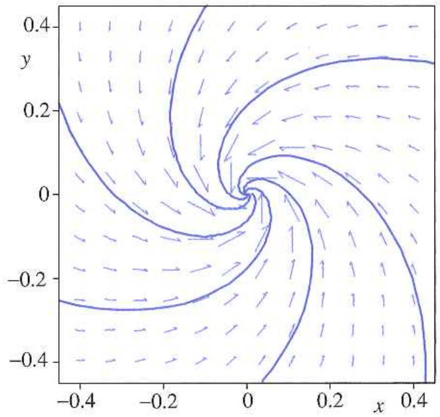


Figure 4.30 Flow lines $r = e^{-\theta+D}$ of the field **H** of Example 4.49.

and, after integration, $\theta = -\frac{1}{2}\ln(-2t + C) + D$, where D is another constant. Eliminating t, we get

$$\theta = -\tfrac{1}{2}\ln r^2 + D,$$

that is, $\theta = -\ln r + D$ and $\ln r = -\theta + D$. It follows that the flow lines of **H** are the spirals represented (in polar coordinates) by $r = e^{-\theta+D}$; see Figure 4.30. ◀

▶ EXERCISES 4.5

1. Find the flow lines of the vector field $\mathbf{F}(x, y) = (x, 2y)$. Compare with the flow lines of the vector fields $\mathbf{F}_1(x, y) = (3x, 6y)$ and $\mathbf{F}_2(x, y) = (-2x, -4y)$.

2. Show that the curve $\mathbf{c}(t) = \left(\frac{3}{5}\cos t + \frac{4}{5}\sin t\right)\mathbf{i} + \left(-\frac{3}{5}\sin t + \frac{4}{5}\cos t\right)\mathbf{j}$ is the flow line of the vector field $\mathbf{F}(x, y) = y\mathbf{i}/\sqrt{x^2+y^2} - x\mathbf{j}/\sqrt{x^2+y^2}$ going through the point $(4/5, -3/5)$.

3. Find the flow line of the vector field $\mathbf{F}(x, y) = 2\|\mathbf{r}\|^{-1}\mathbf{r}$ (where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$) going through the point $(3, 2)$.

4. Find the flow line of the constant vector field $\mathbf{F}(x, y) = 3\mathbf{i} - 4\mathbf{j}$ that goes through the point $(-2, 1)$. (*Hint:* There is no need to solve differential equations.)

5. Find the flow line of the constant vector field $\mathbf{F}(x, y) = a\mathbf{i} + b\mathbf{j}$ (a, b are real numbers, $a \neq 0$, and/or $b \neq 0$) that goes through the origin. (*Hint:* There is no need to solve differential equations.)

Exercises 6 to 10: Sketch the vector field **F** and several of its flow lines.

6. $\mathbf{F}(x, y) = y\mathbf{i} - 2x\mathbf{j}$

7. $\mathbf{F}(x, y) = (x, x^2)$

8. $\mathbf{F}(x, y) = x\mathbf{i} + \mathbf{j}$

9. $\mathbf{F}(x, y) = (-2x, y)$

10. $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$

11. Find a vector field for which the curve $\mathbf{c}(t) = (t^2, 2t, t)$, $t \in \mathbb{R}$ is a flow line.

12. Show that the curve $\mathbf{c}(t) = (e^t, 2\ln t, t^{-1})$, $t > 0$ is a flow line of the vector field $\mathbf{F}(x, y, z) = (x, 2z, -z^2)$.

13. Assume that $\mathbf{c}(t)$, $t \in [a, b]$, is a flow line of **F**. Show that $\boldsymbol{\gamma}(t) = \mathbf{c}(a + b - t)$, $t \in [a, b]$, is a flow line of $-\mathbf{F}$. Interpret this fact geometrically.

SOLUTION In Example 2.40 in Section 2.4, it was shown that

$$\nabla V(x, y, z) = \frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r} = -\mathbf{F},$$

where **F** is the gravitational force field. Therefore,

$$\Delta V = div(\nabla V) = -div\,\mathbf{F}.$$

The fact that $div\,\mathbf{F} = 0$ (see Example 4.52) completes the proof.

▶ EXAMPLE 4.66 Laplace Operator Describes Diffusion

The concentration of a liquid changes ("diffuses") when some chemical is dissolved in it. The heat of a solid "diffuses," "flowing" from warmer regions toward cooler ones. Such processes of "transport" (or "transfer") are described by a *flux density vector field* **F**. In the case of heat transfer, $\mathbf{F} = -k\nabla T$, where T is the temperature (see Example 2.90). Heat transfer is a special case of *Fick's Law,* which states that the flux vector **F** is always parallel (and of the opposite direction) to the gradient of the "species" concentration:

$$\mathbf{F}(x, y, z) = -k\nabla f(x, y, z);$$

see Figure 4.42. For example, $f(x, y, z)$ could be the concentration of bacteria in air or the concentration of acid in a water solution. The symbol k ($k > 0$) denotes a constant, whose name (*conductivity, diffusivity*) depends on the process considered. The minus sign in Fick's Law indicates that the direction of the flow is always *away* from regions of higher concentration.

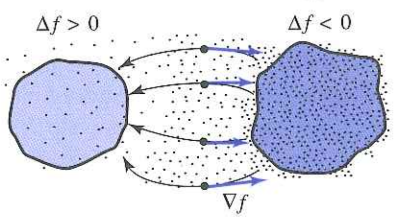


Figure 4.42 The Laplace operator describes a diffusion process.

The divergence of **F** is

$$div\,\mathbf{F}(x, y, z) = -k\,div\,(\nabla f(x, y, z)) = -k\Delta f(x, y, z).$$

We have seen that the divergence measures the net outflow of the "species" (i.e., "species that go out"—"species that go in"). At a point where the Laplacian Δ is negative, the outflow is positive, and the "species" must "go away" from that point; that is, the concentration decreases. Similarly, if $\Delta f(x, y, z) > 0$, then $div\,\mathbf{F} < 0$, and therefore, the inflow is larger than the outflow, and the concentration increases. (It is assumed, of course, that there are no outside "sources" or "sinks.") The equilibrium for a diffusion process is attained when the concentration "evens out" or "averages out"—in that case, the flow "stops" and the Laplacian of f is zero.

▶ EXERCISES 4.6

Exercises 1 to 6: Let f be a scalar function and let **F** and **G** be vector fields in $\mathbb{R}^3$. State whether each expression is a scalar function, a vector field, or meaningless.

1. $grad\,(grad\,f)$
2. $curl\,(grad\,f) - \mathbf{G}$
3. $curl\,(\mathbf{F} - \mathbf{G}) \times grad\,(div\,\mathbf{F})$
4. $div\,(div\,\mathbf{F})$
5. $div\,(curl\,(grad\,f))$
6. $grad\,f^2 \times grad\,(\mathbf{F} \cdot \mathbf{G})$

Exercises 7 to 10: Find an example of a vector field (write down a formula, or make a sketch) that satisfies the following requirements.

7. $curl\,\mathbf{F} = \mathbf{0}$ and $div\,\mathbf{F} = 0$

8. $curl\,\mathbf{F} \neq \mathbf{0}$ and $div\,\mathbf{F} = 0$

9. $curl\,\mathbf{F} = \mathbf{0}$ and $div\,\mathbf{F} \neq 0$

10. $curl\,\mathbf{F} \neq \mathbf{0}$ and $div\,\mathbf{F} \neq 0$

11. Sketch a vector field in $\mathbb{R}^2$ whose divergence is positive at all points.

12. Sketch a vector field in $\mathbb{R}^2$ whose divergence is zero at all points.

Exercises 13 to 16: Find the curl and divergence of the vector field **F**.

13. $\mathbf{F}(x, y, z) = y^2z\mathbf{i} - xz\mathbf{j} + xyz\mathbf{k}$

14. $\mathbf{F}(x, y, z) = (\ln z + xy)\mathbf{k}$

15. $\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)(3\mathbf{i} + \mathbf{j} - \mathbf{k})$

16. $\mathbf{F}(x, y, z) = e^{xy}\mathbf{i} + e^{yz}\mathbf{j} + e^{xz}\mathbf{k}$

17. Let $\mathbf{F}(x, y) = (f(x), 0)$, where $f(x)$ is a differentiable function of one variable. Show that the total outflow from a rectangle R with sides Δx and Δy placed in the flow (as in Figure 4.31) is given by $O(\Delta t) = (f(x + \Delta x) - f(x))\Delta t \Delta y$. Conclude that $\frac{1}{\Delta x \Delta y}\frac{O(\Delta t)}{\Delta t} \approx f'(x) = div\mathbf{F}$. Repeat the calculation with the vector field $\mathbf{F}(x, y) = (0, g(y))$ (g is a differentiable function of one variable) and show that the total outflow is again approximately equal to $div\mathbf{F}$.

18. What are the flow lines of the vector field $\mathbf{F}(x, y) = (-x, -y)$? Determine geometrically the sign of its divergence.

19. It can be easily checked that $curl\,\mathbf{r} = \mathbf{0}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Interpret this result physically, by visualizing **r** as the velocity vector field of a fluid.

20. Consider the vector fields $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, $\mathbf{G} = \mathbf{F}/\sqrt{x^2 + y^2}$, and $\mathbf{H} = \mathbf{F}/(x^2 + y^2)$. Compare their divergences and curls. Show that circles centered at the origin are the flow lines for all three vector fields. Describe their differences in physical terms.

Exercises 21 to 25: It will be shown in the next chapter that a vector field **F** defined on all of $\mathbb{R}^3$ (or all of $\mathbb{R}^2$) is conservative if and only if $curl\,\mathbf{F} = \mathbf{0}$. Determine whether the vector field **F** is conservative or not. If it is, find its potential function (i.e., find a real-valued function V such that $\mathbf{F} = -grad\,V$).

21. $\mathbf{F}(x, y, z) = \cos y\mathbf{i} + \sin x\mathbf{j} + \tan z\mathbf{k}$

22. $\mathbf{F}(x, y, z) = -y^2z\mathbf{i} + (3y^2/2 - 2xyz)\mathbf{j} - xy^2\mathbf{k}$

23. $\mathbf{F}(x, y) = 3x^2y\mathbf{i} + (x^3 + y^3)\mathbf{j}$

24. $\mathbf{F}(x, y, z) = x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$

25. $\mathbf{F}(x, y, z) = -y\mathbf{i} - x\mathbf{j} - 3\mathbf{k}$

26. Check whether the vector field $\mathbf{F}(x, y) = \mathbf{i}/(x \ln xy) + \mathbf{j}/(y \ln xy)$ is conservative for $x, y > 0$, and if so, find all functions f such that $\mathbf{F} = grad\,f$.

27. Verify that $curl\,(grad\,f) = 0$ for the function $f(x, y, z) = (x^2 + y^2 + z^2)^{-1}$.

28. Verify that $\partial(curl\,\mathbf{F})_1/\partial x + \partial(curl\,\mathbf{F})_2/\partial y + \partial(curl\,\mathbf{F})_3/\partial z = 0$ for the vector field $\mathbf{F}(x, y, z) = 3x^3y^2\mathbf{i} + yx^2\mathbf{j} - x^3z^3\mathbf{k}$, where $(curl\,\mathbf{F})_1$, $(curl\,\mathbf{F})_2$, and $(curl\,\mathbf{F})_3$ are the components of $curl\,\mathbf{F}$.

29. Is there a C^2 vector field **F** such that $curl\,\mathbf{F} = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$? Explain.

30. Is there a C^2 vector field **F** such that $curl\,\mathbf{F} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$? If so, find such a field.

31. A vector field **F** is irrotational if $curl\,\mathbf{F} = \mathbf{0}$. Show that any vector field of the form $\mathbf{F}(x, y, z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$, where f, g, and h are differentiable real-valued functions of one variable, is irrotational.

32. A vector field **F** is incompressible if $div\,\mathbf{F} = 0$. Show that any vector field of the form $\mathbf{F}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k}$, where f, g, and h are differentiable real-valued functions of two variables, is incompressible.

33. (For those familiar with complex numbers.) Show that the real and imaginary parts of the complex-valued function $z = (x - iy)^3$ (taken as the **i** and **j** components of a vector field whose **k** component is equal to 0) define an incompressible and irrotational vector field.

34. Find constants a, b, and c so that the vector field $\mathbf{F} = (3x - y + az)\mathbf{i} + (bx - z)\mathbf{j} + (4x + cy)\mathbf{k}$ is irrotational. Find the scalar function f so that $\mathbf{F} = \mathit{grad}\, f$.

35. Show that if the function f is harmonic (i.e., $\Delta f = 0$), then $\mathit{grad}\, f$ is not only an irrotational vector field but also an incompressible vector field.

36. Find the most general differentiable function $f(\|\mathbf{r}\|)$ defined on $\mathbb{R}^2$ such that the vector field $f(\|\mathbf{r}\|)\mathbf{r}$ is incompressible.

37. Show that the vector field $\mathbf{F} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$ is not incompressible but the vector field $\mathbf{G} = xyz^2\mathbf{F}$ is incompressible.

