▶ EXERCISES 2.4

Exercises 1 to 6: Determine which of the following sets are open.

1. $U = \{(x, y) \mid 2 < x^2 + y^2 < 3\} \subseteq \mathbb{R}^2$

2. $U = \{(x, y, z) \mid x \geq 0\} \subseteq \mathbb{R}^3$

3. $U = \{(x, y) \mid x + y = 2\} \subseteq \mathbb{R}^2$

4. $U = \{(x, y) \mid x + y < 2\} \subseteq \mathbb{R}^2$

5. $U = \{(x, y, z) \mid xyz > 0\} \subseteq \mathbb{R}^3$

6. $U = \{(x, y, z) \mid x \neq 0, y > 0\} \subseteq \mathbb{R}^3$

7. Consider the function $f(x, y)$ whose contour diagram is shown in Figure 2.45.

(a) Determine the sign of $(\partial f/\partial x)(5, 3)$.

(b) Which of the two numbers, $(\partial f/\partial x)(10, 3)$ or $(\partial f/\partial x)(10, 5)$, is larger?

8. Draw a contour diagram of a function $f(x, y)$ that satisfies $(\partial f/\partial x)(x, y) > 0$ and $(\partial f/\partial y)(x, y) < 0$ for all (x, y).

Exercises 9 to 18: Find the indicated partial derivatives.

9. $f(x, y) = x^y + y \ln x$; f_x, f_y

10. $f(x, y, z) = xe^{yz^2}$; f_x, f_y, f_z

11. $f(x, y, z) = \ln(x + y + z^2)$; f_x, f_z

12. $f(x, y) = \arctan(x/y)$; f_x, f_y

13. $f(x, y) = e^{xy} \cos x \sin y$; f_x, f_y

14. $f(x, y, z) = x\sqrt{y\sqrt{z}}$; f_x, f_y, f_z

15. $f(x_1, \dots, x_m) = \sqrt{x_1^2 + \cdots + x_m^2}$; $\partial f/\partial x_i$, $i = 1, \dots, m$

16. $f(x_1, \dots, x_m) = e^{x_1 \cdots x_m}$; $\partial f/\partial x_i$, $i = 1, \dots, m$

17. $f(x, y) = \int_0^x te^{-t^2} dt$; f_x, f_y

18. $f(x, y) = \int_{\ln y}^0 (t + 1)^2 \, dt$; f_x, f_y

Exercises 19 to 22: The function $z(x, y)$ is defined in terms of two differentiable real-valued functions f and g of one variable. Compute z_x and z_y.

19. $z = f(x) + g(y)$

20. $z = f(x)g(y)$

21. $z = f(x)/g(y)$

22. $z = f(x)^{g(y)}$

23. A hiker is standing at the point $(2, 1, 11)$ on a hill whose shape is given by the graph of the function $z = 14 - (x - 3)^2 - 2(y - 2)^4$. Assume that the x-axis points east and the y-axis points north. In which of the two directions (east or north) is the hill steeper?

24. The volume of a certain amount of gas is determined by $V = 0.12TP^{-1}$, where T is the temperature and P is the pressure. Compute and interpret $\partial V/\partial P$ and $\partial V/\partial T$ when $P = 10$ and $T = 370$.

25. Consider the function $f(x, y) = -xe^{-x^2-2y^2}$.

(a) Compute $f_y(2, 3)$.

(b) Find the curve that is the intersection of the graph of f and the vertical plane $x = 2$ and compute the slope of its tangent at $y = 3$.

(c) Using (a) and (b), give a geometric interpretation of $f_y(2, 3)$.

26. Let $u(x, y, t) = e^{-2t} \sin(3x) \cos(2y)$ denote the vertical displacement of a vibrating membrane from the point (x, y) in the xy-plane at the time t. Compute $u_x(x, y, t)$, $u_y(x, y, t)$, and $u_t(x, y, t)$ and give physical interpretations of your results.

Exercises 27 to 31: Compute the derivative of the function $\mathbf{F}$ at the point $\mathbf{a}$.

27. $\mathbf{F}(x, y) = (y, x, 11)$, $\mathbf{a} = (0, 0)$

28. $\mathbf{F}(x, y) = (e^{xy}, x^2 + y^2)$, $\mathbf{a} = (a_1, a_2)$

29. $\mathbf{F}(x, y, z) = (\ln(x^2 + y^2 + z^2), 2xy + z)$, $\mathbf{a} = (1, 1, 0)$

30. $\mathbf{F}(x, y) = (x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2})$, $\mathbf{a} = (a_1, a_2) \neq (0, 0)$

31. $f(x, y, z) = ||x\mathbf{i} + y\mathbf{j} + z\mathbf{k}||^2$, $\mathbf{a} = (a_1, a_2, a_3)$

32. Compute $\nabla f(2, 1, -1)$ if $f(x, y, z) = xy \ln(z^2 + xy)$.

33. The electrostatic force field $\mathbf{F}(\mathbf{r})$ and the electrostatic potential $V(\mathbf{r})$ were defined in Example 2.11. Show that $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$. Compare with Example 2.40.

34. Let $f(x, y, z) = xyz(x^2 + y^2 + z^2)^{-2}$. Compute $\nabla f(x, y, z)$ for $(x, y, z) \neq (0, 0, 0)$.

35. Define $f\colon \mathbb{R}^3 \to \mathbb{R}$ by $f(\mathbf{x}) = ||\mathbf{x}||$. Find $\nabla f(\mathbf{x})$ and state its domain.

Exercises 36 to 42: Find the linear approximation of the function f at the point $\mathbf{a}$.

36. $f(x, y) = e^{-x^2-y^2}$, $\mathbf{a} = (0, 0)$

37. $f(x, y) = \ln(3x + 2y)$, $\mathbf{a} = (2, -1)$

38. $f(x, y) = xy(x^2 + y^2)^{-1}$, $\mathbf{a} = (0, 1)$

39. $f(x, y) = x^2 - xy + y^2/2 + 3$, $\mathbf{a} = (3, 2)$

40. $f(x, y, z) = \ln(x^2 - y^2 + z)$, $\mathbf{a} = (3, 3, 1)$

41. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $\mathbf{a} = (0, 1, 1)$

42. $f(x, y) = \int_x^y e^{-t^2} dt$, $\mathbf{a} = (1, 1)$

43. Verify that $xy(x + y)^{-1} \approx \frac{6}{5} + \frac{9}{25}(x - 2) + \frac{4}{25}(y - 3)$, for (x, y) sufficiently close to $(2, 3)$.

44. Prove that $\ln(2x^2 + 3y - 4) \approx 4x + 3y - 7$, for (x, y) sufficiently close to $(1, 1)$.

45. Assume that $f(x, y)$ is differentiable at (a, b) and let $\overline{L}(x, y) = f(a, b) + m(x - a) + n(y - b)$ be a linear function that satisfies (2.17), that is,

$$\lim_{(x,y)\to(a,b)} \frac{\left| f(x, y) - \overline{L}(x, y) \right|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

(a) Substitute $y = b$ into the above formula to show that $m = (\partial f/\partial x)(a, b)$.

(b) Prove that $n = (\partial f/\partial y)(a, b)$ and conclude that $\overline{L}$ must be equal to the linear approximation $L_{(a,b)}$.

46. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$ (see Example 2.44) and assume that it has a linear approximation $L_{(0,0)}(x, y)$ at $(0, 0)$.

(a) Explain why $L_{(0,0)}(x, y) = mx + ny$ for some real numbers m and n.

(b) Use (2.17) to show that f is differentiable at the origin if and only if

$$\lim_{(x,y)\to(0,0)} \left(1 - \frac{mx + ny}{\sqrt{x^2 + y^2}} \right) = 0.$$

(c) Use the approach $x \to 0$ and $y = 0$ to show that the above limit is not equal to 0. Conclude that f is not differentiable at the origin.

Exercises 47 to 51: Approximate the value of the given expression and compare it (except in Exercise 51) with the calculator value.

47. $\sqrt{0.99^3 + 2.02^3}$

48. $-0.09\sqrt{4.11^3 - 14.98}$

49. $7.95 \ln 1.02$

50. $\sin(\pi/50) \cos(49\pi/50)$

51. $\int_{0.995}^{1.02} e^{-t^2} dt$

Now, replacing $h(r, \theta)$ by $f(r, \theta)$, following the usual convention, we obtain

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta,$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta.$$

◀

▶ EXAMPLE 2.74

Let $x = r\cos\theta$, $y = r\sin\theta$, and $f(x, y) = xe^{x^2+y^2}$. Find $\partial f/\partial r$ and $\partial f/\partial\theta$ directly, and then using the chain rule.

SOLUTION

Since $f(x, y) = xe^{x^2+y^2} = r\cos\theta e^{r^2}$, we get $f(r, \theta) = r\cos\theta e^{r^2}$ (both functions are called f—recall the notational convention!) and hence $\partial f/\partial r = (e^{r^2} + 2r^2e^{r^2})\cos\theta$ and $\partial f/\partial\theta = -re^{r^2}\sin\theta$. Using the result of the previous example, we obtain

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = (e^{x^2+y^2} + 2x^2e^{x^2+y^2})\cos\theta + 2xye^{x^2+y^2}\sin\theta$$

$$= (e^{r^2} + 2r^2\cos^2\theta e^{r^2})\cos\theta + 2r^2e^{r^2}\sin\theta\cos\theta\sin\theta = (e^{r^2} + 2r^2e^{r^2})\cos\theta.$$

The expression for $\partial f/\partial\theta$ is obtained similarly. ◀

Notice that in this case the direct computation was faster (and easier). However, there are situations where not only does the chain rule provide a more efficient way, but the direct computation cannot be applied at all; see Exercise 27.

▶ EXERCISES 2.6

1. Assume that g is a differentiable, real-valued function of two variables and let $f(x, y) = g(x^2 - y^2, y^2 - x^2)$. Prove that $x(\partial f/\partial y) + y(\partial f/\partial x) = 0$.

