

2.1 Real Valued and Vector Valued Functions of Several Variables

Definition 1. A function whose domain is a subset U of \mathbb{R}^m , and whose range is contained in \mathbb{R}^n is called a *real-valued function of m variables* if $n = 1$, and a *vector-valued function of m variables* if $n > 1$,

Definition 2. The *component functions* of a vector-valued function are the individual coordinates of the output. For example, if

$$f(r, \theta) = (r \cos(\theta), r \sin(\theta)),$$

the component functions are

$$\begin{aligned} f_1(r, \theta) &= r \cos(\theta) \\ f_2(r, \theta) &= r \sin(\theta) \end{aligned}$$

Sometimes vector valued functions are denoted by $\mathbf{f}(\mathbf{x})$ or $\vec{f}(\vec{x})$.

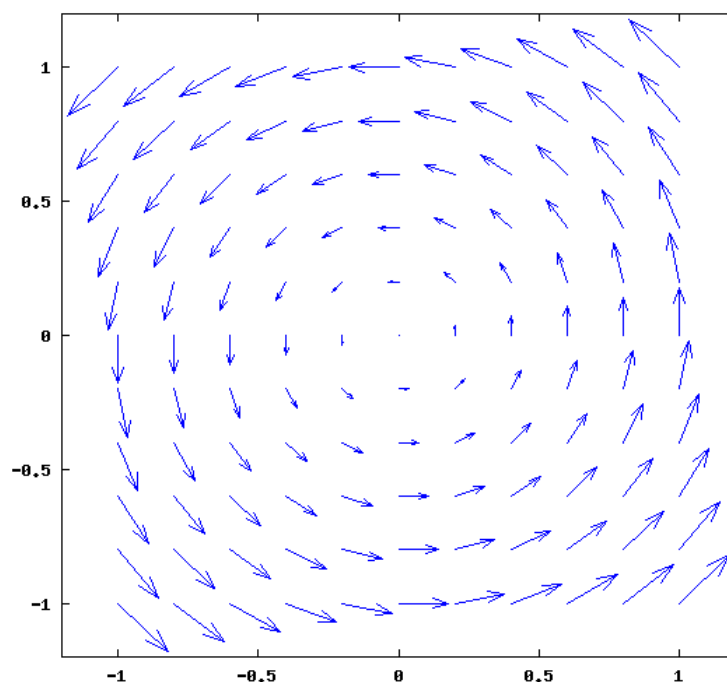
Example 1.

- *linear function:* $f(x) = ax + by + c$
- *distance function:* $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
- *projection function:* $f(x, y, z) = (x, y)$

And more practical examples:

- *Cobb-Douglas Production function:* $P(L, K) = bL^\alpha K^{1-\alpha}$
- *Wind Chill Index:* $W(T, \nu) = 13.12 + 0.6215T - 11.37\nu^{0.16} + 0.3965T\nu^{0.16}$

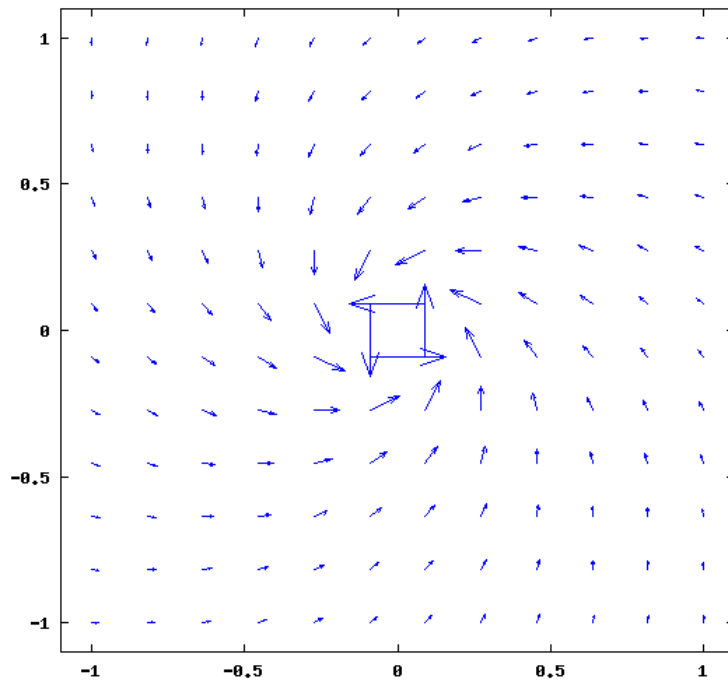
Definition 3. A *vector field* is a vector-valued function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Pictures make it clearer:



Example 2. *Model of a Hurricane:* A *fluid flow* is a vector field whose vectors describe the velocity of the fluid at a point. In hurricanes, a rough approximation of the fluid flow is

$$H(x, y) = \left(\frac{-x - y}{x^2 + y^2}, \frac{x - y}{x^2 + y^2} \right),$$

which has as its vector field



2.2 Graph of a Function of Several Variables

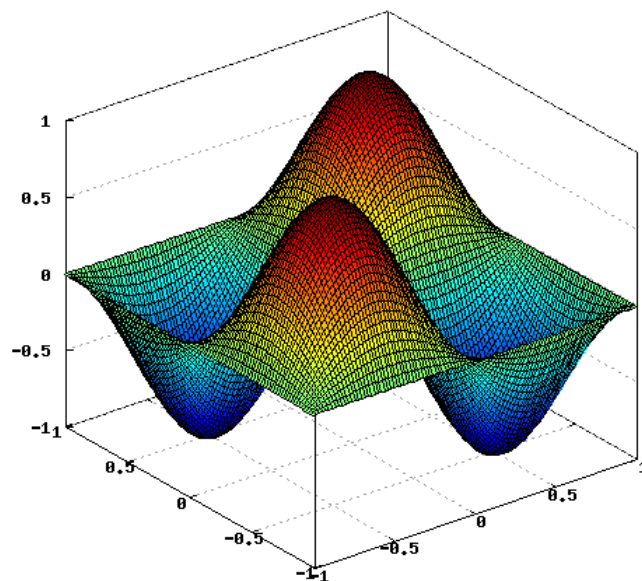
Definition 4. The graph of a real-valued function of one variable is a curve in the xy -plane. It is described by

$$\text{Graph}(f) = \{(x, y) : y = f(x)\}.$$

To generalize to functions of two variables, we define the *graph* by

$$\text{Graph}(f) = \{(x, y, z) : z = f(x, y)\}.$$

This gives a *surface* in three dimensions. Here we have $f(x, y) = \sin(\pi x) \sin(\pi y)$:



This is sometimes hard to visualize, so sometimes people use *level set curves*:

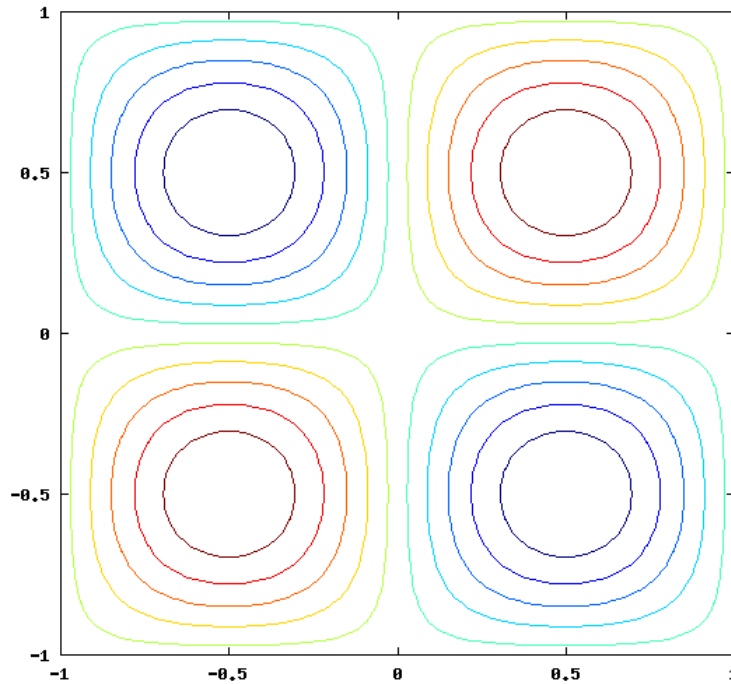
Definition 5. If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, the *level curve of value c* is the set of points

$$\{(x, y) : f(x, y) = c\}.$$

Similarly, for a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, the *level surface of value c* is the set of points

$$\{(x, y, z) : f(x, y, z) = c\}.$$

For example, if $f(x, y) = \sin(\pi x) \sin(\pi y)$:



Level set curves are useful for figuring out what exactly the surface graph looks like:

Example 3. Suppose $f(x, y) = 9 - x^2 - y^2$. Setting $f(x, y) = c$, we have that the level set curves are of the form

$$x^2 + y^2 = 9 - c.$$

Then since $x^2 + y^2 \geq 0$, c must be less than or equal to 9. If $c = 9$, the level set curve is just the point $(0, 0)$. If $c < 9$, the level set curves are circles of radius $\sqrt{9 - c}$, so the surface is a paraboloid.

How do computers plot surfaces? For real-valued functions of one variable, the computer picks points (usually equidistant) in some interval, computes the values of the function at those points, and then “connects the dots”. Similarly, for real-valued functions of two variables, the computer picks points for x and y in some intervals, computes the values of the function at those points, and then constructs the surface.