38. Prove that $\mathbf{F} \times \mathbf{G}$ is incompressible if the vector fields $\mathbf{F}$ and $\mathbf{G}$ are irrotational.

39. If f is a differentiable function of one variable, show that $f(\|\mathbf{r}\|)\mathbf{r}$ is an irrotational vector field.

Exercises 40 to 48: Prove the following identities, assuming that the functions and vector fields involved are differentiable as many times as needed. State those assumptions in each case. The vector $\mathbf{r}$ is the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

40. $\mathit{div}\,(f\mathbf{F}) = f\,\mathit{div}\,\mathbf{F} + \mathbf{F} \cdot \mathit{grad}\, f$

41. $\mathit{curl}\,(f\mathbf{F}) = f\,\mathit{curl}\,\mathbf{F} + (\mathit{grad}\, f) \times \mathbf{F}$

42. $\mathit{curl}\,\mathbf{r} = \mathbf{0}$

43. $\mathit{div}\,\mathbf{r} = 3$

44. $\mathit{grad}\,\|\mathbf{r}\| = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$

45. $\mathit{div}\,(\|\mathbf{r}\|\mathbf{r}) = 4\|\mathbf{r}\|$

46. $\mathit{div}\,(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \mathit{curl}\,\mathbf{F} - \mathbf{F} \cdot \mathit{curl}\,\mathbf{G}$

47. $\mathit{div}\,(\mathit{grad}\, f \times \mathit{grad}\, g) = 0$

48. $\Delta\|\mathbf{r}\|^3 = 12\|\mathbf{r}\|$

49. Evaluate the expression $\mathit{div}\,(\mathbf{F} \times \mathbf{r})$ if $\mathit{curl}\,\mathbf{F} = \mathbf{0}$, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

50. Evaluate the expression $\mathit{curl}\,(f(\|\mathbf{r}\|)\mathbf{r})$, where f is a differentiable scalar function and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

▶ 4.7 IMPLICIT FUNCTION THEOREM

In this section, we state a general version of the Implicit Function Theorem. We have seen the importance of its special case in Section 3.1, where we studied curves defined by the equation $F(x, y) = 0$, for a C^1 function $F\colon \mathbb{R}^2 \to \mathbb{R}$.

We start by giving a straightforward generalization (without proof) of Theorem 3.1 from Section 3.1 to functions of many variables. We will denote points in $\mathbb{R}^{m+1}$ $(m \geq 1)$ by $(\mathbf{x}, z)$, where $\mathbf{x} \in \mathbb{R}^m$ and $z \in \mathbb{R}$.

THEOREM 4.14 Implicit Function Theorem, Special Case

Assume that a function $F\colon \mathbb{R}^{m+1} \to \mathbb{R}$ is of class C^1, $F(\mathbf{x}_0, z_0) = 0$, and $(\partial F/\partial z)(\mathbf{x}_0, z_0) \neq 0$ at a point $(\mathbf{x}_0, z_0)$ in its domain. Then:

(a) There exist an open ball $U \subseteq \mathbb{R}^m$ containing $\mathbf{x}_0$ and an open interval V containing z_0 such that there is a unique function $z = g(\mathbf{x})$, defined on U with values in V, satisfying

$$F(\mathbf{x}, g(\mathbf{x})) = 0$$

[i.e., $g(\mathbf{x})$ solves the equation $F(\mathbf{x}, z) = 0$ locally near $(\mathbf{x}_0, z_0)$].

▶ EXERCISES 5.1

Exercises 1 to 7: State whether it is possible for the map ϕ to be a reparametrization of a path.

1. $\phi\colon [0, 1] \to [0, \ln 2]$, $\phi(t) = \ln(t + 1)$

2. $\phi\colon [-1, 1] \to [0, 1]$, $\phi(t) = t^2$

3. $\phi\colon [-8, 1] \to [-2, 1]$, $\phi(t) = t^{1/3}$

4. $\phi\colon [1, 2] \to [0, 3]$, $\phi(t) = t^2 - 1$

5. $\phi\colon [0, 1] \to [1, e]$, $\phi(t) = e^t$

6. $\phi\colon [-1, 1] \to [-\pi/4, \pi/4]$, $\phi(t) = \arctan t$

7. $\phi\colon [-2, 1] \to [0, 2]$, $\phi(t) = |t|$

8. Let $\mathbf{c}(t) = (t - 2, 3 - t - t^2)$, $t \in [0, 1]$. Is the reparametrization $\phi\colon [0, 3] \to [0, 1]$, given by $\phi(t) = 1 - t/3$, orientation-preserving or orientation-reversing?

9. Using the Mean Value Theorem, show that a differentiable function $\phi\colon [\alpha, \beta] \to [a, b]$ is one-to-one if $\phi'(t) > 0$ for all $t \in (\alpha, \beta)$ (or $\phi'(t) < 0$ for all $t \in (\alpha, \beta)$).

10. Explain why a continuous and bijective function $\phi\colon [\alpha, \beta] \to [a, b]$ must map endpoints to endpoints. Show that this statement is no longer true if ϕ is not continuous.

Exercises 11 to 16: Check whether the curve $\mathbf{c}(t)$ is simple or not, closed or not, simple closed or not.

11. $\mathbf{c}(t) = (\sin t, \cos t, (t - 2\pi)^2)$, $t \in [-2\pi, 6\pi]$

12. $\mathbf{c}(t) = (\sin t, \cos t, (t - 2\pi)^2)$, $t \in [-2\pi, 4\pi]$

13. $\mathbf{c}(t) = (t \sin t, t \cos t)$, $t \in [0, 2\pi]$

14. $\mathbf{c}(t) = (\sin 2t, t \cos t)$, $t \in [0, \pi/2]$

15. $\mathbf{c}(t) = (t - t^{-1}, t + t^{-1})$, $t \in [1, 2]$

16. $\mathbf{c}(t) = (t^2 - t, 3 - \sqrt{t^2 - t})$, $t \in [0, 1]$

17. Find a parametrization of the part of the curve $y = \sqrt{x^2 + 1}$ from $(-1, \sqrt{2})$ to $(1, \sqrt{2})$. Is your parametrization continuous? Differentiable? Piecewise C^1? C^1?

18. Find a parametrization of the curve $x^{2/3} + y^{2/3} = 1$. Is your parametrization continuous? Differentiable? Piecewise C^1? C^1?

19. Consider the following parametrizations of the straight-line segment from $(-1, 1)$ to $(1, 1)$. State which parametrizations are continuous, piecewise C^1 and C^1.

(a) $\mathbf{c}_1(t) = (t, 1)$, $-1 \le t \le 1$

(b) $\mathbf{c}_2(t) = \begin{cases} (-t^2, 1) & \text{if } -1 \le t \le 0 \\ (t^2, 1) & \text{if } 0 \le t \le 1 \end{cases}$

(c) $\mathbf{c}_3(t) = (t^{1/3}, 1)$, $-1 \le t \le 1$

(d) $\mathbf{c}_4(t) = (t^3, 1)$, $-1 \le t \le 1$

(e) $\mathbf{c}_5(t) = \begin{cases} (t, 1) & \text{if } -1 \le t \le 0 \\ (1 - t, 1) & \text{if } 0 \le t \le 1 \end{cases}$

20. Consider the curve $\mathbf{c}$ in $\mathbb{R}^2$ given by $\mathbf{c}(t) = (t, t^2)$, $t \in [-1, 2]$. State which of the following maps ϕ are reparametrizations of $\mathbf{c}$. Describe the curve $\mathbf{c}(\phi(t))$ for those ϕ that are reparametrizations:

(a) $\phi\colon [-1, \sqrt{3}] \to [-1, 2]$, $\phi(t) = t^2 - 1$

(b) $\phi\colon [-1/2, 1] \to [-1, 2]$, $\phi(t) = 2t$

(c) $\phi\colon [-1, 8] \to [-1, 2]$, $\phi(t) = t^{1/3}$

(d) $\phi\colon [-2/3, 1/3] \to [-1, 2]$, $\phi(t) = -3t$

21. Let $\mathbf{c}(t) = (t^2, 2 - t^2)$, $t \in [1, 3]$. Reparametrize $\mathbf{c}$ so that its speed is constant.

22. Let $\mathbf{c}(t) = (\cos 2\pi t, \sin 2\pi t, t)$, $0 \le t \le 1$.

(a) Reparametrize $\mathbf{c}$ so that its speed equals 1.

(b) Reparametrize $\mathbf{c}$ so that it takes 3 units of time to trace it.

(c) Reparametrize $\mathbf{c}$ so that it is traced in the opposite direction.

23. Let **c** be the circle $x^2 + y^2 = 1$, oriented clockwise. Find an orientation-preserving parametrization of **c** of constant speed S. Find an orientation-reversing parametrization of **c** of constant speed 1.

24. Assuming that the units are kilometers and hours, check that the speed of the path $\mathbf{c}(t) = (5\cos t, 5\sin t, 12t)$ is 13 km/h. Reparametrize **c** so that its speed is 13 mph.

▶ 5.2 PATH INTEGRALS OF REAL-VALUED FUNCTIONS

To motivate the definition of a path integral, let us first recall the construction of the definite integral of a function of one variable.

Definite Integral of a Real-Valued Function of One Variable

Assume that $y = f(x)$ is a continuous, positive function defined on an interval $[a, b]$. The graph of f, the vertical lines $x = a$ and $x = b$, and the x-axis define a region R in the xy-plane (called the *region below* f *over* $[a, b]$). We would like to find a way to compute the area of R.

Subdivide the interval $[a, b]$ into n subintervals $[a = t_1, t_2], [t_2, t_3], \ldots, [t_n, t_{n+1} = b]$ and construct rectangles $R_1, \ldots, R_n$ in the following way: the base of R_i, $i = 1, \ldots, n$, is the ith subinterval $[t_i, t_{i+1}]$ and its height is the value $f(t_i^*)$ of f at some point t_i^* in $[t_i, t_{i+1}]$; see Figure 5.11.

The area of R_i is $f(t_i^*)(t_{i+1} - t_i) = f(t_i^*)\Delta t_i$, where $\Delta t_i = t_{i+1} - t_i$. The rectangles $R_1, \ldots, R_n$ approximate the region R, and the sum of their areas

$$A_n = \sum_{i=1}^{n} f(t_i^*)\Delta t_i$$

approximates the area of R. It can be proven that the more rectangles we use, the better approximation we get; consequently, as $n \to \infty$, the sequence A_n of approximations of the area of R will approach the area of R; that is,

$$\text{area}(R) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\Delta t_i.$$

We *define* the definite integral of f on $[a, b]$ as

$$\int_a^b f(x)\,dx = \text{area}\,(R) = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\Delta t_i,$$

provided that the limit exists.

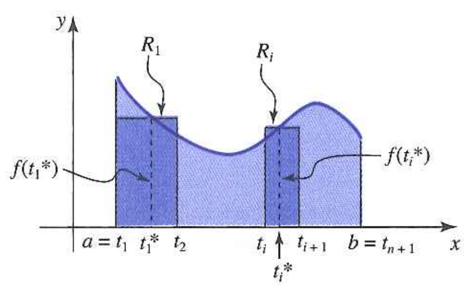


Figure 5.11 Approximating rectangles for the region R.

Hence,

$$\overline{T} = \frac{1}{\sqrt{101}\pi}\int_0^{10\pi}\left(1+\tfrac{1}{10}t\right)\frac{\sqrt{101}}{10}\,dt = \frac{1}{10\pi}\left(t+\tfrac{1}{20}t^2\right)\Big|_0^{10\pi} = 1+\frac{\pi}{2}.$$ ◀

It is worth repeating that in order to compute $\int_{\mathbf{c}} f\,ds$, it suffices to know the values of the function at the points on the curve only [that is the $f(\mathbf{c}(t))$ term in the path integral]. In light of this fact, we notice that Example 5.12 contains more data than needed—the temperature function was defined at all points in $\mathbb{R}^3$.

Further applications of path integrals are discussed in Section 7.5.

A curve in $\mathbb{R}^2$ (or $\mathbb{R}^3$) can be defined in various ways. For example, it can be described as the image of a map $\mathbf{c}\colon [a, b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$), or as the graph of a function $f\colon \mathbb{R} \to \mathbb{R}$. Alternatively, we can use geometric terms, such as a "straight-line segment from A to B," or "circle of radius 4 centered at the origin," or the "intersection of the paraboloid $z = x^2 + 3y^2$ and the plane $-2x - y + 3z = 1$," etc.

Let $\mathbf{c}$ be a curve described in any of the ways given above, or in some other way. Assume that it is either a simple curve or a simple closed curve, endowed with an orientation (see Definitions 5.4, 5.5, and 5.6 at the end of Section 5.1). We would like to define an integral of a function along $\mathbf{c}$.

In order to compute a path integral, we need a parametrization. But how do we decide which one to use? The answer is—it does not matter! We define the integral of a real-valued function f along $\mathbf{c}$ as the path integral of f with respect to *any* smooth parametrization of $\mathbf{c}$. Here is why it works: it can be proved that any two one-to-one, C^1 maps (i.e., paths that parametrize a curve as a simple or a simple closed curve) that have the same image (i.e., represent the same curve) are reparametrizations of each other. And according to Theorem 5.2, the path integral does not depend on the parametrization used. Example 5.10 serves as an illustration of this fact.