2. Assume that g is a differentiable real-valued function of one variable, such that $g(1) = 2$ and $g'(1) = 3$.

(a) If $f(x, y) = g(x) + g(x^2)g(y)$, find $(\partial f/\partial x)(x, y)$ and $(\partial f/\partial x)(1, 1)$.

(b) If $f(x, y) = g(x)^{g(y)}$, find $(\partial f/\partial x)(1, 1)$ and $(\partial f/\partial y)(1, 1)$.

3. Find $g'(t)$ if $g(t) = f(t\sin t, t\cos t, t)$, where f is a differentiable function.

4. Assume that f is a differentiable function and let $g(t) = \sin(f(-t, t, 2t))$. Find $g'(t)$.

Exercises 5 to 7: In each case, compute $(f \circ \mathbf{c})'(t)$ in two different ways: by computing the composition first and then differentiating, and by using formula (2.25).

5. $f(x, y) = x^2y$, $\mathbf{c}(t) = (\sin t, \cos t)$.

6. $f(x, y) = ye^{xy}$, $\mathbf{c}(t) = (t, \ln t)$.

7. $f(x, y, z) = xy + \cos(x^2 + z^2)$, $\mathbf{c}(t) = (t\sin t, t, t\cos t)$.

8. Assume that f is a differentiable function of two variables, and $D_1 f(2, 2) = -2$ and $D_2 f(2, 2) = 4$.

(a) Find $g'(2)$ if $g(x) = f(x, 2)$.

(b) Let $g(x) = f(x, x)$. Find $g'(2)$.

(c) Let $g(x) = f(x^2, x^3)$. Find $g'(x)$.

9. Let $f(x, y) = g(x^2y, 2x + 5y, x, y)$, where g is a differentiable function of four variables. Find f_x and f_y.

10. Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x, y) = (h(x), g(y), k(x, y))$, where h, g, and k are differentiable functions of variables indicated. Find Df.

11. Let $F(x, y) = f(h(x), g(y), k(x, y))$, where $f: \mathbb{R}^3 \to \mathbb{R}$, and all functions involved are assumed to be differentiable. Find F_x and F_y.

12. Let $z = f(r)$, where $r = \sqrt{x^2 + y^2}$ and f is a differentiable function. Prove that $yz_x - xz_y = 0$ for all $(x, y) \neq (0, 0)$.

13. Let $f(x, y) = x^2 + xy$ and $g(x, y) = \ln x + \ln y$. Compute $\nabla(fg)(x, y)$ and $\nabla(f/g)(2, 2)$.

14. Let $\mathbf{G}(x, y) = (2xy, y^2 - x^2)$. Compute $D\mathbf{G}(x, y)$ and $D\mathbf{G}(3, 0)$.

15. Let $f(x, y, z) = x^2 + \sin(yz) - 3$. Find $D(f/x)(1, \pi, -1)$ and $D(x^2yf)(2, 0, 1)$.

16. Let $w = f(x, y, z)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Find $\partial w/\partial r$, $\partial w/\partial\theta$, and $\partial w/\partial z$.

17. Let $w = f(x, y, z)$, where $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. Find $\partial w/\partial\rho$, $\partial w/\partial\theta$, and $\partial w/\partial\phi$.

18. Let $\mathbf{v}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}$ and $\mathbf{w}(t) = \mathbf{i} - 2t\mathbf{j} + e^t\mathbf{k}$. Compute $(\mathbf{v} \cdot \mathbf{w})'(t)$ directly (i.e., by computing the dot product first and then differentiating) and then check your answer using the product rule.

19. Let $\mathbf{v}(t) = t^3\mathbf{i} + te^t\mathbf{k}$ and $\mathbf{w}(t) = -2t\mathbf{j}$. Compute $(\mathbf{v} \times \mathbf{w})'(t)$ directly (i.e., by computing the cross product first) and then check your answer using the product rule.

20. Let $\mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}$, $\mathbf{v}(t) = \mathbf{i} + t\mathbf{j} + \mathbf{k}$, and $\mathbf{w}(t) = t^3(\mathbf{i} + \mathbf{j} + \mathbf{k})$. Compute $(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}))'(t)$.

21. The function $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^3$ is given by $\mathbf{F}(x, y) = (e^x, xy, e^y)$. Compute $D(g \circ \mathbf{F})(0, 0)$, where $g: \mathbb{R}^3 \to \mathbb{R}$ is given by $g(u, v, w) = uw + v^2$.

22. Let $f: \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{c}: \mathbb{R} \to \mathbb{R}^3$ be given by $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ and $\mathbf{c}(t) = (\cos t, \sin t, 1)$. Compute $(f \circ \mathbf{c})'(t)$ and $(f \circ \mathbf{c})'(0)$.

23. Compute $\partial w/\partial x$ and $\partial w/\partial z$ if $w = f(x, y, z)$ and $y = g(x, z)$ are differentiable functions.

24. Let $w = \ln(r^2 + 1)$, where $r = \sqrt{x^2 + y^2}$. Find $\partial w/\partial y$.

25. Define a function $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(\mathbf{x}) = A \cdot \mathbf{x}$, where A is a 2×2 matrix, and the dot indicates matrix multiplication. Compute $D\mathbf{F}(\mathbf{x})$. Prove that $\mathbf{F}$ is differentiable at any point $(a, b) \in \mathbb{R}^2$.

26. Let A and B be 2×2 matrices. Define $\mathbf{F}, \mathbf{G}: \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(\mathbf{x}) = A \cdot \mathbf{x}$ and $\mathbf{G}(\mathbf{x}) = B \cdot \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^2$, and the dot indicates matrix multiplication. Find $D(\mathbf{G} \circ \mathbf{F})(\mathbf{x})$.

27. Let $f(x, y) = x^3y$, where $x^3 + tx = 8$ and $ye^y = t$. Find $(df/dt)(0)$.

28. In Examples 2.46 and 2.47 in Section 2.4, we studied the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Let $\mathbf{c}(t) = (t, t^2)$.

(a) Compute the composition $(f \circ \mathbf{c})(t)$ and show that $(f \circ \mathbf{c})'(0) = 1$.

for $V(x, y, z)$ into $\partial V/\partial y$, thus obtaining

$$-zx + \frac{\partial C(y, z)}{\partial y} = -xz + 1,$$

which implies that $\partial C(y, z)/\partial y = 1$ and $C(y, z) = y + C(z)$, by integration with respect to y (the variable z was kept fixed, so the integration "constant" might still depend on z). Hence,

$$V(x, y, z) = -xyz + y + C(z).$$

Finally, substituting this expression into the equation for $\partial V/\partial z$, we get

$$-xy + C'(z) = -xy,$$

so that $C(z) = C$ after integrating with respect to z. (C is a real number, not a function any longer.) It follows that any function of the form

$$V(x, y, z) = -xyz + y + C$$

(where C is a real number) is a potential function for the given vector field. ◀

▶ EXERCISES 2.7

Exercises 1 to 5: Consider a contour diagram of a function $f(x, y)$ in Figure 2.69. Estimate the directional derivative $D_{\mathbf{v}} f(a, b)$ at the given point in the given direction.

1. $(a, b) = (2, 1)$, $\mathbf{v} = \mathbf{j}$

2. $(a, b) = (2, 1)$, $\mathbf{v} = -\mathbf{i} + \mathbf{j}$

3. $(a, b) = (3, 2)$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$

4. $(a, b) = (3, 2)$, $\mathbf{v} = \mathbf{i}$

5. $(a, b) = (4, 1)$, $\mathbf{v} = -\mathbf{i}$

6. Consider the contour diagram in Figure 2.69. Draw gradient vectors at several points on the level curve $f(x, y) = 16$.

Exercises 7 to 11: Find the directional derivative of the function f at the point $\mathbf{p}$ in the direction of the vector $\mathbf{v}$.

7. $f(x, y) = e^{xy}(\cos x + \sin y)$, $\mathbf{p} = (\pi/2, 0)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$

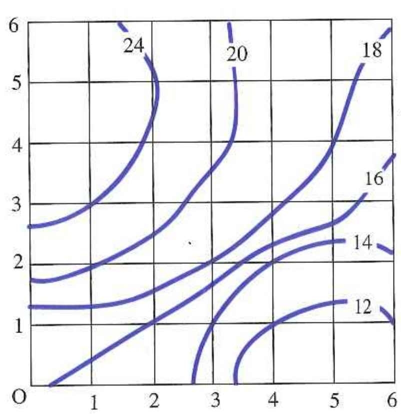


Figure 2.69 Contour diagram used in Exercises 1 to 6.

8. $f(x, y) = x^3y + 2x^2y^2 - xy^3$, $\mathbf{p} = (2, 3)$, $\mathbf{v} = (1, -1)$

9. $f(x, y, z) = e^{-x^2-y^2-z^2}$, $\mathbf{p} = (0, -1, 2)$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

10. $f(x, y) = \arctan(y/x)$, $\mathbf{p} = (1, 1)$, $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$

11. $f(x, y) = x \ln y^2 + 2y - 3$, $\mathbf{p} = (1, 2)$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

Exercises 12 to 14: Find the directional derivative of the function f at the point $\mathbf{p}$ in the direction given by the angle θ, measured from the positive direction of the x-axis in the counterclockwise direction.

12. $f(x, y) = xy^4 + x^2y^2 - 2$, $\mathbf{p} = (0, -1)$, $\theta = \pi/4$

13. $f(x, y) = e^{xy}$, $\mathbf{p} = (0, 1)$, $\theta = \pi/2$

14. $f(x, y) = \cos(2x + y)$, $\mathbf{p} = (2, -3)$, $\theta = -\pi/3$

15. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Compute $D_{\mathbf{u}}f(0, 0)$, where $\mathbf{u} = (u, v)$ is a unit vector in $\mathbb{R}^2$.

