x}_2(t) \,\middle|\, \cdots \,\middle|\, \vec{x}_n(t)\right]$$

Ex: The system

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{bmatrix} \vec{x}(t)$$

has linearly independent solutions

$$\vec{x}_1(t) = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_2(t) = \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_3(t) = \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix}$$

so the fundamental matrix is

$$X(t) = \begin{bmatrix} 0 & 2e^{2t} & e^{t} \\ 0 & e^{2t} & e^{t} \\ e^{3t} & e^{2t} & 0 \end{bmatrix}$$

- the general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = X(t)\vec{c},$$

where $X(t)$ is the fundamental matrix, and $\vec{c}$ is a vector of arbitrary constants.

Practice Problems: # 5-8, 17-22

• 6.2 - Linear Systems w/ Real Eigenvalues

- For a 2x2 system $\vec{x}' = A\vec{x}$ with linearly independent e-vectors $\vec{v}_1, \vec{v}_2$, the general solution is

$$\boxed{\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2}$$

- For a 2x2 system $\vec{x}' = A\vec{x}$ w/ only <u>one</u> e-vector $\vec{v}$, first find a generalized e-vector: by solving

$$(A - \lambda I)\vec{u} = \vec{v}$$

Then the general solution is

$$\boxed{\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(t\vec{v} + \vec{u})}$$

<u>Practice Problems</u>: #9-32

• <u>6.3 Linear Systems w/ Nonreal E-values</u>

- Nonreal e-values always come in pairs:

$$\lambda = \alpha \pm i\beta$$

the corresponding e-vectors also come in pairs:

$$\vec{v} = \vec{p} \pm i\vec{q}$$

- In the 2x2 case, the solution to $\vec{x}' = A\vec{x}$ is

$$\boxed{\begin{aligned}\vec{x}(t) = c_1 e^{\alpha t}(\cos\beta t\,\vec{p} - \sin\beta t\,\vec{q}) \\ + c_2 e^{\alpha t}(\sin\beta t\,\vec{p} + \cos\beta t\,\vec{q})\end{aligned}}$$

- Check out the example in the lecture notes from July 24

<u>Practice Problems</u>: #1-16

How to find the general solution to $\vec{x}' = A\vec{x}$ (2x2)

Find the e-stuff
$\lambda_1, \lambda_2 ; \vec{v}_1, \vec{v}_2$

Is λ_1, λ_2 real, or imaginary?

real

One, or 2 e-vectors?

one

Find a generalized e-vector by solving $(A - \lambda I)\vec{u} = \vec{v}$

The general solution is
$$\boxed{\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(t\vec{v} + \vec{u})}$$

two

The general solution is
$$\boxed{\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t} v_2}$$

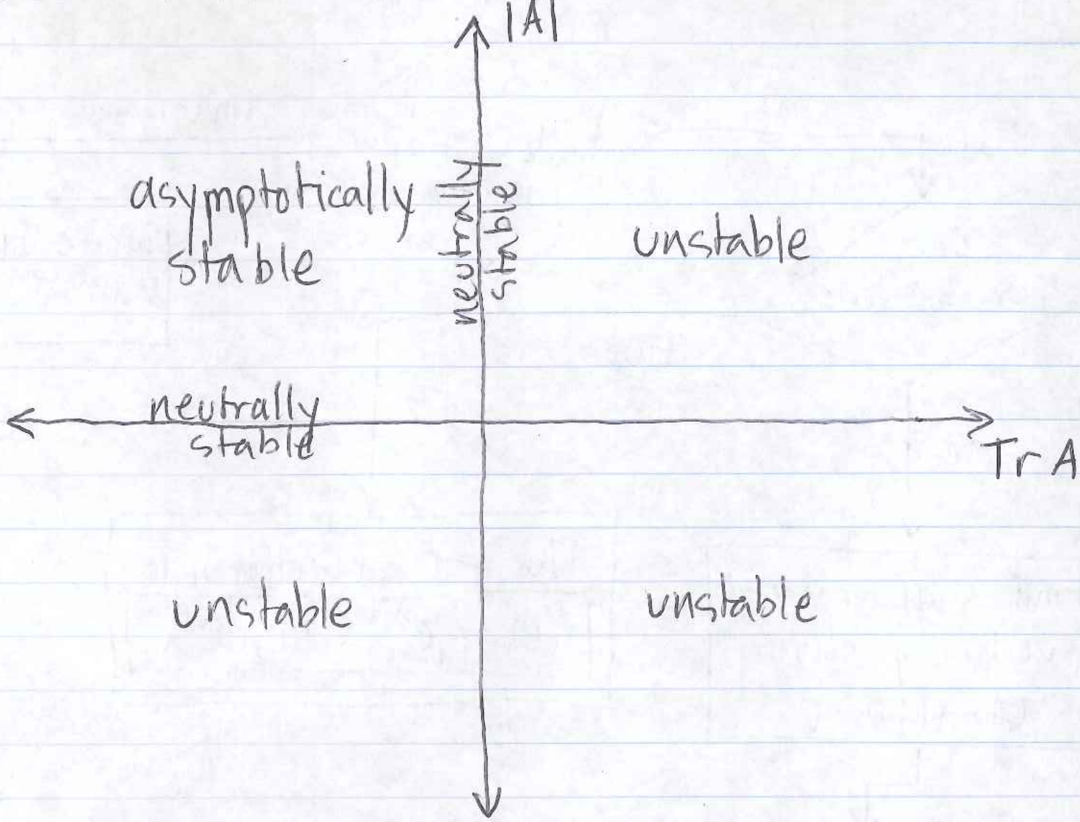
imaginary

write the e-vectors as $\vec{v} = \vec{p} \pm i\vec{q}$

The general solution is
$$\boxed{\begin{aligned}\vec{x}(t) = {} & c_1 e^{\alpha t}(\cos\beta t\,\vec{p} - \sin\beta t\,\vec{q}) \\ & + c_2 e^{\alpha t}(\sin\beta t\,\vec{p} + \cos\beta t\,\vec{q})\end{aligned}}$$

6.4 Stability and Linear Classification

- know how to determine stability for 2x2 systems: Using the trace (Tr A) and determinant (|A|)



- in terms of the general solution, ~~[illegible]~~
  - if all e terms have negative exponent $\Rightarrow$ asymptotically stable
  - if <u>any</u> e term has positive exponent $\Rightarrow$ unstable
  - if all e terms have 0 exponent $\Rightarrow$ neutrally stable
- know the classification of equilibrium points: spirals, nodes, saddles; etc.