A consequence of Theorem 5.2 states that when we integrate a *scalar* function along a curve, the orientation does not play any role. This sounds reasonable: for example, the average temperature of the wire should not depend on the way (i.e., on the direction in which) we measure the temperature at the points of the wire. The analogous statement does not hold for integrals of vector-valued functions, as we will witness in the next section.

However, the path integral *does* depend on the path used, as shown in Example 5.9. There is an important class of functions whose path integrals depend only on the endpoints, and not on the curve that joins them. Section 5.4 is devoted to a study of such functions.

▶ EXERCISES 5.2

1. Level curves of a linear function $f(x, y)$ are shown in Figure 5.15. Find the path integral of $f(x, y)$ along

(a) The line segment perpendicular to the level curves, from A to B

(b) The line segment that crosses all level curves at the angle of $\pi/4$, from C to D.

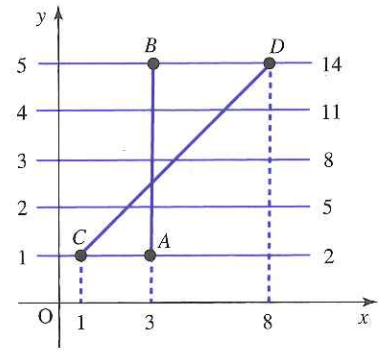


Figure 5.15 Level curves of Exercise 1.

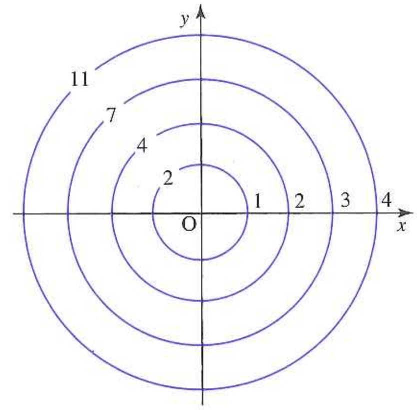


Figure 5.16 Level curves of Exercise 2.

2. The level curves of a function $f(x, y)$ are concentric circles centered at the origin; see Figure 5.16. Compute the path integral of $f(x, y)$ along

(a) The semicircle $x^2 + y^2 = 4$, $y \geq 0$

(b) Quarter-circle $x^2 + y^2 = 9$, $x \leq 0$, $y \leq 0$.

Exercises 3 to 11: Compute $\int_{\mathbf{c}} f\, ds$.

3. $f(x, y) = 2x - y$, $\mathbf{c}(t) = (e^t + 1, e^t - 2)$, $0 \leq t \leq \ln 2$

4. $f(x, y, z) = xy$, $\mathbf{c}(t) = (2\cos t, 3\sin t, 5t)$, $0 \leq t \leq \pi/2$

5. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1}$, $\mathbf{c}(t) = (t, t, t)$, $1 \leq t < \infty$ (*Hint:* Take $1 \leq t \leq b$ and then compute the limit as b approaches ∞.)

6. $f(x, y) = x^3 + y^3$, $\mathbf{c}$ is the part of the curve $x^{2/3} + y^{2/3} = 1$ in the first quadrant

7. $f(x, y, z) = y - z^2$, $\mathbf{c}(t) = t^2\mathbf{i} + \ln t\mathbf{j} + 2t\mathbf{k}$, $1 \leq t \leq 4$

8. $f(x, y) = x^2 + 3y^2 - xy$, $\mathbf{c}$ is the circular arc of radius 3 in the xy-plane, from $(0, 3)$ to $(-3, 0)$

9. $f(x, y, z) = xyz$, $\mathbf{c}$ is the helix given by $\mathbf{c}(t) = (2\sin t, 4t, 2\cos t)$, $0 \leq t \leq 6\pi$

10. $f(x, y, z) = (x + y + z)/(x^2 + y^2 + z^2)$, $\mathbf{c}$ is the straight-line segment joining $(1, 1, 1)$ and (a, a, a), where $a \neq 1$

11. $f(x, y) = e^{x+3y}$, $\mathbf{c}$ is the line segment in $\mathbb{R}^2$ from $(0, 0)$ to $(3, -4)$

12. Compute $\int_{\mathbf{c}} f\, ds$, where $f(x, y, z) = x + 2y - z^2$, and $\mathbf{c}$ consists of the parabolic path $t\mathbf{i} + t^2\mathbf{j}$ from $(0, 0, 0)$ to $(1, 1, 0)$, followed by the straight line to $(1, -1, 1)$.

13. Compute $\int_{\mathbf{c}} f\, ds$, where $f(x, y, z) = x - 4y + z$, and $\mathbf{c}$ consists of the straight line from $(4, 2, 0)$ to $(0, 2, 0)$, followed by the circular path in the yz-plane (and above the xy-plane) with its center at the origin, from $(0, 2, 0)$ to $(0, -2, 0)$.

14. Let $f(x, y, z) = x - 3y^2 + z$ and let $\mathbf{c}$ be the straight-line segment from the origin to the point $(1, 1, 1)$. The four paths $\mathbf{c}_1(t) = (t, t, t)$, $t \in [0, 1]$, $\mathbf{c}_2(t) = (1 - t, 1 - t, 1 - t)$, $t \in [0, 1]$, $\mathbf{c}_3(t) = (e^t - 1, e^t - 1, e^t - 1)$, $t \in [0, \ln 2]$, and $\mathbf{c}_4(t) = (\ln t, \ln t, \ln t)$, $t \in [1, e]$, parametrize the given line segment.

(a) Describe their differences in terms of orientation and speed.

(b) Compute $\int_{\mathbf{c}_i} f\, ds$, $i = 1, \ldots, 4$.

15. Suppose that a continuous function f is integrated along two different paths joining the points $(1, 2)$ and $(3, -5)$, and two different answers are obtained. Is that possible, or has an error been made in the evaluation of integrals?

16. Compute the integral of $f(x, y) = xy - x - y + 1$ along the following curves connecting the points $(1, 0)$ and $(0, 1)$:

(a) $\mathbf{c}_1$: circular arc $\mathbf{c}_1(t) = (\cos t, \sin t)$, $0 \le t \le \pi/2$

(b) $\mathbf{c}_2$: straight-line segment $\mathbf{c}_2(t) = (1 - t, t)$, $0 \le t \le 1$

(c) $\mathbf{c}_3$: from $(1, 0)$ horizontally to the origin, then vertically to $(0, 1)$

(d) $\mathbf{c}_4$: from $(1, 0)$ vertically to $(1, 1)$, then horizontally to $(0, 1)$

(e) $\mathbf{c}_5$: circular arc $\mathbf{c}_5(t) = (\cos t, -\sin t)$, $0 \le t \le 3\pi/2$.

17. Compute the area of the part of the cylinder $x^2 + y^2 = 4$ between the xy-plane and the plane $z = y + 2$.

18. Compute the area of the part of the surface $y^2 = x$ defined by $0 \le x \le 2$, $0 \le z \le 2$.

19. Compute the area of the part of the surface $y = \sin x$, $0 \le x \le \pi/2$, above the xy-plane and below the surface $z = \sin x \cos x$.

20. Let $\mathbf{c}$ be the straight-line segment joining $(1, 0, 0)$ and $(0, 2, 0)$. Use a geometric argument (i.e., do not evaluate the integral) to find $\int_{\mathbf{c}} (x + 3y)\, ds$.

21. Use a geometric argument to find $\int_{\mathbf{c}} e^{x^2+y^2} ds$, where $\mathbf{c}$ is the circle centered at the origin of radius 4.

22. Argue geometrically that $\int_{\mathbf{c}} \sin(x^3)\, ds \ge 0$, where $\mathbf{c}$ is the graph of $y = \tan x$, $-\pi/4 \le x \le \pi/4$.

23. Is it possible that the average value of $f(x, y) = \sin x \cos y$ along some curve $\mathbf{c}$ is equal to 5?

24. Write down the version of the statement of Theorem 5.2 in the case where $\mathbf{c}$ is a piecewise C^1 path and prove it.

25. Find the average value of the function $f(x, y, z) = -\sqrt{x^2 + z^2}$ along the curve $\mathbf{c}(t) = (3\cos t)\mathbf{j} + (3 \sin t)\mathbf{k}$, $0 \le t \le 2\pi$.

26. Find the average value $\overline{f}$ of the function $f(x, y, z) = 2x^2 - y^2$ along the unit circle in the xy-plane. Identify all points on $\mathbf{c}$ where the value of f is equal to $\overline{f}$.

27. Assume that $\mathbf{c}(t)\colon [a, b] \to \mathbb{R}^3$ represents a metal wire and that its density at a point (x, y, z) is given by the function $\rho(x, y, z)$. Explain how to use a path integral to compute the mass of the wire.

28. The density at a point (x, y) on a metal wire in the shape of a quarter-circle $x^2 + y^2 = 1$, $x, y \ge 0$, is given by $\rho(x, y) = 3 + 2xy$ g/cm (assume that the units along the coordinate axes are centimeters). Compute the mass of the wire.

29. Assume that a path $\mathbf{c}$ is given in polar coordinates by $r = r(\theta)$, $\theta_1 \le \theta \le \theta_2$. Show that $\int_{\mathbf{c}} f\, ds = \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

30. Compute the path integral of the function $f(x, y) = x^2 + y^2$ along the curve $r = \sin\theta$, where $0 \le \theta \le \pi$.

▶ 5.3 PATH INTEGRALS OF VECTOR FIELDS

In this section, we are going to introduce one of the most important and useful concepts in vector calculus (and its applications), that of an integral of a vector field along a curve. It will be defined as the limiting case of a summation, in much the same way as the path

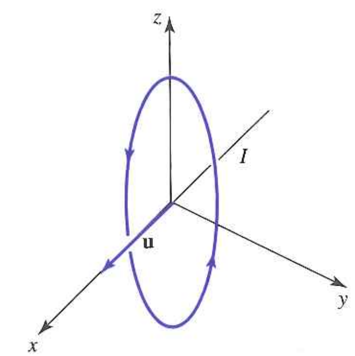


Figure 5.24 Filament of Example 5.23.

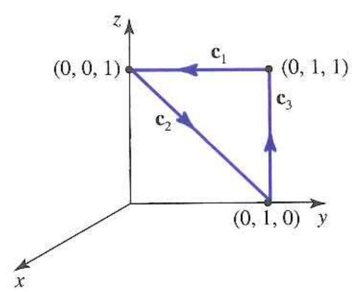


Figure 5.25 Piecewise C^1 curve.

Let us check, with an example, the physical fact that the magnetic circulation $\mathcal{B}$ vanishes along any closed curve that does not enclose the current filament. Consider the piecewise C^1 curve shown in Figure 5.25.

Parametrize the curve $\mathbf{c}_1$ by $\mathbf{c}_1(t) = (0, 1, 1) + t(0, -1, 0) = (0, 1 - t, 1)$, $t \in [0, 1]$. From (5.7), it follows that [use $\int 1/(x^2 + 1)\,dx = \arctan x$]

$$\int_{\mathbf{c}_1} \mathbf{B} \cdot d\mathbf{s} = \frac{\mu_0 I}{2\pi} \int_0^1 \frac{(0, -1, 1-t)}{(1-t)^2 + 1} \cdot (0, -1, 0)\,dt$$

$$= \frac{\mu_0 I}{2\pi} \int_0^1 \frac{1}{(1-t)^2 + 1}\,dt = \frac{\mu_0 I}{2\pi} \arctan(t - 1)\Big|_0^1 = \frac{\mu_0 I}{2\pi}\frac{\pi}{4} = \frac{\mu_0 I}{8}.$$

Parametrize $\mathbf{c}_2$ by $\mathbf{c}_2(t) = (0, 0, 1) + t(0, 1, -1) = (0, t, 1 - t)$, $t \in [0, 1]$. Then

$$\int_{\mathbf{c}_2} \mathbf{B} \cdot d\mathbf{s} = \frac{\mu_0 I}{2\pi} \int_0^1 \frac{(0, -1 + t, t)}{(-1 + t)^2 + t^2} \cdot (0, 1, -1)\,dt = \frac{\mu_0 I}{2\pi} \int_0^1 \frac{-1}{2t^2 - 2t + 1}\,dt$$

$$= \frac{\mu_0 I}{2\pi} \int_0^1 \frac{-1}{\frac{1}{2}\left((2t-1)^2 + 1\right)}\,dt = -\frac{\mu_0 I}{2\pi} \arctan(2t - 1)\Big|_0^1 = -\frac{\mu_0 I}{2\pi}\frac{\pi}{2} = -\frac{\mu_0 I}{4}.$$

Parametrize $\mathbf{c}_3$ by $\mathbf{c}_3(t) = (0, 1, 0) + t(0, 0, 1) = (0, 1, t)$, $t \in [0, 1]$. Then

$$\int_{\mathbf{c}_3} \mathbf{B} \cdot d\mathbf{s} = \frac{\mu_0 I}{2\pi} \int_0^1 \frac{(0, -t, 1)}{1 + t^2} \cdot (0, 0, 1)\,dt$$

$$= \frac{\mu_0 I}{2\pi} \int_0^1 \frac{1}{1 + t^2}\,dt = \frac{\mu_0 I}{2\pi} \arctan t\Big|_0^1 = \frac{\mu_0 I}{2\pi}\frac{\pi}{4} = \frac{\mu_0 I}{8}.$$

Hence,

$$\mathcal{B} = \int_{\mathbf{c}_1} \mathbf{B} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{B} \cdot d\mathbf{s} + \int_{\mathbf{c}_3} \mathbf{B} \cdot d\mathbf{s} = 0.$$ ◀

▶ EXERCISES 5.3

1. Consider the vector field $\mathbf{F}$ and the curves $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ in Figure 5.26.