16. Show that the function $f(x, y) = x^{1/3}y^{1/3}$ is continuous at $(0, 0)$ and has partial derivatives f_x and f_y at $(0, 0)$, but the directional derivative of f in any other direction does not exist.

Exercises 17 to 21: Determine the maximum rate of change of the function f at the point $\mathbf{p}$, and the direction in which it occurs.

17. $f(x, y) = \sec x \tan y$, $\mathbf{p} = (\pi/4, \pi/4)$

18. $f(x, y) = 2ye^x + e^{-y}$, $\mathbf{p} = (0, 0)$

19. $f(x, y, z) = xy^{-1} + yz^{-1} + zx^{-1}$, $\mathbf{p} = (1, 2, -1)$

20. $f(x, y, z) = \sqrt{xyz}$, $\mathbf{p} = (3, 3, 2)$

21. $f(x, y) = |xy|$, $\mathbf{p} = (3, -2)$

22. The temperature inside an object is given by $T(x, y, z) = 30(x^2 + y^2 + z^2)^{-1}$, at all points $(x, y, z) \neq (0, 0, 0)$.

(a) Find the rate of change of the temperature at the point $(1, 2, 0)$ inside the object in the direction toward the point $(2, -1, -1)$.

(b) Find the direction of the largest rate of increase in temperature at the point $(0, 0, 1)$.

(c) Find the direction of the most rapid decrease in temperature at a point (x, y, z) inside the object, if $(x, y, z) \neq (0, 0, 0)$.

23. The pressure $P(x, y)$ at a point $(x, y) \in \mathbb{R}^2$ on a metal membrane is given by the function $P(x, y) = 100e^{-x^2-2y^2}$.

(a) Find the rate of change of the pressure at the point $\mathbf{p} = (0, 1)$ in the direction $\mathbf{i} + \mathbf{j}$.

(b) In what direction away from the point $\mathbf{p}$ does the pressure increase most rapidly? Decrease most rapidly?

(c) Find the maximum rate of increase of pressure at $\mathbf{p}$.

(d) Locate the direction(s) at $\mathbf{p}$ in which the rate of change of pressure is zero.

24. Let $f(x, y) = e^x \cos(2x - y)$. Find the directional derivative of f at the point $(0, 1)$ in the direction of the line $y = 3x + 1$, for increasing values of x.

25. Consider the function $f(x, y) = 2xy$. In what directions at the point $(1, 2)$ is the directional derivative of f equal to 4?

▶ EXAMPLE 3.22 Torque Equals the Rate of Change of Angular Momentum

Let $\mathbf{r}(t)$ be the position vector of a particle moving in $\mathbb{R}^3$. The *angular momentum* is defined as the vector

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t),$$

where $\mathbf{p}(t) = m\mathbf{v}(t)$ is the momentum vector [m is the mass of the particle and $\mathbf{v}(t)$ is its velocity]. The *torque* of a force $\mathbf{F}(t) = m\mathbf{a}(t)$ exerted at the point $\mathbf{r}(t)$ is

$$\mathbf{T}(t) = \mathbf{r}(t) \times \mathbf{F}(t).$$

Since (with the use of the product rule for the cross product of vector functions)

$$\begin{aligned} \frac{d\mathbf{L}(t)}{dt} &= \frac{d}{dt}(\mathbf{r}(t) \times \mathbf{p}(t)) = \frac{d\mathbf{r}(t)}{dt} \times (m\mathbf{v}(t)) + \mathbf{r}(t) \times \frac{d(m\mathbf{v}(t))}{dt} \\ &= m\mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times m\frac{d\mathbf{v}(t)}{dt} \\ &= \mathbf{r}(t) \times m\mathbf{a}(t) = \mathbf{r}(t) \times \mathbf{F}(t) = \mathbf{T}(t), \end{aligned}$$

it follows [because $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$ and $\mathbf{v}'(t) = \mathbf{a}(t)$] that the torque $\mathbf{T}(t)$ on a particle equals the rate of change of the particle's angular momentum. ◀

▶ EXERCISES 3.2

1. Compute the velocity and speed of the cycloid defined in Example 3.2 in Section 3.1. Identify points where the speed is the largest.

2. Show that the speed of the logarithmic spiral $\mathbf{c}(t) = (e^{at}\cos t, e^{at}\sin t)$, $a \neq 0$, $t \geq 0$, is equal to $e^{at}\sqrt{1+a^2}$. Compute the angle that $\mathbf{c}(t)$ makes with the velocity vector and give an interpretation of your answer.

3. Show that the speed of the helix $\mathbf{c}(t) = (a\cos t, a\sin t, bt)$, $a > 0$, $b > 0$, is constant. Compute the dot product $\mathbf{c}'(t) \cdot \mathbf{c}''(t)$ and give a physical interpretation.

4. Find the maximum speed of the projectile in Example 3.14 and the time when it is reached.

Exercises 5 to 9: Find the velocity $\mathbf{v}(t)$ and the position $\mathbf{c}(t)$ of a particle, given its acceleration $\mathbf{a}(t)$, initial velocity, and initial position.

5. $\mathbf{a}(t) = (-1, 1, 0)$, $\mathbf{v}(0) = (1, 2, 0)$, $\mathbf{c}(0) = (0, 2, 0)$

6. $\mathbf{a}(t) = -9.8\mathbf{k}$, $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$, $\mathbf{c}(0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

7. $\mathbf{a}(t) = (t, 1, 1)$, $\mathbf{v}(0) = (0, 1, 0)$, $\mathbf{c}(0) = (-2, 0, 3)$

8. $\mathbf{a}(t) = e^t(1, 0, 1)$, $\mathbf{v}(0) = (1, 0, -2)$, $\mathbf{c}(0) = (0, 1, 0)$

9. $\mathbf{a}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $\mathbf{v}(0) = 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{c}(0) = 4\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$

10. Find a parametrization of the circle $x^2 + y^2 = 1$ of nonconstant speed. Find another parametrization $\mathbf{c}(t)$ such that $\|\mathbf{c}''(t)\|$ is nonconstant and $\|\mathbf{c}''(t)\| \neq 0$ for all t.

11. The position of a particle is given by $\mathbf{c}(t) = (t^{-1}, 1, t^2)$, where $t \in [1, 4]$. When and where does the particle reach its maximum speed?

12. The position of a particle is given by $\mathbf{c}(t) = (3e^{-t}\cos t, 3e^{-t}\sin t)$, where $t \in [1, 3]$. When and where does the particle reach its maximum speed?

13. A particle moves with acceleration $\mathbf{a}(t) = (3, 0, 1)$, where $0 \le t \le 12$. Assuming that the particle is initially located at the origin and its initial velocity is $(1, 3, 2)$, find the time needed for the particle to reach its highest position.

14. Prove that if a particle moves with constant speed, then its velocity and acceleration vectors are always perpendicular.

15. A projectile is fired from the origin with an initial speed of 700 m/s at an angle of elevation of 60^o. Find the range of the projectile, the maximum height reached, the time needed to reach it, and the speed and the time of impact. Assume that no forces other than gravity act on the projectile.

16. An object is thrown upward from a point 10 m above the ground at an angle of 30^o and with an initial speed of 100 m/s. Find a parametric equation of the path of the object. When does it reach its highest point? Where and when does it hit the ground? Assume that no forces other than gravity act on the object.

17. Find a parametrization of the line tangent to the ellipse $x^2 + 4y^2 = 3$ at the point where $x = \sqrt{3}$.

Exercises 18 to 21: Find a parametrization of the line tangent to the given curve $\mathbf{c}(t)$ at the point indicated.

18. $\mathbf{c}(t) = (2t, t^3, 0)$, at the point $(4, 8, 0)$

19. $\mathbf{c}(t) = (3\cos t, 3\sin t, 4t)$, at the point $(0, 3, 2\pi)$

20. $\mathbf{c}(t) = (t, t^2, t^3)$, at the point $(1, 1, 1)$

21. $\mathbf{c}(t) = (-\cosh t, 1 + \sinh t)$ at the point $(-1, 1)$

22. Let $\mathbf{c}(t) = (x(t), y(t))$ be a curve in $\mathbb{R}^2$ and let $\mathbf{c}(t_0) = (x_0, y_0)$. Find an equation and the slope of the tangent line at (x_0, y_0).

23. Let $\mathbf{c}(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t))$ be the trajectory of a particle moving in $\mathbb{R}^2$. Show that its velocity and acceleration are given by the expressions (drop the notation for the dependence on t) $\mathbf{v} = r'(\cos\theta, \sin\theta) + r\theta'(-\sin\theta, \cos\theta)$ and $\mathbf{a} = (r'' - r(\theta')^2)(\cos\theta, \sin\theta) + (2r'\theta' + r\theta'')(-\sin\theta, \cos\theta)$.

24. Show that if a particle moves along the spiral $(2e^t\cos t, 2e^t\sin t)$, then the angle between its position and velocity vectors is constant.

25. The curve $\mathbf{c}(t) = (e^{-t}\cos t, e^{-t}\sin t)$, $0 \le t \le 3\pi$, represents the trajectory of a particle moving in $\mathbb{R}^2$. Compute its velocity and find all points at which the velocity is horizontal or vertical.

26. Let $\mathbf{F}(x, y) = (-y, x)$. Compute the curve that is the image under $\mathbf{F}$ of $\mathbf{c}(t) = (\sin t, \cos t)$, $t \in [0, \pi]$. Describe in words the map $D\mathbf{F}$.

27. Define the map $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (x^2y - x^3, ye^x - 2)$. Find the tangent vector to the image of the curve $\mathbf{c}(t) = (\sin t, t^2 - t)$ under $\mathbf{F}$ at $t = 0$.

28. Let $\mathbf{c}(t)$ be a curve such that $\mathbf{c}(0) = (1, 1)$ and $\mathbf{c}'(0) = (2, -1)$. Find the tangent vector to the image of $\mathbf{c}$ under the map $\mathbf{F} = (-y/\sqrt{x^2 + y^2}, x/\sqrt{x^2 + y^2})$ at $t = 0$.

29. Define the map $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(\mathbf{x}) = A \cdot \mathbf{x}$, where A is a nonzero 2×2 matrix and the dot denotes matrix multiplication. Find the tangent vector to the image under $\mathbf{F}$ of the curve $\mathbf{c}(t)$ at $t = 0$ such that $\mathbf{c}(0) = (0, 0)$ and $\mathbf{c}'(0) = (c_1, c_2)$.

30. Assume that the curve given in polar coordinates by $r = A/(1 + B\cos\theta)$, where A and B are positive constants, represents the orbit of a planet revolving around the Sun.

▶ EXERCISES 3.3

1. Consider the curve parametrized by $\mathbf{c}(t) = (t, 1/t)$, $t \in [1, 2]$. Divide $[1, 2]$ into 5 subintervals of equal length, and sketch the curve $\mathbf{c}$ and polygonal path p_5 that approximates it. Approximate the length of p_5 using formula (3.13).

2. Consider the curve parametrized by $\mathbf{c}(t) = (t, e^{2t})$, $t \in [0, 1]$. Divide $[0, 1]$ into 4 subintervals of equal length, and sketch the curve $\mathbf{c}$ and polygonal path p_4 that approximates it. Approximate the length of p_4 using formula (3.13). Compare your approximation with the length of p_4 computed using the formula for the distance between two points.

3. Assume that $\mathbf{c}(t)\colon [a, b] \to \mathbb{R}^2$ is a differentiable path in $\mathbb{R}^2$, and consider the partition $[a = t_1, t_2], [t_2, t_3], \ldots, [t_n, t_{n+1} = b]$ of $[a, b]$, as in the beginning of the section. Let $\ell(c_i) = \|\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)\|$ be the length of the line segment from $\mathbf{c}(t_i)$ to $\mathbf{c}(t_{i+1})$.

(a) Show that $\ell(c_i) = \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2}$.

(b) Apply the Mean Value Theorem from one-variable calculus to show that there exist t_i^*, t_i^{**} in $[t_i, t_{i+1}]$ such that $x(t_{i+1}) - x(t_i) = x'(t_i^*)\Delta t$ and $y(t_{i+1}) - y(t_i) = y'(t_i^{**})\Delta t$, where $\Delta t = t_{i+1} - t_i$. Conclude that $\ell(c_i) = \sqrt{(x'(t_i^*))^2 + (y'(t_i^{**}))^2}\,\Delta t$.

(c) Using (b), find a formula for the length of the polygonal path p_n [see (3.13)]. Compute the limit as $n \to \infty$ to obtain the formula from Definition 3.3.

4. Two students run around a circular track, given by $\mathbf{c}(t) = (50 \sin t, 50 \cos t)$, $t \in [0, 2\pi]$. Student A runs according to $\mathbf{c}_A(t) = (50 \sin(t/5), 50 \cos(t/5))$, $t \in [0, 30\pi]$, and student B according to $\mathbf{c}_B(t) = (50 \sin(t/4), 50 \cos(t/4))$, $t \in [0, 32\pi]$. Note that both A and B start and end at the point $(0, 50)$.

(a) What is the length of the track?

(b) Which student is running faster? Compute the distance covered by each student.

5. Consider the curve that is the graph of the function $y = x^{2/3}$ on $[-1, 1]$.

(a) Show that y is not differentiable at 0. Conclude that the parametrization $\mathbf{c}(t) = (t, t^{2/3})$, $t \in [-1, 1]$, is not differentiable.

(b) Show that $\mathbf{c}(t) = (\cos^3 t, \cos^2 t)$, $t \in [-\pi, \pi]$, is a differentiable parametrization of the given curve.

(c) Prove that the parametrization in (b) is not smooth.

6. Prove that the statement we made in Example 3.27 is true; that is, by "unfolding" the helix, we obtain a straight-line segment (see Figure 3.27).

Exercises 7 to 14: Find the length of the path $\mathbf{c}(t)$.

7. $\mathbf{c}(t) = (\sin 2t, \cos 2t)$, $t \in [0, \frac{\pi}{2}]$

8. $\mathbf{c}(t) = (2t^{3/2}, 2t)$, from $(0, 0)$ to $(2, 2)$

9. $\mathbf{c}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j}$, $0 \le t \le \pi$

10. $\mathbf{c}(t) = t^3\mathbf{i} + t^2\mathbf{j}$, $-2 \le t \le 1$

11. $\mathbf{c}(t) = ((1 + t), (1 + t)^{3/2})$, $t \in [0, 1]$

12. $\mathbf{c}(t) = (e^{2t}, e^{-2t}, \sqrt{8}t)$, $t \in [0, 1]$

13. $\mathbf{c}(t) = (2t - t^2)\mathbf{i} + \frac{8}{3}t^{3/2}\mathbf{j} + \mathbf{k}$, from $t = 1$ to $t = 3$

14. $\mathbf{c}(t) = \cos^2 t\mathbf{i} + \sin^2 t\mathbf{j}$, from $t = 0$ to $t = 2\pi$

15. Show that the length of a logarithmic spiral $\mathbf{c}(t) = (e^{at} \cos t, e^{at} \sin t)$, where $a < 0$ and $t \ge 0$, is finite.

16. Find the length of the catenary curve given by $\mathbf{c}(t) = (t, a \cosh(t/a))$, $a > 0$, $t \in [-a, a]$.

17. Is it true that the curve $y = 2 \sin x$, $x \in [0, 2\pi]$ is twice as long as $y = \sin x$, $x \in [0, 2\pi]$?

▶ EXAMPLE 3.42

Find the curvature and osculating circle of the parabola $y = x^2$ at the point (0, 0).

SOLUTION

All computations that are needed here have been already done in Example 3.39. The osculating circle has radius of $1/\kappa = 1/2$, since the curvature of $y = x^2$ at the origin is $\kappa = 2$. It has to lie above (i.e., "inside") the parabola, since the normal $\mathbf{N}(0) = \mathbf{T}'(0)/\|\mathbf{T}'(0)\| = (0, 1) = \mathbf{j}$ points that way; see Figure 3.34. The center of the osculating circle has to lie on the line normal to the tangent at (0, 0); that is, on the y-axis. It follows that the equation $x^2 + (y - 1/2)^2 = 1/4$ represents the desired osculating circle.

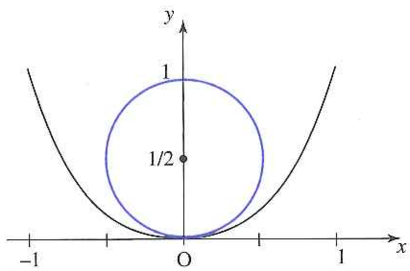


Figure 3.34 The parabola $y = x^2$ and its osculating circle at the origin.

We are now ready to give a geometric interpretation of the normal acceleration $\mathbf{a}_N$. Start with $\mathbf{a}_N = \|\mathbf{c}'(t)\|\,\mathbf{T}'(t)$, and divide and multiply the right side by $\|\mathbf{c}'(t)\|\,\|\mathbf{T}'(t)\|$ to get

$$\mathbf{a}_N = \|\mathbf{c}'(t)\|^2 \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{c}'(t)\|} \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

We recognize the first factor as the square of the speed $\|\mathbf{c}'(t)\| = ds/dt$. The second factor is the curvature $\kappa(t)$, and the third is the principal normal vector. Hence,

$$\mathbf{a}_N = \left(\frac{ds}{dt}\right)^2 \kappa(t)\mathbf{N}(t);$$

that is, the magnitude of the normal component of acceleration is the product of the square of the speed and the curvature.

▶ EXERCISES 3.4

Exercises 1 to 6: Find the tangential and normal components of acceleration for the motion of a particle described by its position vector $\mathbf{c}(t)$.

1. $\mathbf{c}(t) = (t^2, t, t^2)$

2. $\mathbf{c}(t) = (e^t, \sqrt{2}t, e^{-t})$

3. $\mathbf{c}(t) = 5t\mathbf{i} + 12\sin t\mathbf{j} + 12\cos t\mathbf{k}$

4. $\mathbf{c}(t) = 2t\mathbf{i} + 2\sin^2 t\mathbf{j} - 2\cos^2 t\mathbf{k}$

5. $\mathbf{c}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$

6. $\mathbf{c}(t) = (\cos 3t, \sin 3t, 4t)$

7. Let $\mathbf{c}(t) = (x(t), y(t))$ be a smooth C^2 parametrization of a curve $\mathbf{c}$ in $\mathbb{R}^2$. Show that its curvature is $\kappa(t) = |x'(t)y''(t) - x''(t)y'(t)|/[(x'(t))^2 + (y'(t))^2]^{3/2}$.

8. Using the formula from Exercise 7, compute the curvature of $\mathbf{c}_1(t) = (t\sin t, t\cos t)$ and $\mathbf{c}_2(t) = (t, t^2)$.

9. Compute the curvature of $\mathbf{c}(t) = (2 - 2t^3, t^3 + 1)$, $t \in \mathbb{R}$. Identify the curve, thus checking your answer.

10. Compute the curvature of $\mathbf{c}(t) = (t, \sin t)$, $t \in \mathbb{R}$, and plot the curve and its curvature function using the same coordinate system. Identify the points (if any) where the curvature is zero and where it is largest.

11. Find the curvature of the plane curve $\mathbf{c}(t) = (t^2, 3 - t)$. Identify the point(s) where the curvature is largest. What happens as $t \to \infty$?

Exercises 12 to 17: For each parametrization, find the unit tangent and the unit normal vector, the curvature, and the normal component of acceleration.