(a) Explain why $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} < 0$.

(b) Assume that $\mathbf{c}_2$ and $\mathbf{c}_3$ have the same speed. Which of the path integrals $\int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$ or $\int_{\mathbf{c}_3} \mathbf{F} \cdot d\mathbf{s}$ is larger? Why?

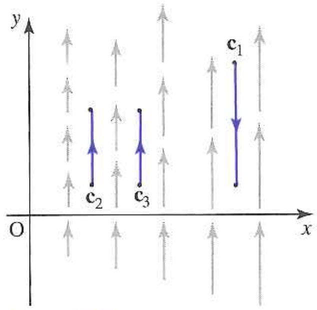


Figure 5.26 Diagram for Exercise 1.

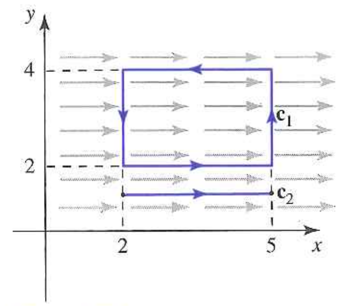


Figure 5.27 Diagram for Exercise 2.

2. Figure 5.27 shows a constant vector field **F**.

(a) Compute $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s}$ along the closed path $\mathbf{c}_1$.

(b) Assuming that $\|\mathbf{F}\| = 3/2$, compute $\int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$

3. Let us compute the work W done by the force $\mathbf{F} = x\mathbf{i} + \mathbf{j}$ along the straight-line segment $\mathbf{c}(t) = (t, 1)$, $1 \le t \le 4$, using Riemann sums W_n defined in (5.1).

(a) Check that when the interval $[1, 4]$ is divided into n subintervals of equal length, the subdivision points are $\mathbf{c}(t_i) = (1 + 3(i-1)/n, 1)$, $i = 1, \ldots, n+1$.

(b) Show that $W_n = 3 + 9(n-1)/(2n)$.

(c) Conclude that $W = 15/2$. Check your answer by computing W using a path integral.

4. Using Riemann sums, as in Exercise 3, compute the work done by the force $\mathbf{F} = x\mathbf{i} + \mathbf{j}$ along the straight-line segment $\mathbf{c}(t) = (t, t)$, $1 \le t \le 2$. Verify your answer by computing the work using a path integral.

Exercises 5 to 12: Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.

5. $\mathbf{F}(x, y) = y^2\mathbf{i} - x^2\mathbf{j}$, $\mathbf{c}$ is the part of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$

6. $\mathbf{F}(x, y) = x^2y\mathbf{i} + (y-1)\mathbf{j}$, $\mathbf{c}$ is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(1, 1)$, oriented counterclockwise

7. $\mathbf{F}(x, y) = e^{x+y}\mathbf{i} - \mathbf{j}$, $\mathbf{c}$ is the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, oriented clockwise

8. $\mathbf{F}(x, y, z) = (yz^2, xyz, 2x^2z)$, $\mathbf{c}$ consists of straight-line segments from $(-1, 2, -2)$ to $(-1, -2, -2)$, then to $(-1, -2, 0)$ and then to $(0, -2, 0)$

9. $\mathbf{F}(x, y, z) = (x^2, xy, 2z^2)$, $\mathbf{c}(t) = (\sin t, \cos t, t^2)$, $0 \le t \le \pi/2$

10. $\mathbf{F}(x, y) = e^{x+y}\mathbf{i} + e^{x-y}\mathbf{j}$, $\mathbf{c}$ is the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$, oriented counterclockwise

11. $\mathbf{F}(x, y) = 2xy\mathbf{i} + e^y\mathbf{j}$, $\mathbf{c}(t) = 4t^3\mathbf{i} + t^2\mathbf{j}$, $t \in [0, 1]$

12. $\mathbf{F}(x, y, z) = (xy, yz, xz)$, $\mathbf{c}$ consists of the straight-line segments from the origin to $(1, 0, 1)$, and then to $(1, 1, 0)$

13. Let $\mathbf{c}$ be an oriented C^1 path. A vector field $\mathbf{F}$ of constant magnitude $\|\mathbf{F}\| = k$ is tangent to $\mathbf{c}$ (at all points of $\mathbf{c}$) and points in the direction of $\mathbf{c}$. Find $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.

14. Let $\mathbf{F}$ be a continuous vector field defined on all of $\mathbb{R}^2$, and let $\mathbf{c}_1$ be a C^1 path from a point P to a point Q in $\mathbb{R}^2$. Define the piecewise C^1 path $\mathbf{c}_2$ as follows: $\mathbf{c}_2$ has the same image as $\mathbf{c}_1$; $\mathbf{c}_2$ starts at

P and stops at some point Q_1 before it reaches Q; then it moves back and stops at a point P_1 between P and Q_1. Finally, it moves from P_1 to Q. Explain why $\int_{\mathbf{c}_1} \mathbf{F}\, ds = \int_{\mathbf{c}_2} \mathbf{F}\, ds$.

15. Compute the work done when the force $\mathbf{F}(x, y) = x^3\mathbf{i} + (x + y)\mathbf{j}$ acts on a particle that moves from $(0, 0)$ to $(1, \pi^2/4)$ along the curve $\mathbf{c}(t) = \sin t\mathbf{i} + t^2\mathbf{j}$.

16. Assume that $\mathbf{F}$ is a constant force field acting in $\mathbb{R}^2$. Show that $\mathbf{F}$ does zero work on a particle that moves counterclockwise once around a circle in the xy-plane with constant speed.

17. Assume that the force $\mathbf{F} = C(x\mathbf{i} + y\mathbf{j})$ (C is a constant) acts on a particle moving in $\mathbb{R}^2$. Show that $\mathbf{F}$ does zero work if the particle moves counterclockwise once around a circle with constant speed.

18. The force between two positive electric charges [one, of charge ρ, is placed at the origin, and the other, of charge 1 C, is placed at (x, y)] is given by the formula $\mathbf{F} = \rho\mathbf{r}/\|\mathbf{r}\|^3$. How much work is needed in order to move the 1-C charge along the straight line from $(1, 0)$ to $(-1, 2)$ if the other charge remains at the origin?

19. Consider the force field $\mathbf{F}(x, y) = (y, 0)$. Compute the work done on a particle by the force $\mathbf{F}$ if the particle moves from $(0, 0)$ to $(1, 1)$ in each of the following ways:

(a) Along the x-axis to $(1, 0)$, then vertically up to $(1, 1)$

(b) Along the parabolic path $y = x^2$

(c) Along the path $y = x^4$

(d) Along the straight line

(e) Along the path $y = \sin(\pi x/2)$

(f) Along the y-axis to $(0, 1)$, then horizontally to $(1, 1)$

Interpret your results.

20. Compute $\int_{\mathbf{c}} 3(x + y)dx$ along the path $\mathbf{c}(t) = (e^t + 1, e^t - 2), 0 \le t \le 1$.

21. Compute $\int_{\mathbf{c}} (ydx + xdy)/(x^2 + y^2)$, where $\mathbf{c}$ is the circle centered at the origin of radius 2, oriented counterclockwise.

22. Compute $\int_{\mathbf{c}} xydx + ye^x dy$, where $\mathbf{c}$ is the rectangle with vertices $(0, 0), (1, 0), (1, 1)$, and $(0, 1)$, oriented counterclockwise.

23. Consider $\int_{\mathbf{c}_i} xydx + 2ydy$, where $\mathbf{c}_i$ is the straight-line segment joining the points $(0, 0)$ and $(1, 1)$ parametrized in the following ways:

(a) $\mathbf{c}_1(t) = (t, t), t \in [0, 1]$

(b) $\mathbf{c}_2(t) = (\sin t, \sin t), t \in [0, \pi/2]$

(c) $\mathbf{c}_3(t) = (\cos t, \cos t), t \in [0, \pi/2]$

Are all the results the same? Explain why or why not.

24. Compute $\int_{\mathbf{c}} M(x, y, z)dx$, where M is a continuous function and $\mathbf{c}$ is any curve contained in a plane parallel to the yz-plane.

25. Show that the assumption "$\mathbf{c}$ is a C^1 curve" in Theorem 5.3 can be replaced by "$\mathbf{c}$ is a piecewise C^1 curve."

Exercises 26 to 29: In $\mathbb{R}^2$, the flux (flow) of a vector field $\mathbf{F}$ across a smooth closed curve $\mathbf{c}$ is defined as $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{n}\, ds$, where $\mathbf{n}$ denotes the outward unit normal vector field along $\mathbf{c}$. The circulation of $\mathbf{F}$ is given by $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$. Compute the flux and the circulation for the vector field $\mathbf{F}$ and the curve $\mathbf{c}$.

26. $\mathbf{F}(x, y) = 4x\mathbf{i} - 2y\mathbf{j}$, $\mathbf{c}$ is a circle of radius r, oriented clockwise

27. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$, $\mathbf{c}$ is a circle of radius r, oriented counterclockwise

28. $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$, $\mathbf{c}$ is the semicircle of radius r from $(r, 0)$ to $(-r, 0)$, followed by the straight-line segment back to $(r, 0)$, oriented counterclockwise

29. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$, $\mathbf{c}$ is the curve from Exercise 28

30. Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a continuous vector field. Show that its outward flux is given by $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{n}\, ds = \int_{\mathbf{c}} P(x, y)dy - Q(x, y)dx$.

Another way to solve parts (b) and (c) would be to compute V from the system of differential equations

$$\frac{\partial V}{\partial x} = Cx(1+y^2) \qquad \text{and} \qquad \frac{\partial V}{\partial y} = Cx^2 y$$

with the initial condition $V(0,0)=0$. Integrating the first equation with respect to x, we get $V(x,y) = Cx^2(1+y^2)/2 + D(y)$, where $D(y)$ depends (possibly) on y. Computing the derivative of $V(x,y)$ with respect to y and combining with $\partial V/\partial y = Cx^2 y$ yield

$$C\frac{x^2}{2}2y + \frac{dD(y)}{dy} = Cx^2 y;$$

hence, $dD(y)/dy = 0$ and $D(y) = D$ (D is a real number). Therefore, $V = Cx^2(1+y^2)/2 + D$. The condition $V(0,0)=0$ implies that $D=0$, and thus, $V = Cx^2(1+y^2)/2$. ◀

▶ EXERCISES 5.4

Exercises 1 to 10: State which of the following sets are connected, simply-connected, and/or star-shaped:

1. $\mathbb{R}^2$, with the circle $x^2+y^2=1$ removed

2. The set $\{(x,y) | y < |x|\}$ in $\mathbb{R}^2$

3. $\mathbb{R}^3$, with the circle $x^2+y^2=1$, $z=1$ removed

4. The set $\{(x,y,z) | x^2+y^2+z^2 \leq 1 \text{ and } z \neq 1\}$ in $\mathbb{R}^3$

5. $\mathbb{R}^3$, with the sphere $x^2+y^2+z^2=1$ removed

6. $\mathbb{R}^3$, with the ball $x^2+y^2+z^2 \leq 1$ removed

7. $\mathbb{R}^3$, with the helix $\mathbf{c}(t) = (\cos t, \sin t, t)$, $t \in [0,\pi]$ removed

8. The set $\{(x,y) | x^2+y^2 < 1 \text{ or } x^2+y^2 > 2\}$ in $\mathbb{R}^2$

9. The region inside the polygonal line joining the points $(0,5)$, $(2,0)$, $(2,3)$, $(4,3)$, $(4,5)$, and $(0,5)$ (in that order)

10. The set $\{(x,y) | x^2-y^2 < 0\}$ in $\mathbb{R}^2$

11. Explain why the sphere $\{(x,y,z) | x^2+y^2+z^2=1\}$ and the sphere without the "North Pole" $\{(x,y,z) | x^2+y^2+z^2=1 \text{ and } z<1\}$ are simply-connected sets.

12. Let $\mathbf{F} = (F_1, F_2, F_3)$ be a C^1 vector field defined on a star-shaped set U, such that $curl\,\mathbf{F} = \mathbf{0}$. Define the function $f(x,y,z)$ as in the proof of Theorem 5.4.

(a) Show that $f(x,y,z) = \int_0^1 (F_1(tx,ty,tz)x + F_2(tx,ty,tz)y + F_3(tx,ty,tz)z)\,dt$.