12. $\mathbf{c}(t) = (\sin 2t, \cos 2t, 5t)$

13. $\mathbf{c}(t) = (e^t \sin t, 0, e^t \cos t)$

14. $\mathbf{c}(t) = (3 + 2t, -t, t - 3)$

15. $\mathbf{c}(t) = (t, \cos t, 1 - \sin t)$

16. $\mathbf{c}(t) = (1, 0, t^2/2)$

17. $\mathbf{c}(t) = (e^{-t} \cos t, e^{-t} \sin t, e^{-t})$

18. Find equations of the lines tangent and normal to the curve $\mathbf{c}(t) = (t^3/3 - t)\mathbf{i} + t^2\mathbf{j}$ at the point $(0, 3)$.

19. Find an equation of the osculating circle of $\mathbf{c}(t) = t^3\mathbf{i} + t\mathbf{j}$ at the point $(8, 2)$.

20. Find an equation of the osculating plane of the curve $\mathbf{c}(t) = (1/t + 1, 1/t - 1, t)$ at a point $\mathbf{c}(t_0)$, $t_0 \neq 0$.

21. Find an equation of the osculating plane of the helix $\mathbf{c}(t) = (2\cos t, 2\sin t, t)$ at the point $\mathbf{c}(\pi/2)$.

22. Prove that the curvature of the graph of a C^2 function $y = f(x)$ is given by the formula $\kappa(x) = |f''(x)|/(1 + (f'(x))^2)^{3/2}$. Show that this formula is a special case of the formulas in Exercises 7 and 28.

Exercises 23 to 26: Use the formula of Exercise 22 to solve the following problems.

23. Find the curvature of $y = x^2$ at a point (x_0, y_0).

24. Find the curvature of $y = x + \ln x$ at $(1, 1)$. What happens to the curvature as $x \to \infty$?

25. Where does the graph of $y = \ln x$ have maximum curvature?

26. Find the curvature of the graph of $y = \sin x$ at $(\pi/2, 1)$ and at $(\pi, 0)$.

27. Find the equation of the osculating plane of the curve $\mathbf{c}(t) = (t, 1 - t^2, 2t^2)$ at the point $\mathbf{c}(1)$.

28. Prove that the curvature of a smooth C^2 curve $\mathbf{c}(t)$ in $\mathbb{R}^3$ can be computed from the formula $\kappa(t) = \|\mathbf{c}'(t) \times \mathbf{c}''(t)\|/\|\mathbf{c}'(t)\|^3$. Show that the equation of Exercise 7 is a special case of this formula.

29. Find the equation of the osculating circle of the graph of $y = \sin x$ at $(\pi/2, 1)$.

30. Find the equation of the osculating circle of the graph of $y = e^x$ at $(1, e)$.

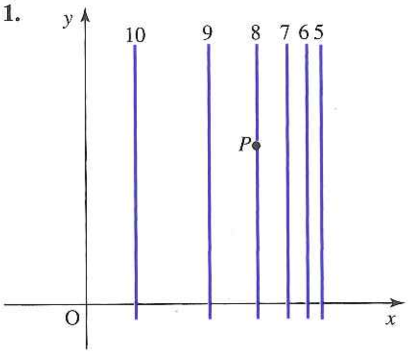
▶ 3.5 INTRODUCTION TO DIFFERENTIAL GEOMETRY OF CURVES

The material we have covered so far in this chapter shows that a large amount of information can be represented visually as a curve (or algebraically as its parametric representation). This is certainly a good reason to study curves in more depth. In this section, we only indicate one possible approach, the so-called differential geometry of curves. In Section 4.5, we will relate curves to vector fields by defining a flow line of a vector field. Concepts relevant to integration along paths will be discussed at the beginning of Chapter 5.

▶ EXERCISES 4.1

Exercises 1 to 4: Looking at the level curves of a function $f(x, y)$, determine whether the partial derivatives $f_x(P)$, $f_y(P)$, $f_{xx}(P)$, $f_{xy}(P)$, and $f_{yy}(P)$ are positive, negative, or zero.

1.



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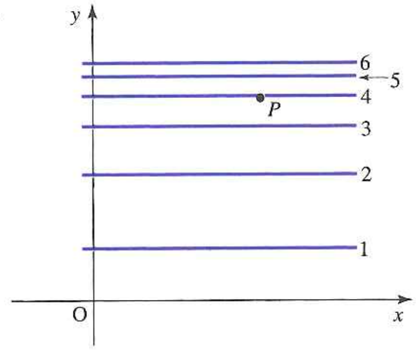

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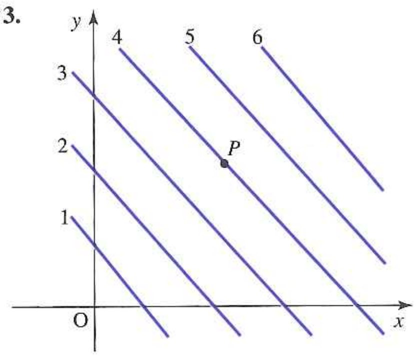

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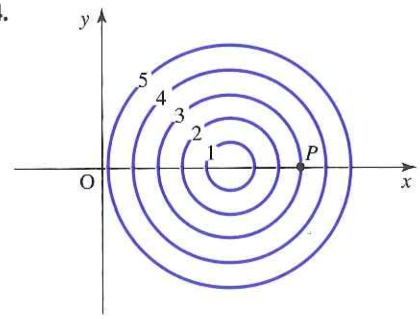


Exercises 5 to 13: Find the indicated second (or higher-order) partial derivatives of the given function.

5. $z = e^{xy} + \ln(x^2y^3)$; $z_{xx}, z_{xy}, z_{yx}, z_{yy}$

6. $z = x^y + (\ln y)^x$; $z_{xx}, z_{xy}, z_{yx}, z_{yy}$

7. $z = (x^2 + y^2)^{5/2}$; $z_{xx}, z_{xy}, z_{yx}, z_{yy}$

8. $z = x \arctan(y/x)$; $z_{xx}, z_{xy}, z_{yx}, z_{yy}$

9. $z = \sin^2(x + y)$; $z_{xx}, z_{xy}, z_{yx}, z_{yy}$

10. $z = f(x)g(y)$; $z_{xx}, z_{xy}, z_{yx}, z_{yy}$ (f and g are differentiable real-valued functions)

11. $z = f(ax + by) + g(ax/y)$; $z_{xx}, z_{xy}, z_{yx}, z_{yy}$ (f and g are differentiable real-valued functions of one variable and a and b are constants)

12. $z = e^{xy}$; $z_{xx}, z_{xxx}, z_{xxxx}, z_{yyyy}$

13. $w = y^3 \ln(x^2 + 3x + e^y) + x^3y^2z^4$; w_{xyzx}

14. A differential equation of the form $u_t = cu_{xx}$ where $u = u(x, t)$ and c is a constant, is called a *diffusion equation*.

(a) Show that $u(x, t) = e^{ax+bt}$ (a and b are constants) satisfies the diffusion equation with $c = b/a^2$.

(b) Show that $u(x, t) = t^{-1/2}e^{-x^2/t}$ satisfies the diffusion equation with $c = 1/4$.

15. Show that $z = xe^y + ye^x$ satisfies the equation $z_{xxx} + z_{yyy} = xz_{xyy} + yz_{yxx}$.

16. Explain why there is no C^2 function $f(x, y)$ such that $f_x(x, y) = e^x + xy$ and $f_y(x, y) = e^x + xy$.

Generalizing (4.22), we obtain the Hessian matrix of a function $f = f(x_1, x_2, \ldots, x_m)$ of m variables:

$$Hf(\mathbf{x}_0) = \begin{bmatrix} f_{x_1x_1}(\mathbf{x}_0) & f_{x_1x_2}(\mathbf{x}_0) & \ldots & f_{x_1x_m}(\mathbf{x}_0) \\ f_{x_2x_1}(\mathbf{x}_0) & f_{x_2x_2}(\mathbf{x}_0) & \ldots & f_{x_2x_m}(\mathbf{x}_0) \\ \vdots & \vdots & & \vdots \\ f_{x_mx_1}(\mathbf{x}_0) & f_{x_mx_2}(\mathbf{x}_0) & \ldots & f_{x_mx_m}(\mathbf{x}_0) \end{bmatrix}. \tag{4.23}$$

In order to obtain certain form of the remainder, we will need to assume that the function f in Theorem 4.4 is C^2 (see Exercise 19).

THEOREM 4.5 Second-Order Taylor Formula for Functions of *m* Variables

Assume that $f: U \subseteq \mathbb{R}^m \to \mathbb{R}$ has continuous second partial derivatives at $\mathbf{x}_0 \in U$. Then

$$f(\mathbf{x}_0 + \mathbf{h}) = T_2(\mathbf{x}_0, \mathbf{h}) + R_2(\mathbf{x}_0, \mathbf{h}),$$

where

$$T_2(\mathbf{x}_0, \mathbf{h}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h}^t Hf(\mathbf{x}_0)\mathbf{h}.$$

The second-order remainder $R_2(\mathbf{x}_0, \mathbf{h})$ satisfies $|R_2(\mathbf{x}_0, \mathbf{h})|/\|\mathbf{h}\|^2 \to 0$ as $\mathbf{h} \to \mathbf{0}$.

The main reason why we developed second-order Taylor formula is to analyze extreme values of functions of several variables (see Section 4.3).

Formula (4.17) can be generalized to

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n)}(0)}{n!} + R_n(0, 1).$$

Therefore, it is possible to compute nth-order Taylor formula (for $n \geq 1$) for a function of any number of variables (see Exercise 17). Since we will not use it in this book, we do not state it here. In Exercises 19 and 20, we derive formulas for remainders for the first-order and second-order Taylor formulas.

In Theorem 4.5 we assumed that f is C^2. However, in order to obtain certain forms of the remainder, we need to assume that f is C^3; see Exercises 16 and 20.

▶ EXERCISES 4.2

1. In computing the estimate for $R_1(x_0, h)$ in (4.10), we used the formula $|\int_{x_0}^{x_0+h} f(t)\,dt| \leq \int_{x_0}^{x_0+h} |f(t)|\,dt$. Explain why this formula works for $h \geq 0$ only. If $h < 0$, then $x_0 + h < x_0$, so we start the estimate by $|R_1(x_0, h)| = \left|\int_{x_0}^{x_0+h} (x_0 + h - t) f''(t)\,dt\right| = \left|\int_{x_0+h}^{x_0} (x_0 + h - t) f''(t)\,dt\right|$. Explain why this step is correct. Proceed as in (4.10) to complete the estimate.

2. Assuming that $|f'''(t)| \leq M$ for all $t \in [x_0, x_0 + h]$, prove that the second-order remainder (4.14) satisfies $|R_2(x_0, h)| \leq M|h|^3/2$. What condition(s) must f satisfy so that $|f'''(t)| \leq M$ for all $t \in [x_0, x_0 + h]$?

3. Apply integration by parts [with $u = f'''(t)$ and $dv = (x_0 + h - t)^2 dt$] to the formula (4.12) to obtain the third-order Taylor formula. Find an integral formula for the remainder $R_3(x_0, h)$, and show that $|R_3(x_0, h)|/|h|^3 \to 0$ as $h \to 0$.

4. Check that $T_3(x) = x - x^3/6$ is the third-order Taylor polynomial of $\sin x$ at $x_0 = 0$. Find an estimate for the error if $T_3(x)$ is used to compute $\sin x$ for $-0.1 \le x \le 0.1$.

5. Find the second-order Taylor polynomial for the function $f(x) = \sqrt{x}$ at $x_0 = 3$. Find an estimate for the error when $2 \le x \le 4$.

6. Find Taylor polynomials $T_2(x)$, $T_3(x)$, and $T_4(x)$ for the function $f(x) = e^x \cos x$ at $x_0 = 0$. Graph $f(x)$ and all three polynomials on $[-\pi/2, \pi/2]$.

Exercises 7 to 10: Find the second-order Taylor formula for the function $f(x)$ at the given point x_0. Give the remainder in integral form.

7. $f(x) = \sin x$, $x_0 = \pi/4$

8. $f(x) = \cos x$, $x_0 = \pi/3$

9. $f(x) = \ln x$, $x_0 = 4$

10. $f(x) = \sqrt{1 + x^2}$, $x_0 = 1$

11. Using the second-order Taylor polynomial, give an estimate for $\sin 0.1 + \cos 0.1$. Estimate the same expression using the third-order Taylor polynomial, and compare the two approximations.

12. Using the second-order Taylor polynomial, give an estimate for $0.087 \ln(1.087)$. Estimate the same expression using the first-order Taylor polynomial, and compare the two approximations.

13. Show that the equation $F(x, y) = y^3 - 4y + x^2 = 0$ defines y implicitly as a function of x, $y = g(x)$, near the point $x = 0$, $y = 2$. Find the second-order Taylor polynomial of $g(x)$ at $x = 0$.

14. Show that the equation $F(x, y) = \cos y - xy = 0$ defines implicitly, near the point $x = 0$, $y = \pi/2$, the function $y = g(x)$. Find the second-order Taylor polynomial of $g(x)$ at $x = 0$.

15. Find an approximation of $0.2e^{-0.2}$ using the second-order Taylor polynomial at $x_0 = 0$. Estimate the error term $R_2(0, 0.2)$.

16. Assume that $f = f(x, y)$ is of class C^3. Continuing the calculations preceding the statement of Theorem 4.3, obtain a formula for $F'''(t)$. Using (4.18), show that $R_2(\mathbf{x}_0, \mathbf{h}) = \frac{1}{2}\int_0^1 (t-1)^2 G(t)\,dt$, where $G(t) = f_{xxx}(\mathbf{x}_0 + t\mathbf{h})h_1^3 + 3f_{xxy}(\mathbf{x}_0 + t\mathbf{h})h_1^2 h_2 + 3f_{xyy}(\mathbf{x}_0 + t\mathbf{h})h_1 h_2^2 + f_{yyy}(\mathbf{x}_0 + t\mathbf{h})h_2^3$. Show that $|R_2(\mathbf{x}_0, \mathbf{h})| \le C\|\mathbf{h}\|^3$, where C is a positive constant.

17. Assume that $f = f(x, y, z)$ is of class C^2, and let $\mathbf{x}_0 = (x_0, y_0, z_0)$ and $\mathbf{h} = (h_1, h_2, h_3)$. Write down the formula for $T_2(\mathbf{x}_0, \mathbf{h})$. If $f = f(x_1, x_2, \dots, x_m)$ is of class C^2, and $\mathbf{h} = (h_1, h_2, \dots, h_m)$, show that $\mathbf{h}^t Hf(\mathbf{x}_0)\,\mathbf{h} = \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\, h_i h_j$. Write down the formula for $T_2(\mathbf{x}_0, \mathbf{h})$ in this case.

18. If $f = f(x_1, x_2, \dots, x_m)$ is of class C^2, how many entries in its Hessian matrix $Hf(\mathbf{x}_0)$ are repeated?

19. Assume that $f\colon U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function.

(a) Imitate the derivation of the formula (4.17) to obtain the first-order remainder $R_1(\mathbf{x}_0, \mathbf{h}) = R_1(0, 1) = \int_0^1 (1-t)F''(t)\,dt$. Show that $R_1(\mathbf{x}_0, \mathbf{h}) = \int_0^1 (1-t)G(t)\,dt$, where $G(t) = f_{xx}(\mathbf{x}_0 + t\mathbf{h})h_1^2 + 2f_{xy}(\mathbf{x}_0 + t\mathbf{h})h_1 h_2 + f_{yy}(\mathbf{x}_0 + t\mathbf{h})h_2^2$.

(b) Recall the Second Mean-Value Theorem for integrals: if g and h are continuous functions and $h(t) \ge 0$ on $[a, b]$, then $\int_a^b g(t)h(t)\,dt = g(c)\int_a^b h(t)\,dt$, where c is a number in $[a, b]$. Use this theorem to show that $R_1(\mathbf{x}_0, \mathbf{h}) = \frac{1}{2}\left(f_{xx}(\mathbf{c}_{11})h_1^2 + 2f_{xy}(\mathbf{c}_{12})h_1 h_2 + f_{yy}(\mathbf{c}_{22})h_2^2\right)$, where $\mathbf{c}_{11}$, $\mathbf{c}_{12}$, and $\mathbf{c}_{22}$ lie on the line joining $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{h}$.

(c) Generalize (b) to obtain the formula for $R_1(\mathbf{x}_0, \mathbf{h})$ in the case of a differentiable function of m variables.

20. Apply the Second Mean-Value Theorem for integrals [see (b) in Exercise 19] to the remainder from Exercise 16 to obtain a formula for $R_2(\mathbf{x}_0, \mathbf{h})$ for a function of two variables. Show that, if $f = f(x_1, x_2, \ldots, x_m)$ is of class C^3, and $\mathbf{h} = (h_1, h_2, \ldots, h_m)$, then $R_2(\mathbf{x}_0, \mathbf{h}) = \frac{1}{3!}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\mathbf{c}_{ijk})\, h_i h_j h_k$, where all $\mathbf{c}_{ijk}$ lie on the line joining $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{h}$.

21. Derive the third-order Taylor formula for a C^4 function $f(x, y)$ of two variables.

Exercises 22 to 27: Find the second-order Taylor formula for the function f at the given point $\mathbf{x}_0$.

22. $f(x, y) = e^{-x}\sin y$, $(2, 0)$

23. $f(x, y) = x^2 + y^2 - 2xy + 1$, $(1, 1)$

24. $f(x, y) = \ln(x^2 + y^2 + 1)$, $(0, 1)$

25. $f(x, y) = \sin x + \sin 2y$, $(0, \pi/2)$

26. $f(x, y) = (x-2)^2(y+4)^2$, $(0, 0)$

27. $f(x, y) = (xy)^{-1}$, $(1, 2)$

28. Find the first-order and second-order Taylor polynomials of the function $f(x, y) = \arctan(xy)$ at $(1, 1)$. Compare the two approximations of $f(1.15, 0.93)$ with the value of the function.

29. Find the first-order and second-order Taylor polynomials of the function $f(x, y) = \sqrt{x + 4y - 1}$ at $(5, 3)$. Compare the two approximations of $f(4.9, 3.1)$ with the value of the function.

30. Compute linear and quadratic approximations of $f(x, y) = (x + y + 3)^{-1}$ at $(0, 0)$. Compare the values of the two approximations at $(0.1, 0.04)$ with the value $f(0.1, 0.04)$.

31. Find the second-order Taylor polynomial of the function $f(x, y) = y \sin x$ at $(0, 1)$ and use it to draw an approximation of the contour diagram of $f(x, y)$ near $(0, 1)$.

32. Find the second-order Taylor polynomial of the function $f(x, y) = ye^{x^2}$ at $(0, 0)$ and use it to draw an approximation of the contour diagram of $f(x, y)$ near $(0, 0)$.

Exercises 33 to 36: Use the quadratic approximation (i.e., the second-order Taylor formula) to give estimates for the following expressions.

33. $e^{0.03^2 - 0.95^2}$

34. $0.98 \ln 1.03$

35. $3.98 \arctan 0.02$

36. $\sin 0.96 \cos 0.04$

▶ 4.3 EXTREME VALUES OF REAL-VALUED FUNCTIONS

In the mid-18th century, French mathematician and astronomer Pierre Louis Moreau de Maupertius formulated a so-called "metaphysical principle" that could serve as a guiding mechanism driving the laws of nature. The principle states that every "action" of nature is actually an attempt to minimize or maximize a certain quantity. An animal, sleeping in the snow, curls up—in doing so, it minimizes the surface area of its body that is exposed to cold temperatures and loses the least amount of heat. A river, flowing down a mountain, follows the curve of steepest descent (provided that there are no obstructions in its way). A large number of celestial objects have the (approximate) shape of a ball—and a ball can be shown to possess a remarkable number of minimizing and maximizing properties. Moreover, a "bottom line" in economics is usually a synonym for a maximum profit or a minimum loss. Our daily actions sometimes follow "the line of least resistance."