(b) Derive the formula $\partial f/\partial x = \int_0^1 A\,dt$, where

$$A = F_1(tx,ty,tz) + D_1F_1(tx,ty,tz)tx + D_2F_1(tx,ty,tz)ty + D_3F_1(tx,ty,tz)tz.$$

(c) Show that the integrand in (b) is equal to $(\partial/\partial t)(tF_1(tx,ty,tz))$ and conclude that $\partial f/\partial x = F_1$.

13. Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = y^2\cos x\,\mathbf{i} + 2y\sin x\,\mathbf{j}$, and $\mathbf{c}$ is any path starting at $(1,1)$ and ending at $(1,3)$.

14. Compute the path integral $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = (\cos(xy) - xy\sin(xy))\mathbf{i} - x^2\sin(xy)\mathbf{j}$, and $\mathbf{c}(t) = (e^t\cos t, e^t\sin t)$, $0 \leq t \leq \pi$.

15. Check that the vector field $\mathbf{F} = (-y/(x^2+y^2), x/(x^2+y^2), 1)$ satisfies $curl\,\mathbf{F} = \mathbf{0}$ in $\mathbb{R}^3 - \{(0,0,0)\}$, but is not conservative in $\mathbb{R}^3$. In order to show that $\mathbf{F}$ is not conservative, compute path integrals of **F** along the curves $\mathbf{c}_1(t) = (\cos t, \sin t, 0)$, $0 \le t \le \pi$, and $\mathbf{c}_2(t) = (\cos t, -\sin t, 0)$, $0 \le t \le \pi$, joining (1, 0) and (−1, 0). Explain why this does not contradict Theorem 5.8.

16. Let $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$.

(a) Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ along the circular path from (1, 0) counterclockwise to (0, −1), then along the y-axis from (0, −1) to (0, 2), and then along the straight line from (0, 2) to (1, 0).

(b) Show that $\mathbf{F}(x, y)$ is a gradient vector field and use this fact to check your answer in (a).

Exercises 17 to 21: Determine whether **F** is a gradient vector field, and if so, specify its domain U and find all functions f such that $\mathbf{F} = \nabla f$.

17. $\mathbf{F} = (4x^2 - 4y^2 + x)\mathbf{i} + (7xy + \ln y)\mathbf{j}$

18. $\mathbf{F} = (3x^2 \ln x + x^2)\mathbf{i} + x^3 y^{-1}\mathbf{j}$

19. $\mathbf{F} = 2x \ln y\mathbf{i} + (2y + x^2/y)\mathbf{j}$

20. $\mathbf{F} = (yz + e^x \sin z)\mathbf{i} + (xz + y^2 - e^y)\mathbf{j} + (xy + e^x \cos z)\mathbf{k}$

21. $\mathbf{F} = y \cos(xy)\mathbf{i} + (x \cos(xy) - z \sin y)\mathbf{j} + \cos y\mathbf{k}$

Exercises 22 to 24: Evaluate the following integrals:

22. $\displaystyle\int_{(0,1,0)}^{(3,3,1)} (4xy - 2xy^2z^2)\,dx + (2x^2 - 2x^2yz^2)\,dy - 2x^2y^2z\,dz$

23. $\displaystyle\int_{(0,0,0)}^{(\pi,\pi/2,\pi/3)} \cos x \tan z\,dx + dy + \sin x \sec^2 z\,dz$

24. $\displaystyle\int_{(1,2,1)}^{(2,2,2)} x^{-2}dx + z^{-1}dy + yz^{-2}\,dz$

25. Provide omitted detail in the proof of Theorem 5.7: parametrize the path $\overline{\mathbf{c}}$ in Figure 5.36(b) to obtain the formula (5.18). Then show that $\partial f/\partial x(x, y) = F_1(x, y)$.

26. Let $\mathbf{F}(x, y, z) = (x, y, z)/(x^2 + y^2 + z^2)^{3/2}$. Show that $curl\,\mathbf{F} = \mathbf{0}$. Show that $\nabla(x^2 + y^2 + z^2)^{-1/2} = -\mathbf{F}$ (see Example 2.40 in Section 2.4).

27. Consider the vector field $\mathbf{F}(x, y) = -(1 + x)ye^x\mathbf{i} - xe^x\mathbf{j}$.

(a) Show that $\mathbf{F}(x, y)$ is conservative.

(b) Using the Fundamental Theorem of Calculus, find the potential energy $V(x, 0)$ along the x-axis if $V(0, 0) = 0$.

(c) Using the Fundamental Theorem of Calculus, find the potential energy $V(0, y)$ along the y-axis if $V(0, 0) = 0$.

28. Let $\mathbf{F}(x, y, z) = x^3y\mathbf{i} + z^2\mathbf{k}$. Does there exist a function f such that $\mathbf{F} = \nabla f$?

29. Find $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = 2xye^y\mathbf{i} + x^2e^y(1 + y)\mathbf{j}$ and **c** is the straight-line segment from (0, 0) to (3, −2),

(a) Using a parametrization for **c**.

(b) Using the fact that $curl\,\mathbf{F} = \mathbf{0}$.

30. Check that $\mathbf{F}(x, y, z) = 2xy^2\mathbf{i} + (2x^2y + e^z)\mathbf{j} + ye^z\mathbf{k}$ is a conservative force field in $\mathbb{R}^3$. Find the work done by **F** on an object that moves from (0, 2, −1) to (3, 2, 0).

31. Show that the vector field $\mathbf{F}(\mathbf{r}) = \|\mathbf{r}\|^2\mathbf{r}$ is a gradient vector field ($\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$), and find a function f such that $\nabla f = \mathbf{F}$.

32. Show that the vector field $\mathbf{F}(\mathbf{r}) = \|\mathbf{r}\|\mathbf{r}$ is a gradient vector field ($\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$), and find a function f such that $\nabla f = \mathbf{F}$.

33. Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = (\ln(x + y^2) + x/(x + y^2))\mathbf{i} + (2xy/(x + y^2))\mathbf{j}$, and **c** is the part of the curve $y = x^3$ from (1, 1) to (2, 8). Use the fact (check it!) that **F** is a gradient vector field.

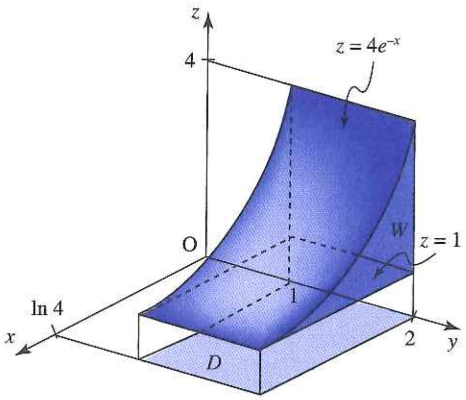


Figure 6.32 Solid W of Example 6.18.

SOLUTION The surface $z = 4e^{-x}$ and the plane $z = 1$ intersect when $4e^{-x} = 1$, that is, when $x = -\ln(1/4) = \ln 4$ (in words, they intersect along the line $x = \ln 4$ in the plane $z = 1$).

The region D of integration is the rectangle $0 \le x \le \ln 4$, $1 \le y \le 2$, and the volume of W is given by $v(W) = \iint_D (4e^{-x} - 1)\, dA$, since, on D, $4e^{-x} \ge 1$. It follows that

$$
\begin{aligned}
v(W) &= \iint_D (4e^{-x} - 1)\, dA = \int_1^2 \left(\int_0^{\ln 4} (4e^{-x} - 1)\, dx \right) dy \\
&= \int_1^2 \left(-4e^{-x} - x \Big|_0^{\ln 4} \right) dy = \int_1^2 (3 - \ln 4)\, dy \\
&= 3 - \ln 4 \approx 1.61.
\end{aligned}
$$

◀

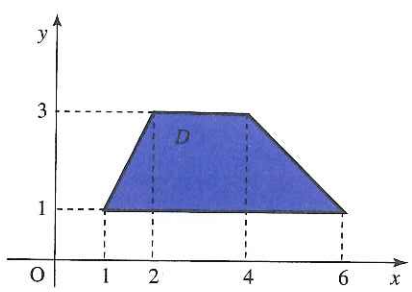
Additional examples and some techniques for computing double integrals will be presented in the following two sections.

▶ EXERCISES 6.2

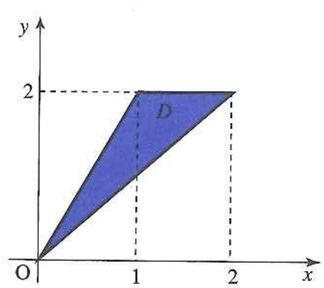
Exercises 1 to 4: Identify the regions below as type 1, type 2, or type 3 [for a region of type 1, state explicitly the functions $y = \phi(x)$ and $y = \psi(x)$ and the values of a and b; likewise, give all necessary detail for a region of type 2].

1. Disk $D = \{(x, y) \mid x^2 + y^2 \le 1\}$

2.



3.



4. The region D bounded by the curves $y = 1 - x^2$ and $y = -3$

5. Suppose that $f(x, y) \leq 0$ for all $(x, y) \in D \subseteq \mathbb{R}^2$. What is a geometric meaning of $\iint_D f\,dA$? If $f(x, y) \geq g(x, y)$ for all $(x, y) \in D$, what geometric interpretation could be given to $\iint_D (f - g)\,dA$? (Note that we do not assume that f and g are positive.)

6. Using Cavalieri's principle, find the volume of the solid W in Figure 6.33.

7. Using Cavalieri's principle, find the volume of a cone of radius r and height h.

8. Find the volume of the solid obtained by rotating the graph of $y = \ln x$, $1 \leq x \leq 2$, about the x-axis. Now imagine that the same graph is rotated about the y-axis. Find the volume of the solid thus obtained.

9. Using Cavalieri's principle, find the volume of the solid W in Figure 6.34.

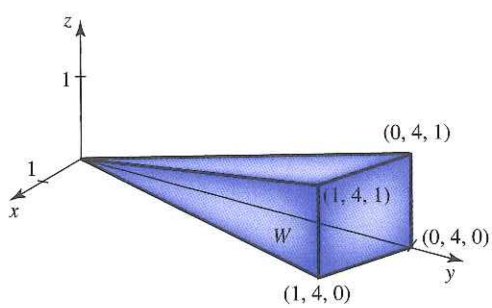


Figure 6.33 Solid W of Exercise 6.

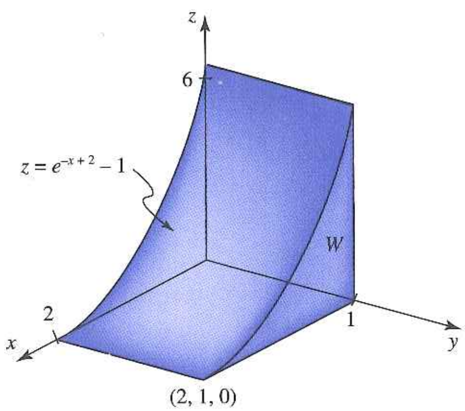


Figure 6.34 Solid W of Exercise 9.

10. Consider the double integral of Example 6.12. Show that, when D is viewed as a type-2 region, $\iint_D e^{2x+y}\,dA = \int_1^2 \left(\int_1^y e^{2x+y}\,dx\right) dy + \int_2^4 \left(\int_{y/2}^2 e^{2x+y}\,dx\right) dy$. Evaluate this integral, thus checking the result of Example 6.12.

11. Consider the double integral of Example 6.13. Show that, when D is viewed as a type-1 region, $\iint_D 2y\,dA = \int_0^4 \left(\int_{-\sqrt{x}}^{\sqrt{x}} 2y\,dy\right) dx + \int_4^9 \left(\int_{x-6}^{\sqrt{x}} 2y\,dy\right) dx$. By evaluating this integral, check the result of Example 6.13.

Exercises 12 to 17: Evaluate the following iterated integrals:

12. $\displaystyle\int_0^1 \left(\int_0^x \cos(x^2)\,dy\right) dx$

13. $\displaystyle\int_0^2 \left(\int_{2-x}^{x+1} (xe^y - 2y - 1)\,dy\right) dx$

14. $\displaystyle\int_{-1}^1 \left(\int_0^{3x} e^{x+3y}\,dy\right) dx$

15. $\displaystyle\int_0^\pi \left(\int_0^{\cos\theta} \rho^2 \sin\theta\,d\rho\right) d\theta$

16. $\displaystyle\int_1^2 \left(\int_0^{y/2} x\sqrt{x^2 + y^2}\,dx\right) dy$

17. $\displaystyle\int_0^{\pi/2} \left(\int_0^{\sin y} x \cos y\,dx\right) dy$

Exercises 18 to 24: Evaluate $\iint_D f\,dA$ for the function f and the region $D \subseteq \mathbb{R}^2$.