In this section, we develop tools that will allow us to investigate extreme values of functions of several variables. For simplicity, we focus our investigation on functions of two variables. Most results we state hold, however, for a function of any number of variables. The two problems we study are finding relative (or local) extreme values of a function on its domain and finding absolute (or global) extreme values of a function on a closed and bounded set contained in its domain.

Parametrize $\mathbf{c}_2$ by $\mathbf{c}_2(t) = (\cos t, \sin t)$, $t \in [\pi/2, 2\pi]$. The values of f along $\mathbf{c}_2$ are given by $g_2(t) = e^{-(\cos t)^2 - (\sin t)^2} = e^{-1}$. Thus, f is a constant function when viewed as a function on $\mathbf{c}_2$ only (we say that the restriction of f to $\mathbf{c}_2$ is a constant function).

Consequently, the absolute maximum of f is $f(0, 0) = 1$ and the absolute minimum is e^{-1} (it occurs at all points on $\mathbf{c}_2$, including its endpoints). ◀

Let us mention that results stated in this section generalize to functions of an arbitrary number of variables. The most important result is the generalization of the Extreme Value Theorem (see Theorem 4.11). It states that a continuous function f defined on a closed and bounded set $D \subseteq \mathbb{R}^m$, $m \geq 1$, attains its maximum and minimum values at some points $\mathbf{a}_1$ and $\mathbf{a}_2$ in D. ("Closed" is defined as in Definition 4.4, and "bounded" as in Definition 4.5, by replacing "open ball in $\mathbb{R}^2$" by "open ball in $\mathbb{R}^m$.")

There is an analogue of the Second Derivatives Test (with the same philosophy: second partials are used to determine what is happening at a critical point). Unfortunately, technical intricacies and difficulties increase proportionally to the number of variables.

▶ EXERCISES 4.3

1. Show that if $f: \mathbb{R} \to \mathbb{R}$ is an even differentiable function, then $x = 0$ is a critical point of f. A function $f: \mathbb{R}^2 \to \mathbb{R}$ is called *even* if $f(-x, -y) = f(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Assuming that f is differentiable, show that (0,0) is a critical point of f.

2. We discuss an alternative proof of the Second Derivatives Test, case (a).

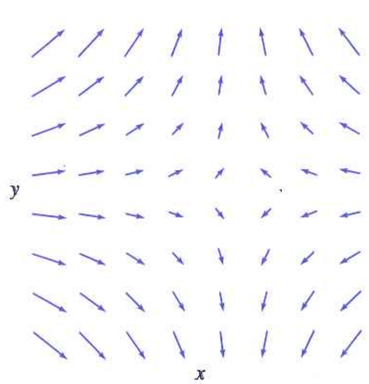
(a) By Theorem 2.8 in Section 2.7, the directional derivative of f in the direction of a unit vector $\mathbf{u} = (u_1, u_2)$ is given by $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = f_x u_1 + f_y u_2$. By the same theorem, $D_{\mathbf{u}}(D_{\mathbf{u}} f) = \nabla(D_{\mathbf{u}} f) \cdot \mathbf{u} = \frac{\partial}{\partial x}(D_{\mathbf{u}} f) u_1 + \frac{\partial}{\partial y}(D_{\mathbf{u}} f) u_2$. Continue this calculation to show that $D_{\mathbf{u}}(D_{\mathbf{u}} f) = f_{xx} u_1^2 + 2 f_{xy} u_1 u_2 + f_{yy} u_2^2$.

(b) Complete the square to get $D_{\mathbf{u}}(D_{\mathbf{u}} f) = f_{xx}\left(u_1 + \frac{f_{xy}}{f_{xx}} u_2\right)^2 + \frac{u_2^2}{f_{xx}}\left(f_{xx} f_{yy} - f_{xy}^2\right)$.

(c) Explain why we can find an open ball B centered at (x_0, y_0) such that $f_{xx}(x, y) < 0$ and $D(x, y) > 0$ for all (x, y) in B. Show that $D_{\mathbf{u}}(D_{\mathbf{u}} f)(x, y) < 0$ for all (x, y) in B. Conclude that $f(x_0, y_0)$ is a local maximum.

Exercises 3 to 6: Looking at the gradient vector field of a differentiable function $f(x, y)$, identify points (or say there are none) where f has a local minimum, local maximum, or saddle point.

3.



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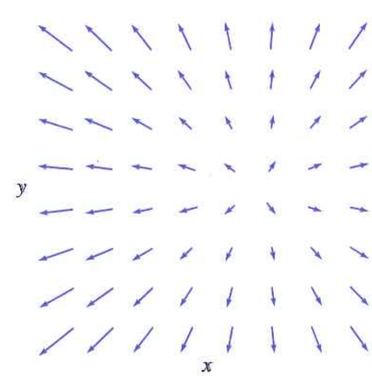

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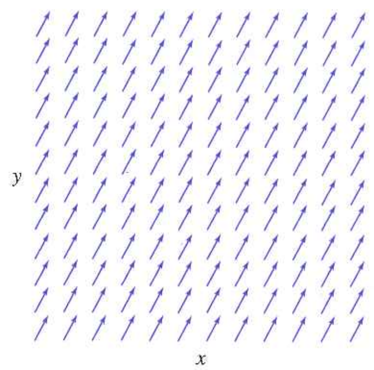

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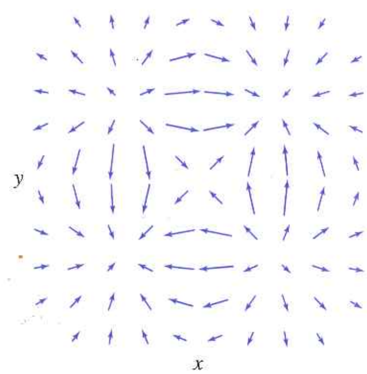


Exercises 7 to 16: Find all critical points (if any) of a given function $f(x, y)$ and determine whether they are local extreme points or saddle points.

7. $f(x, y) = x^2 + y^2 + xy^2$

8. $f(x, y) = x + y + \dfrac{2}{xy}$

9. $f(x, y) = xy + \dfrac{x + y}{xy}$

10. $f(x, y) = x^3 + y^3 + 3x^2y - 3y$

11. $f(x, y) = xye^{-x^2-y^2}$

12. $f(x, y) = e^x \sin y$

13. $f(x, y) = x \cos y$

14. $f(x, y) = \ln(x^2 + y^2 + 2)$

15. $f(x, y) = x \sin(x + y)$

16. $f(x, y) = (x + y)(xy - 1)$

17. Find the shortest distance from the point (2, 0, 3) to the plane $x - y + z = 4$.

18. Find the shortest distance between the surface $z = 1/xy$ and the origin.

19. Find the dimensions of a closed, rectangular box of given volume $V > 0$ that has minimum surface area.

20. Find the point(s) on the surface $xyz + 1 = 0$ that are closest to the origin.

21. Find the volume of the largest (i.e., of maximum volume) rectangular box that can be inscribed into the sphere of radius $R > 0$.

22. Suppose that you have to build a rectangular box (with a lid) using $S > 0$ units^2 of material. Find the dimensions of the box that has the largest possible volume.

23. It was shown that the function $g(x, y) = x^4 - y^4$ of Example 4.28 has a saddle point at (0, 0). Draw the contour curve that goes through (0, 0). Add a few more level curves to your picture.

24. Find all points where the magnitude of the vector field $\mathbf{F} = (x - y)\mathbf{i} + (2x + y + 3)\mathbf{j}$ attains its local minimum.

25. A plane in a three-dimensional space, which is not parallel to any of the three coordinate planes, can be analytically described using the equation $x/a + y/b + z/c = 1$, where a, b, and c are its x-intercept, y-intercept, and z-intercept, respectively. Find the plane that passes through (1, 1, 1) and is such that the solid in the first octant bounded by that plane has the smallest volume.

Exercises 26 to 29: Find the absolute minimum and absolute maximum of a given function $f(x, y)$ on a set D.

26. $f(x, y) = xy - 3x + y$; D is the triangular region with vertices (0, 0), (2, 0), and (0, 2)

27. $f(x, y) = \ln(x^2 + y + 1)$; D is the triangular region with vertices (0, 0), (1, 0), and (1, 1) that satisfy

$$\nabla f(x_0, y_0, z_0) = \lambda_1 \nabla g_1(x_0, y_0, z_0) + \lambda_2 \nabla g_2(x_0, y_0, z_0), \tag{4.30}$$

where λ_1 and λ_2 are real numbers and $g_1(x_0, y_0, z_0) = k_1$ and $g_2(x_0, y_0, z_0) = k_2$. It is assumed that $\nabla g_1(x_0, y_0, z_0)$ and $\nabla g_2(x_0, y_0, z_0)$ are not parallel vectors.

Geometrically, the two constraints $g_1(x, y, z) = k_1$ and $g_2(x, y, z) = k_2$ represent the intersection (call it D) of the surfaces $g_1(x, y, z) = k_1$ and $g_2(x, y, z) = k_2$ in space. If D is a closed and bounded set in $\mathbb{R}^3$, then f must have a maximum and minimum (subject to the given constraints). Otherwise, additional arguments may be needed to determine whether a point (x_0, y_0, z_0) is a minimum point, a maximum point, or neither.

We now illustrate this in an example.

▶ EXAMPLE 4.45

Find the maximum and minimum values of $f(x, y, z) = 2x + y - z$ subject to the constraints $2x + z = 2/\sqrt{5}$ and $y^2 + z^2 = 1$.