18. $f(x, y) = e^{-x-3y}$, $D = [0, \ln 2] \times [0, \ln 3]$

19. $f(x, y) = xy^{-1} - x^2y^2$, $D = [0, 2] \times [3, 4]$

20. $f(x, y) = xye^{x^2}$, $D = [-1, 1] \times [0, 1]$

21. $f(x, y) = 2xy - y$, $D = \{(x, y) \mid 0 \le y \le 1, -y \le x \le 1 + y\}$

22. $f(x, y) = e^x$, $D = \{(x, y) \mid 0 \le x \le 3, x \le y \le 2x^2\}$

23. $f(x, y) = x^{-2/3}$, D is the region in the first quadrant bounded by the parabolas $y = x^2$ and $y = 4 - x^2$

24. $f(x, y) = \ln(xy)$, D is the triangular region bounded by the lines $y = 1$, $y = x$, and $x = 0$

25. Find an upper bound and a lower bound for $\iint_D e^{-x-y}\,dA$, where $D = [-1, 1] \times [0, 2]$.

26. Find an upper bound and a lower bound for the double integral $\iint_D x^2 \sin(x^2 - y)\,dA$, where D is the disk $x^2 + y^2 \le 1$.

Exercises 27 to 30: Find the volume of the solid in $\mathbb{R}^3$.

27. The solid under the plane $x + y/2 + z = 6$ and above the rectangle $[-1, 1] \times [0, 2]$

28. The solid in the first octant bounded by the cylindrical sheet $z = -y^2 + 9$ and the plane $x = 2$

29. The solid between the planes $x + y + z = 1$ and $x + y + 2z = 1$ in the first octant

30. The solid below $z = 9 - x^2 - y^2$ and above the triangle in the xy-plane with the vertices $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 2, 0)$

31. Let $D \subseteq \mathbb{R}^2$ be an elementary region and let f and g be continuous, real-valued functions on D. Show that if $\iint_D f\,dA = \iint_D g\,dA$ then there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = g(x_0, y_0)$.

Exercises 32 to 36: Find the area of the region $D \subseteq \mathbb{R}^2$.

32. Bounded by $y = 2x$, $y = 5x$, and $x^2 + y^2 = 1$

33. The ellipse with the semiaxes $a > 0$ (in the x-direction) and $b > 0$ (in the y-direction)

34. Below $y = x^{-1}$, between $x = a$ and $x = b$, where $a, b > 0$

35. Between $y = x^2$ and $y = 4 - x^2$, to the right of the y-axis

36. Inside the disk $x^2 + y^2 \le 2$ and outside the square $[-1, 1] \times [-1, 1]$

37. Let $f(x, y) = k(x^2 + y^2)$ describe the temperature ($k > 0$ is a constant) at points on a rectangular metal plate $R = [0, 1] \times [0, 2]$. Find all points (x_0, y_0) in R that satisfy the conclusion of the Mean Value Theorem.

38. Find the point (x_0, y_0) from the Mean Value Theorem if $f(x, y) = x^2$ and D is the triangle defined by the coordinate axes and the line $x + y = 1$.

39. Assume that a function $y = f(x)$ is continuous on an interval $[a, b]$.

(a) Explain why there exist real numbers m and M such that $m \le f(x) \le M$ for all x in $[a, b]$.

(b) By integrating the inequality in (a), prove that $m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$.

(c) Explain why there exists a number x_0 in $[a, b]$ satisfying $f(x_0) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

(d) The expression on the right side of the equation in (c) is called the *average value* of f on $[a, b]$. Explain in words the meaning of the formula in (c).

▶ EXAMPLE 6.28

Compute

$$\int_0^1 \left(\int_x^1 \frac{\cos y}{y} \, dy \right) dx.$$

SOLUTION The integral of $\cos y/y$ cannot be computed (exactly, as a compact formula), so we reverse the order of integration. The region of integration, given by the inequalities $0 \le x \le 1$ and $x \le y \le 1$, is the triangle with sides $y = x$, $y = 1$, and the y-axis shown in Figure 6.41. It follows that

$$\int_0^1 \left(\int_x^1 \frac{\cos y}{y} \, dy \right) dx = \int_0^1 \left(\int_0^y \frac{\cos y}{y} \, dx \right) dy$$

$$= \int_0^1 \left(\frac{\cos y}{y} x \Big|_{x=0}^{x=y} \right) dy = \int_0^1 \cos y \, dy = \sin y \Big|_0^1 = \sin 1.$$

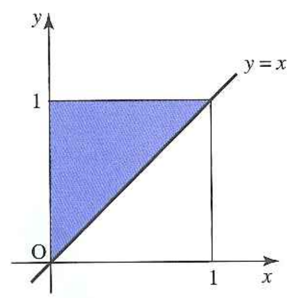


Figure 6.41 Region of Example 6.28.

▶ EXERCISES 6.3

Exercises 1 to 4: Evaluate $\iint_D f \, dA$ for the function f and the region $D \subseteq \mathbb{R}^2$.

1. $f(x, y) = ye^x$, D is the triangular region bounded by the lines $y = 1$, $y = 2x$, and $x = 0$

2. $f(x, y) = y(x^2 + y^2)^{3/4}$, D is the disk $x^2 + y^2 \le 9$

3. $f(x, y) = y^2$, D is the triangular region bounded by the lines $y = 2x$, $y = 5x$, and $x = 2$

4. $f(x, y) = (2x - x^2)^{-1/2}$, D is the triangular region bounded by the lines $y = -x + 1$, $y = 0$, and $x = 0$

Exercises 5 to 9: Find the volume of the solid in $\mathbb{R}^3$.

5. The solid bounded by the cylinder $y^2 + z^2 = 4$ and the plane $x + y = 2$ in the first octant

6. The solid under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the parabola $y = x^2$ and the line $y = x$

7. The solid bounded by the planes $y = 3x$, $y = 0$, $z = 0$, and $x + y + z = 4$

8. The solid bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$

9. The solid under the surface $z = xy$ and above the triangle in the xy-plane with vertices $(0, 1)$, $(1, 1)$, and $(1, 2)$

10. Compute $\int_0^{\pi/3} \left(\int_0^{\pi/2} \cos(x + y) \, dx \right) dy$ using $\cos(x + y) = \cos x \cos y - \sin x \sin y$ and separation of variables. Check your result by direct evaluation.

11. Compute $\int_0^1 \left(\int_0^2 (1 - x - y + xy)\,dx\right) dy$ using separation of variables. Check your result by direct evaluation.

Exercises 12 to 15: Sketch the region of integration and reverse the order of integration. Do not solve the integrals.

12. $\displaystyle\int_0^{\pi} \left(\int_0^{\sin(x/2)} x^3 y^2\,dy\right) dx$

13. $\displaystyle\int_0^1 \left(\int_{\arctan x}^{\pi/4} (y^2 - x)\,dy\right) dx$

14. $\displaystyle\int_0^1 \left(\int_{x/2}^{x} x^2 y^2\,dy\right) dx + \int_1^2 \left(\int_{x/2}^{1} x^2 y^2\,dy\right) dx$

15. $\displaystyle\int_1^2 \left(\int_1^{2y} \frac{\ln x}{x}\,dx\right) dy$

Exercises 16 to 22: Evaluate the following integrals by reversing the order of integration.

16. $\displaystyle\int_0^1 \left(\int_y^1 e^{x^2}\,dx\right) dy$

17. $\displaystyle\int_0^1 \left(\int_0^{\arccos y} x\,dx\right) dy$

18. $\displaystyle\int_0^1 \left(\int_{y^{1/3}}^1 e^{x^4}\,dx\right) dy$

19. $\displaystyle\int_0^3 \left(\int_{x^2}^9 x\cos(2y^2)\,dy\right) dx$

20. $\displaystyle\int_{1/2}^1 \left(\int_1^{2y} \frac{\ln x}{x}\,dx\right) dy + \int_1^2 \left(\int_y^2 \frac{\ln x}{x}\,dx\right) dy$

21. $\displaystyle\int_1^2 \left(\int_1^{\sqrt{y}} 5\,dx\right) dy + \int_2^4 \left(\int_{y/2}^{\sqrt{y}} 5\,dx\right) dy$

22. $\displaystyle\int_0^1 \left(\int_0^{\arcsin x} y^2\,dy\right) dx$

23. Compute $\iint_D e^{x^3}\,dA$ over the region bounded by $y = x^2$, $x = 1$, and $y = 0$.

24. Compute the area of the region $x^2 + y^2 \le 9$ that lies to the left of the line $x = 1/5$.

▶ 6.4 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

Sometimes, the evaluation of a double integral $\iint_D f\,dA$ is difficult because either the region D is geometrically complicated, or the function f and/or D give rise to an integrand that is hard to handle. One possible way to solve this problem is to use the change of variables technique.

Let us recall how change of variables (also known as the substitution rule) works for functions of one variable. Consider the definite integral $\int_1^2 e^{5x}\,dx$. Let $u = 5x$, so that $x = u/5$; this means that x is now viewed as a function of u, $x(u) = u/5$. Then $dx = x'(u)\,du = (1/5)\,du$ and

$$\int_1^2 e^{5x}\,dx = \int_5^{10} e^u \tfrac{1}{5}\,du,$$

where the limits of integration have been changed accordingly (when $x = 1$, $u = 5$; when $x = 2$, $u = 10$). One more example: consider the integral $\int_1^2 e^{-5x}\,dx$. Using $u = -5x$, so

(b) Direct computation gives

$$\iint_D (x+y)\,dx\,dy = \int_0^1 \left(\int_0^x (x+y)\,dy \right) dx = \int_0^1 \left(xy + \frac{y^2}{2}\bigg|_{y=0}^{y=x} \right) dx$$

$$= \frac{3}{2}\int_0^1 x^2\,dx = \frac{3}{2}\,\frac{x^3}{3}\bigg|_0^1 = \frac{1}{2}.$$

▶ EXAMPLE 6.40

Using the change of variables $x = v$ and $y = u/v$, transform the integral $\iint_D x^2y^2\,dA$, where D is the region in the first quadrant bounded by the parabolas $y = x^2$ and $y = 2x^2$ and the hyperbolas $xy = 1$ and $xy = 2$.

SOLUTION The change of variables function T is defined by $T(u, v) = (v, u/v)$. First of all, we have to find the region D^* such that $T(D^*) = D$. From $y = x^2$, we get $u/v = v^2$ and $v = u^{1/3}$. From $y = 2x^2$, we get $u/v = 2v^2$ and $v = (u/2)^{1/3}$. Similarly, $xy = 1$ implies $u = 1$ and $xy = 2$ implies $u = 2$. It follows that D^* is the region of type 1 in the uv-plane, defined by $1 \le u \le 2$ and $(u/2)^{1/3} \le v \le u^{1/3}$; see Figure 6.50.

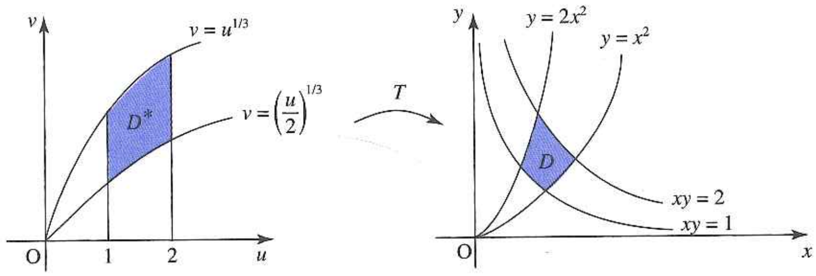


Figure 6.50 The function T maps the region D^* to D.

The function T is C^1 except at $v = 0$ (but that is irrelevant, since D^* is away from the u-axis), and its Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{vmatrix} = \frac{1}{v}.$$

Therefore,

$$\iint_D x^2y^2\,dA = \iint_{D^*} v^2 \left(\frac{u}{v}\right)^2 \left|\frac{1}{v}\right| dA^* = \int_1^2 \left(\int_{(u/2)^{1/3}}^{u^{1/3}} \frac{u^2}{v}\,dv \right) du.$$

Sometimes, it is more or less obvious what T, that is, what change of variables (change of coordinates), to choose; such cases include polar coordinates (as in Example 6.38), or cylindrical and spherical coordinates (see Section 6.5). In general, however, identifying a change of variables that makes a given integration easier could be quite difficult.

▶ EXERCISES 6.4

Exercises 1 to 5: Evaluate the given double integral by converting to polar coordinates.

1. $\int_0^2 \left(\int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy \right) dx$

2. $\iint_D xy\,dA$, where D is the region in the first quadrant bounded by $x^2+y^2=4$, $y=x/\sqrt{2}$, and $y=0$

3. $\int_{-1}^{1}\left(\int_0^{\sqrt{1-x^2}} \arctan(y/x)\,dy\right)dx$

4. $\iint_D \sqrt{x^2+y^2}\,dA$, where D is the semicircle $(x-1)^2+y^2=1$, $y \geq 0$

5. $\iint_D \sqrt{2x^2+2y^2+3}\;dA$, where D is the disk $x^2+y^2 \leq 1$

6. Convert the double integral $\int_0^2\left(\int_0^{\sqrt{4x-x^2}}(x^2+y^2)^{3/4}\,dy\right)dx$ to polar coordinates. Do not evaluate it.

7. Find the volume of the solid under the paraboloid $z=x^2+y^2$, inside the cylinder $x^2+y^2=5$, and above the xy-plane.

8. Find the volume of the solid between the paraboloids $z=3x^2+3y^2$ and $z=12-3x^2-3y^2$.

9. Find the volume of the solid inside the cylinder $x^2+y^2=4$, inside the ellipsoid $4x^2+4y^2+z^2=64$, and above the xy-plane.

10. Using a double integral, compute the volume of a sphere of radius $a>0$.

11. Show that if $T(\mathbf{u})=A\mathbf{u}+\mathbf{b}$ is an affine map, then $\det(A)$ is equal to the Jacobian of T.

12. Consider the map $T(\mathbf{u})=A\mathbf{u}+\mathbf{b}$, where $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix} b_1 \\ b_2\end{bmatrix}$, and assume that $\det(A)\neq 0$.

(a) Show that T is one-to-one. [*Hint*: You will need the fact that if $\det(A)\neq 0$, then the 2×2 system $a_{11}X+a_{12}Y=0$ and $a_{21}X+a_{22}Y=0$ has a unique solution $X=Y=0$.]

(b) Compute the image of the line $\mathbf{l}_1(t)=(w_1+tv_1, w_2+tv_2)$, $t\in\mathbb{R}$, under T. Next, compute the image of the line $\mathbf{l}_2(t)=(\overline{w_1}+tv_1, \overline{w_2}+tv_2)$, $t\in\mathbb{R}$, parallel to $\mathbf{l}_1(t)$. Conclude that parallel lines map to parallel lines.

(c) Consider two lines $\mathbf{l}_1(t)=(tv_1, tv_2)$, $t\in\mathbb{R}$, and $\mathbf{l}_2(t)=(tw_1, tw_2)$, $t\in\mathbb{R}$, that intersect at the origin. The origin is mapped to (b_1, b_2) under T. Compute the images of $\mathbf{l}_1(t)$ and $\mathbf{l}_2(t)$ and find their point of intersection.

(d) Show that $S(\mathbf{x})=A^{-1}\mathbf{x}-A^{-1}\mathbf{b}$ is the inverse map of T, where A^{-1} is the inverse matrix of A [i.e., show that $S\circ T(u,v)=(u,v)$ and $T\circ S(x,y)=(x,y)$].

(e) Using (d), prove statement (d) of Theorem 6.6.

13. Approximate the area of the image of a small rectangle with sides $\Delta u=0.1$ and $\Delta v=0.05$ and one vertex located at $(2,4)$, under the mapping T defined by $T(u,v)=(\sqrt{u^2+v^2}, uv)$.

14. Approximate the area of the image of a small rectangle with sides $\Delta u=0.03$ and $\Delta v=0.1$ and one vertex located at $(-2,1)$, under the mapping T defined by $T(u,v)=(u\sin v, u\cos v)$.

15. Describe in words the map $T\colon \mathbb{R}^2\to\mathbb{R}^2$, $T(u,v)=(au, v+b)$, where $a>1$ and $b>0$. What is the relation between the area of a region D and the area of its image $T(D)$?

16. Consider the map $T\colon \mathbb{R}^2\to\mathbb{R}^2$ given by $T(u,v)=(u+v, v)$. Describe the region to which T maps a square whose sides are parallel to the coordinate axes. Compute the Jacobian of T and interpret the result geometrically.

17. Evaluate $\iint_D(5x+y^2+x^2)\,dA$, where D is the part of the annulus $1\leq x^2+y^2\leq 4$ in the upper half-plane.

18. Using a double integral, find the area of the region enclosed by one loop of the curve $r=\cos 2\theta$. [*Hint*: Sketch the curve.]

19. Find the area of the region inside the cardioid $r = 1 - \sin\theta$, $0 \le \theta \le 2\pi$.

20. Express the volume of the right circular cone of radius r and height h as a double integral in polar coordinates.

21. Find the area of the region in the first quadrant bounded by the curves $r = \theta$ and $r = 2\theta$.

22. Compute the integral $\iint_D (4x + 6y)\,dA$, where D is the region bounded by the lines $4y = x - 3$, $4y = x + 2$, $2x + 3y = 6$, and $2x + 3y = 17$. [*Hint*: Use the change of variables $x = 4u - 3v$, $y = u + 2v$.]

23. Compute the integral $\iint_D (x^2 - y^2)\,dA$, where D is the region bounded by the curves $xy = 1$, $y = x - 1$, and $y = x + 1$. [*Hint*: Use the change of variables $x = u + v$, $y = -u + v$.]

24. Compute the integral $\iint_D (2x - y)\,dA$, where D is the region bounded by the curves $y = 2x$, $x = 2y$, and $x + y = 6$. [*Hint*: Use the change of variables $x = u - v$, $y = u + v$.]

25. Compute the volume of the wedge cut from the cylinder $x^2 + y^2 = 9$ by the planes $z = 0$ and $z = y + 3$.

26. Compute the volume of the solid below the plane $z = y + 4$ and above the disk $x^2 + y^2 \le 1$.

27. Compute the integral $\iint_D xy^3\,dA$, where D is the region in the first quadrant bounded by the lines $x = 1$ and $x = 2$ and the hyperbolas $xy = 1$ and $xy = 3$. [*Hint*: Use the change of variables $x = v$, $y = u/v$.]

28. Compute the integral $\iint_D 5\,dA$, where D is the region inside the ellipse $4x^2 + 2y^2 = 1$. [*Hint*: Define a change of variables so that the region of integration becomes a circle.]

29. Evaluate $\iint_D 5(x + y)\,dA$, where D is the region bounded by the lines $3x - 2y = 5$, $3x - 2y = -2$, $x + y = -2$, and $x + y = 1$ using a suitable change of variables.

30. Evaluate $\iint_D x^2\,dA$, where D is the region $0 \le \frac{1}{9}x^2 + y^2 \le 1$ using a suitable change of variables.

31. Evaluate $\iint_D e^x\,dA$, where D is the region defined by $x + y = 0$, $x + y = 2$, $y = x$, and $y = 2x$.

32. Evaluate $\iint_D (x^2 - y^2)\,dA$, where D is the region in the first quadrant bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $y = 0$, and $y = x/2$, using the change of variables $x = u\cosh v$ and $y = u\sinh v$.

33. Evaluate $\iint_D \sin\frac{x+y}{x-y}\,dA$, where D is the region bounded by the lines $x - y = 1$, $x - y = 5$, and the coordinate axes.

▶ 6.5 TRIPLE INTEGRALS

The definition, properties, and methods of evaluation of triple integrals are analogous to those of double integrals. Nevertheless, for the sake of completeness, we will briefly go through the relevant concepts.

Assume that $f = f(x, y, z)$ is a *bounded* function defined on a closed and bounded solid W in $\mathbb{R}^3$. Recall that a function $f\colon W \to \mathbb{R}$ is bounded if there exists $M \ge 0$ such that $|f(x, y, z)| \le M$ for all $(x, y, z) \in W$. The fact that W is closed means that it contains the surface(s) that constitute(s) its boundary.

Enclose W into a big rectangular box (this is possible due to the assumption that W is bounded) and divide it into n^3 subboxes W_{ijk}, $i, j, k = 1, \ldots, n$, with faces parallel to

It follows that the unit normal (whenever $v \neq 0, \pi$)

$$\frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{-\sin v\ \mathbf{r}(u, v)}{|-\sin v| \cdot \|\mathbf{r}(u, v)\|} = \frac{-\sin v}{|-\sin v|}\mathbf{r}(u, v) = -\mathbf{r}(u, v)$$

points in the direction opposite to $\mathbf{r}(u, v)$ [since $0 < v < \pi$, it follows that $\sin v > 0$ and therefore $-\sin v/|-\sin v| = -\sin v/\sin v = -1$; this explains the appearance of the minus sign in front of $\mathbf{r}(u, v)$]. Therefore, $\mathbf{N}/\|\mathbf{N}\| = -\mathbf{n}$; that is, the above parametrization is orientation-reversing.

On the other hand, the parametrization

$$\mathbf{r}(u, v) = (\cos v \cos u, \cos v \sin u, \sin v), \quad 0 \leq u \leq 2\pi, \quad -\pi/2 \leq v \leq \pi/2,$$

of S given in Examples 7.4 and 7.7 (with $a = 1$) yields the unit normal vector (whenever $v \neq -\pi/2, \pi/2$)

$$\frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{\cos v\ \mathbf{r}(u, v)}{\|\cos v\ \mathbf{r}(u, v)\|} = \frac{\cos v}{|\cos v|}\mathbf{r}(u, v) = \mathbf{r}(u, v).$$

This time, $\mathbf{N}/\|\mathbf{N}\|$ has the same direction as $\mathbf{r}(u, v)$, since $\cos v > 0$ for $-\pi/2 < v < \pi/2$. So, $\mathbf{N}/\|\mathbf{N}\| = \mathbf{n}$, and it follows that this is an orientation-preserving parametrization. ◀

DEFINITION 7.9 Orientation of the Graph of $z = f(x, y)$

The orientation of the graph S of a differentiable function $z = f(x, y)$ is defined as follows: the outside of S is the side away from which the unit normal vector

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}$$

points. ◀

In other words, the positive orientation is determined by the choice of the normal with $\mathbf{k}$ component equal to $+1$, and the negative orientation corresponds to the normal whose $\mathbf{k}$ component is -1 (the normal was computed in Example 7.9).

▶ EXAMPLE 7.15

The normal $\mathbf{N}$ to the graph S of the function $f(x, y) = x^2y - y^3$ (shown in Figure 7.14) is given by $\mathbf{N} = (-\partial f/\partial x, -\partial f/\partial y, 1) = (-2xy, -x^2 + 3y^2, 1)$; see (7.6). The corresponding unit normal field is $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\| = (-2xy, -x^2 + 3y^2, 1)/\|\mathbf{N}\|$.

Thus, $\mathbf{n}(0, 0, 0) = (0, 0, 1) = \mathbf{k}$, and it follows that the outside of S is the side that we see when we look at S from high up on the z-axis. [Note that, in order to determine the orientation (of an orientable surface), we need to find the normal $\mathbf{n}$ at only *one* point on the surface.] ◀

Note that we have used the word "surface" in different contexts: level surface, the graph of a function of two variables, and parametrized surface. In Exercise 6 in the chapter review section (see Review Exercises and Problems), we show that these three concepts coincide.

▶ EXERCISES 7.1

1. Consider the parametric representations of the cylinder and the sphere given in Examples 7.2 and 7.4. Describe how $\mathbf{r}$ maps the boundary of the rectangle D in each case.

2. Consider the parametrization $\mathbf{r}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v)$, $0 \le u \le 2\pi$, $0 \le v \le \pi$ of the sphere of radius a. Give a geometric interpretation of the parameters u and v.

Exercises 3 to 12: Find a parametrization of each surface in $\mathbb{R}^3$.

3. Upper hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$

4. Quarter-sphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$, $x \ge 0$

5. Sphere of radius 2, centered at $(-2, 3, 7)$

6. The part of the upper hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$, cut out by the cone $z^2 = x^2 + y^2$

7. The part of the plane $z - 3y + x = 2$ inside the cylinder $x^2 + y^2 = 4$

8. The graph of $x^2 + y^2 - z^2 = 1$

9. The part of the plane $x + 2y + z = 6$ in the first octant

10. The part of the cone $x^2 + y^2 = z^2$ in the first octant

11. The part of the paraboloid $z = x^2 + y^2$ in the first octant

12. The surface obtained by rotating the circle $(y - 3)^2 + z^2 = 1$, $x = 0$ about the z-axis

Exercises 13 to 20: For each parametrized surface $\mathbf{r}(u, v)$ in $\mathbb{R}^3$,

(a) Find the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ and the normal $\mathbf{N}$, and

(b) Find all points where $\mathbf{r}(u, v)$ is smooth.

13. $\mathbf{r}(u, v) = (2u, u^2 + v, v^2)$, $u, v \ge 0$

14. $\mathbf{r}(u, v) = (u, e^u \sin v, e^u \cos v)$, $0 \le v \le 2\pi$, $u \in \mathbb{R}$

15. $\mathbf{r}(u, v) = (\sin u \cos v, \sin u \sin v, 2 \cos u)$, $0 \le u, v \le 2\pi$

16. $\mathbf{r}(u, v) = (u^2 + v^2, u^2 - v^2, 2uv)$, $0 \le u, v \le 1$

17. $\mathbf{r}(u, v) = ((1 + \cos v) \cos u, (1 + \cos v) \sin u, \sin v)$, $0 \le u, v \le 2\pi$

18. $\mathbf{r}(u, v) = (u, \cos v, \sin v)$, $0 \le u \le 1$, $0 \le v \le \pi$

19. $\mathbf{r}(u, v) = (u, v, 1 - (u^2 + v^2))$, $u, v \ge 0$

20. $\mathbf{r}(u, v) = (u, |u|, v)$, $-1 \le u \le 1$, $0 \le v \le 2$

21. Find an equation of the plane tangent to $\mathbf{r}(u, v) = (e^u, e^v, uv)$ at $(1, 1, 0)$.

22. Let S be the surface $z = 10 - x^2 - 2y^2$. Compute the equation of the plane tangent to it at the point $(1, 2, 1)$ in three different ways:

(a) By using the parametrization $\mathbf{r}(u, v) = (u, v, 10 - u^2 - 2v^2)$

(b) By viewing S as the graph of the function $f(x, y) = 10 - x^2 - 2y^2$

(c) By viewing S as the level surface of $f(x, y, z) = z + x^2 + 2y^2$.