SOLUTION

Label the constraints as $g_1(x, y, z) = 2x + z = 2/\sqrt{5}$ and $g_2(x, y, z) = y^2 + z^2 = 1$. From $\nabla f = (2, 1, -1)$, $\nabla g_1 = (2, 0, 1)$, and $\nabla g_2 = (0, 2y, 2z)$ (notice that ∇g_1 and ∇g_2 are not parallel), using $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$, we get $(2, 1, -1) = \lambda_1(2, 0, 1) + \lambda_2(0, 2y, 2z)$. Thus, we obtain the system

$$2 = 2\lambda_1, \qquad 1 = 2y\lambda_2, \qquad -1 = \lambda_1 + 2z\lambda_2,$$

which, combined with the two constraints $2x + z = 2/\sqrt{5}$ and $y^2 + z^2 = 1$, will give points where extreme values might occur. The constraint g_1 represents a plane and the constraint g_2 represents a cylinder. Their intersection is an ellipse, which is a closed and bounded set—thus, the minimum value and maximum value must exist.

From the first equation, $\lambda_1 = 1$. The second and the third equations imply that $\lambda_2 = 1/2y$ and $\lambda_2 = -1/z$; thus, $z = -2y$. Using the constraint $y^2 + z^2 = 1$, we get $5y^2 = 1$ and $y = \pm 1/\sqrt{5}$. It follows that $z = \mp 2/\sqrt{5}$. If $z = 2/\sqrt{5}$, then from $2x + z = 2/\sqrt{5}$, we get $x = 0$. Similarly, from $z = -2/\sqrt{5}$ (using $2x + z = 2/\sqrt{5}$), we obtain $2x = 4/\sqrt{5}$ and $x = 2/\sqrt{5}$. Thus, there are two candidates: $(2/\sqrt{5}, 1/\sqrt{5}, -2/\sqrt{5})$ and $(0, -1/\sqrt{5}, 2/\sqrt{5})$. It follows that the maximum of f is $f(2/\sqrt{5}, 1/\sqrt{5}, -2/\sqrt{5}) = 7/\sqrt{5}$ and the minimum of f is $f(0, -1/\sqrt{5}, 2/\sqrt{5}) = -3/\sqrt{5}$. ◀

It is possible to further generalize the method of Lagrange multipliers (so that it applies to functions of four, five, and more variables, and to more than two constraints).

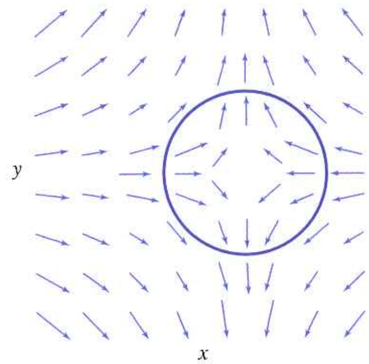
▶ EXERCISES 4.4

1. Explain geometrically (i.e., by sketching level curves) why the function $f(x, y) = (x^2 + y^2)^{-1}$ cannot have a minimum or maximum subject to the constraint $x - y = 0$.

2. Minimize the function $f(x, y) = \sqrt{(x-2)^2 + (y-2)^2}$ subject to $x + y = 0$. Give a geometric interpretation of your answer.

Exercises 3 to 6: Shown is the gradient field of a C^1 function f. Find the approximate locations of the minimum and maximum of f subject to the given constraint curve.

3.



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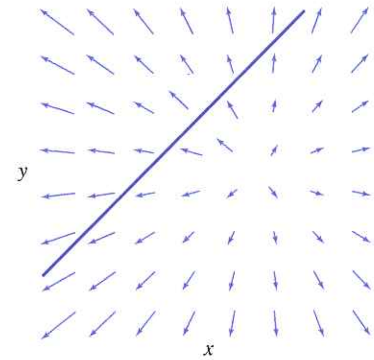

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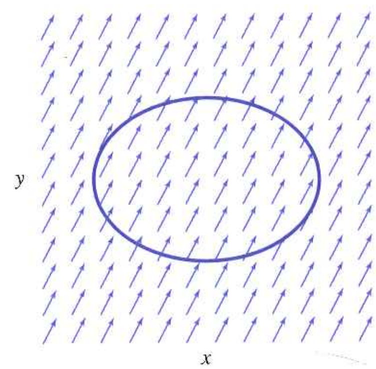

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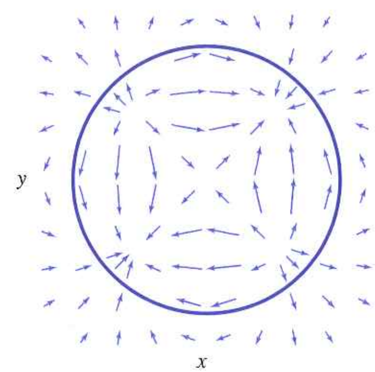

7. Sketch the level curves $f(x, y) = c$ of $f(x, y) = x^2 + y^2$ for $c = 1, 2, 4, 6, 9$, and 10. In the same coordinate system, sketch the graph of the constraint $(x - 1)^2 + y^2 = 4$.

(a) Looking at your picture, identify the minimum and maximum of f subject to the given constraint.

(b) Use the method of Lagrange multipliers to confirm your geometric reasoning.

8. Sketch the level curves $f(x, y) = c$ of $f(x, y) = 2x - y$ for $c = -3, -2, -1, 0, 1, 2$, and 3. In the same coordinate system, sketch the graph of the constraint $x^2 + y^2 = 1$.

(a) Looking at your picture, identify the minimum and maximum of f subject to the given constraint.

(b) Use the method of Lagrange multipliers to obtain the desired constrained minimum and maximum algebraically.

9. State the problem in Example 4.26 in Section 4.3 as a constrained optimization problem and solve it using Lagrange multipliers.

10. Explain why it does not make much sense to develop the Lagrange multipliers method to optimize a function $f(x, y)$ of two variables subject to two constraints $g_1(x, y) = k_1$ and $g_2(x, y) = k_2$.

Exercises 11 to 19: Find the extreme values (if any) of a function f subject to the given constraint.

11. $f(x, y) = 3xy$; $x^2 + y^2 = 4$

12. $f(x, y) = 4 - x^2 - y^2$; $y - 2x = 1$ (*Hint:* Find the maximum; argue that a minimum does not exist.)

13. $f(x, y) = 2x^2 - y^2$; $x^2 + y^2 = 1$

14. $f(x, y, z) = x - y + 4z$; $x^2 + y^2 + z^2 = 2$

15. $f(x, y) = x - y^2$; $x^3 - y^2 = 0$, $-1 \leq y \leq 1$

16. $f(x, y, z) = x^2 + 2y^2$; $x^2 + y^2 - z^2 = 1$ (*Hint:* Find the minimum; argue that a maximum does not exist.)

17. $f(x, y, z) = x + 2y - 4z$; $x^2 + y^2 + 2z^2 = 4$

18. $f(x, y, z) = xyz$; $x^2 + y^2 + z^2 = 9$

19. $f(x, y) = xy + 2y$; $x^2 + 2y^2 = 4$

20. Find all points on the curve $y^2x = 16$ that are closest to the origin.

21. Find the minimum distance from the surface $x^2 + y^2 - z^2 = 4$ to the origin.

22. The temperature at a point (x, y) on a metal plate in the shape of the disk $x^2 + y^2 \leq 50$ is $T(x, y) = 2x^2 - xy + 2y^2 + 10$. Find the coldest point on the rim of the plate.

23. Find the point in the plane $x + y + 2z = 11$ that is closest to the point $(0, 1, 1)$.

24. Find the dimensions of a cylindrical can (with a lid) with a volume of 10 units^3 and minimum surface area.

25. Find the minimum of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $2y + z = 6$ and $x - 2y = 4$.

26. Find the minimum and maximum of $f(x, y, z) = xy + z^2$ subject to the constraints $x + y = 0$ and $x^2 + y^2 + z^2 = 4$.

27. Find the point closest to the origin that belongs to the intersection of the planes $2z - y = 0$ and $x + y - z = 4$.

28. Find the extreme values of the function $f(x, y, z) = x + y + 4z$ along the ellipse that is the intersection of the cylinder $x^2 + y^2 = 82$ and the plane $z = 2x$.

29. Solve Exercise 21 in Section 4.3 using the method of Lagrange multipliers.

30. Solve Exercise 22 in Section 4.3 using the method of Lagrange multipliers.

31. Solve Exercise 25 in Section 4.3 using the method of Lagrange multipliers.

32. Let (x_0, y_0) be the point where a differentiable function $f(x, y)$ attains its maximum subject to the constraint $g(x, y) = k$.

(a) As k changes, so does the location of the constrained maximum; that is, x_0 and y_0 become functions of k. Consequently, $f(x_0, y_0)$ is a function of k. Show that, at the point (x_0, y_0), df/dk satisfies $\frac{df}{dk} = \frac{\partial f}{\partial x}\frac{dx_0}{dk} + \frac{\partial f}{\partial y}\frac{dy_0}{dk}$.

(b) Show that at (x_0, y_0), the right side is equal to $\lambda\left(\frac{\partial g}{\partial x}\frac{dx_0}{dk} + \frac{\partial g}{\partial y}\frac{dy_0}{dk}\right) = \lambda\frac{dg}{dk}$.

(c) Explain why $(dg/dk)(x_0, y_0) = 1$. Conclude that $\lambda = (df/dk)(x_0, y_0)$, and interpret the result.

▶ 4.5 FLOW LINES

Assume that the motion of a fluid is described by a vector field **F** (i.e., the value of **F** at a point gives the velocity of the fluid at that point). One way of visualizing **F** is to isolate a point in the fluid and follow its trajectory under the influence of the field. The path thus obtained is called a flow line.

A familiarity with basic concepts in the theory of ordinary differential equations is needed in this section.