23. Find an equation of the plane tangent to the graph of $y = x^2 + 2xz$ at $(1, 3, 1)$.

24. Find an equation of the plane tangent to the graph of $y = f(x, z)$ at (x_0, y_0, z_0).

25. Show that the plane tangent to the cone $z^2 = x^2 + y^2$ (at any point where it exists) goes through the origin.

26. Consider the following parametrizations:

(a) $\mathbf{r}_1(u, v) = (u, v, 1)$, $-1 \le u, v \le 1$

(b) $\mathbf{r}_2(u, v) = (2u, 3v, 1)$, $-1/2 \le u \le 1/2$, $-1/3 \le v \le 1/3$

(c) $\mathbf{r}_3(u, v) = (u^3, v^3, 1)$, $-1 \le u, v \le 1$

(d) $\mathbf{r}_4(u, v) = (\sin u, \sin v, 1)$, $0 \le u, v \le 2\pi$

Check that the images of $\mathbf{r}_1, \ldots, \mathbf{r}_4$ represent the same set. State which parametrizations are continuous, differentiable, C^1. State which parametrizations are smooth at $(0, 0, 1)$. Compute the tangent plane at $(0, 0, 1)$ for those parametrizations.

27. Find a parametrization of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

28. Let S be the surface $\mathbf{r}(u, v) = (u^2, 2uv, 0)$, $-\infty < u, v < \infty$. Find an orientation-reversing parametrization of S.

29. Find the points (if any) on the surface $\mathbf{r}(u, v) = (u^2 v, uv^2, 1)$ where the tangent plane is parallel to the plane $z = x - y$.

30. Find all points (if any) (x, y, z) on the paraboloid $z = 2 - x^2 - y^2$ where the normal vector is parallel to the vector joining the origin and the point (x, y, z).

31. Consider a differentiable parametrized surface $\mathbf{r}(u, v)\colon D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, and pick a point (u_0, v_0) in the domain of $\mathbf{r}$ where $\mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0) \neq \mathbf{0}$.

(a) Recall that the derivative $D\mathbf{r}(u_0, v_0)$ is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$. Find its matrix representation. Show that the range of $D\mathbf{r}(u_0, v_0)$ is the plane spanned by $\mathbf{T}_u(u_0, v_0)$ and $\mathbf{T}_v(u_0, v_0)$.

(b) Show that the plane tangent to the image of $\mathbf{r}$ at $\mathbf{r}(u_0, v_0)$ can be represented as $\mathbf{p}(u, v) = \mathbf{r}(u_0, v_0) + D\mathbf{r}(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$. Thus (as expected), the derivative $D\mathbf{r}$ enters into the equation of the tangent plane (thought of as the linear approximation).

▶ 7.2 WORLD OF SURFACES

In this section, we study various surfaces, to understand better how parametrizations work, to provide more examples of implicitly defined surfaces and the Implicit Function Theorem, to hint at some (of many) applications of surfaces, and because we will need these surfaces in this chapter and also in Chapter 8. Differential geometry is one of several mathematical disciplines that are dedicated to exploring geometric objects such as surfaces.

▶ EXAMPLE 7.16 Surface of Revolution

As the graph of a differentiable function $y = f(x)$, $x \in [a, b]$, is rotated about the x-axis, it generates a *surface of revolution* S; see Figure 7.19. (Recall Example 6.7 in Section 6.2, where we studied solids of revolution.)

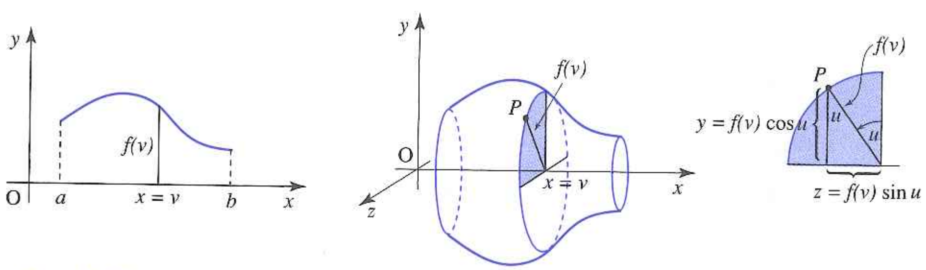


Figure 7.19 Surface of revolution obtained by rotating $y = f(x)$ about the x-axis.

The projection D_1 of S onto the xz-plane is the region bounded by $x=0$, $z=0$, and (substitute $y=0$ into the equation for S) $z=2-x^2/2$. Since $|\mathbf{n}\cdot\mathbf{j}|=1/\sqrt{4x^2+5}$, we get

$$\iint_S f\,dS=\iint_{D_1} f(x,y,z)\frac{dA}{|\mathbf{n}\cdot\mathbf{j}|}=\iint_{D_1} 4xz\frac{dA}{1/\sqrt{4x^2+5}},$$

where dA refers to integration with respect to x and z. Hence,

$$\iint_S f\,dS=\int_0^2\left(\int_0^{2-x^2/2} 4xz\sqrt{4x^2+5}\,dz\right)dx.$$

The projection D_2 of S onto the yz-plane is the triangle bounded by $y=0$, $z=0$, and (substitute $x=0$ into the equation for S) $y+2z=4$. It follows that

$$\iint_S f\,dS=\iint_{D_2} f(x,y,z)\frac{dA}{|\mathbf{n}\cdot\mathbf{i}|},$$

where dA refers to integration with respect to y and z, and $|\mathbf{n}\cdot\mathbf{i}|=|2x/\sqrt{4x^2+5}|=2x/\sqrt{4x^2+5}$, since $x\ge 0$. Thus,

$$\iint_S f\,dS=\iint_{D_2} 4xz\frac{dA}{2x/\sqrt{4x^2+5}}=\iint_{D_2} 2z\sqrt{4x^2+5}\,dA.$$

Since the integration is with respect to y and z, we have to eliminate x. From $x^2+y+2z=4$, we get $x^2=4-y-2z$, and therefore,

$$\iint_S f\,dS=\iint_{D_2} 2z\sqrt{21-4y-8z}\,dA=\int_0^4\left(\int_0^{2-y/2} 2z\sqrt{21-4y-8z}\,dz\right)dy.$$

The projection D_3 of S onto the xy-plane is the region bounded by $x=0$, $y=0$, and $y=4-x^2$. Since $|\mathbf{n}\cdot\mathbf{k}|=2/\sqrt{4x^2+5}$, it follows that

$$\iint_S f\,dS=\iint_{D_3} f(x,y,z)\frac{dA}{|\mathbf{n}\cdot\mathbf{k}|}=\iint_{D_3} 4xz\frac{dA}{2/\sqrt{4x^2+5}},$$

where dA refers to integration with respect to x and y. Eliminating z and setting up the limits of integration, we get

$$\begin{aligned}\iint_S f\,dS&=\iint_{D_3} 2x\left(2-\frac{x^2}{2}-\frac{y}{2}\right)\sqrt{4x^2+5}\,dA\\&=\int_0^2\left(\int_0^{4-x^2} x(4-x^2-y)\sqrt{4x^2+5}\,dy\right)dx.\end{aligned}$$ ◀

▶ EXERCISES 7.3

1. Let $\mathbf{r}\colon D\to\mathbb{R}^3$ be a differentiable parametrization of a surface in $\mathbb{R}^3$; in components, $\mathbf{r}(u,v)=(x(u,v),y(u,v),z(u,v))$, for $(u,v)\in D\subseteq\mathbb{R}^2$.

(a) Explain why $x(u+\Delta u,v)-x(u,v)\approx(\partial x/\partial u)(u,v)\,\Delta u$.

(b) Using similar approximations for the y and z components of $\mathbf{r}$, show that $\mathbf{r}(u+\Delta u,v)-\mathbf{r}(u,v)\approx(\partial\mathbf{r}/\partial u)(u,v)\,\Delta u$.

2. Consider the surface of revolution S obtained when the graph of a C^1 function $y=f(x)$, $x\in[a,b]$, is rotated about the x-axis (see Example 7.16 in Section 7.2).

(a) Show that the surface area is given by $A(S)=2\pi\int_a^b|f(x)|\sqrt{1+(f'(x))^2}\,dx$.

(b) Show that $A(S)$ from (a) is equal to the path integral $\int_{\mathbf{c}} 2\pi|f(x)|\,ds$, where $\mathbf{c}(t) = (t, f(t))$, $t \in [a, b]$. Explain in words how to compute the surface area of a surface of rotation using path integrals.

3. Show that $A(S) = 2\pi \int_a^b |x|\sqrt{1 + (f'(x))^2}\,dx$ gives the surface area of the surface of revolution S obtained when the graph of a C^1 function $y = f(x)$, $x \in [a, b]$, is rotated about the y-axis.

4. The surface S in Example 7.32 is a triangle whose vertices lie on the coordinate axes, as shown in Figure 7.50.

(a) Compute the area of S using Heron's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ for the area of the triangle with sides a, b, and c, where $s = (a + b + c)/2$.

(b) Compute the area of S using vector product.

Exercises 5 to 12: Compute $\iint_S f\,dS$ in each case.

5. $f(x, y, z) = xy$, S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder of radius 2 whose axis of rotation is the z-axis

6. $f(x, y, z) = 2z(x^2 + y^2)$, S is the surface parametrized by $\mathbf{r}(u, v) = (\cos u, \sin u, v)$, $0 \le u \le \pi$, $0 \le v \le 2$

7. $f(x, y, z) = y + x$, S is the tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

8. $f(x, y, z) = x^2 + y^2$, S is the part of the cone $z^2 = x^2 + y^2$ between $z = 1$ and $z = 4$

9. $f(x, y, z) = 2y - x$, S is the part of the cone $x^2 = y^2 + z^2$, $x \le 1$, in the first octant

10. $f(x, y, z) = 8y$, S is the parabolic sheet $z = 1 - y^2$, $0 \le x \le 2$, $0 \le y \le 1$

11. $f(x, y, z) = (4x^2 + 4y^2 + 1)^{-1/2}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane

12. $f(x, y, z) = \sqrt{x^2 + y^2}$, S is the helicoidal surface $\mathbf{r}(u, v) = (u\cos v, u\sin v, v)$, $0 \le u \le 1$, $0 \le v \le 4\pi$

13. Compute the surface area of the part of the surface $\mathbf{r}(u, v) = (2u\cos v, 2u\sin v, v)$, where $0 \le u \le 2$, $0 \le v \le \pi$.

14. Compute the surface area of the part of the cylinder $x^2 + z^2 = 1$, $z \ge 0$, between the planes $y = 0$ and $z = y + 1$.

15. Compute the surface area of a cone of radius r and height h, using surface integrals.

16. Find the area of the triangle with vertices $(1, 2, 0)$, $(3, 0, 7)$, and $(-1, 0, 0)$ using a surface integral. Check your answer using the cross product.

17. Let S be the sphere $x^2 + y^2 + z^2 = a^2$. Find $\iint_S x\,dS$, $\iint_S x^2\,dS$, and $\iint_S x^3\,dS$ without evaluating surface integrals using a parametrization.

18. Compute the surface area of the part of the plane $z = 0$ defined by $-1 \le x \le 1$, $-1 \le y \le 1$ using the following parametrizations:

(a) $\mathbf{r}(u, v) = (u, v, 0)$, $-1 \le u, v \le 1$ (b) $\mathbf{r}(u, v) = (u^3, v, 0)$, $-1 \le u, v \le 1$

(c) $\mathbf{r}(u, v) = (u^{1/3}, v^{1/3}, 0)$, $-1 \le u, v \le 1$

(d) $\mathbf{r}(u, v) = (\sin u, \sin v, 0)$, $0 \le u, v \le 2\pi$

The results in (a), (b), and (c) are the same. Why is the result in (d) different?

19. Let S be the rectangle in the plane $z = my$, $m > 0$, lying directly above the rectangle $R = [0, a] \times [0, b]$, $a, b > 0$, in the xy-plane. Show that (area of S) $= \sqrt{m^2 + 1} \cdot$ (area of R). Let α be the angle between $\mathbf{k}$ and the upward normal to S. Conclude that (area of S) $= \sec\alpha \cdot$ (area of R).

20. Consider the integral $\iint_S f(x, y, z)\,dS$, where S is a surface symmetric with respect to the xz-plane. If $f(x, -y, z) = -f(x, y, z)$, what is the value of $\iint_S f\,dS$? Using your result, recompute the surface integral in Example 7.35.

21. Evaluate $\iint_S f\,dS$, where $f(x, y, z) = 4xy$ and S is the parabolic sheet $z = 1 - y^2$ in the first octant, bounded by the plane $x = 2$.

22. Using the Area Cosine Principle, find the formula for the area of an ellipse with semi-axes a and b.

23. Compute $\iint_S y\,dS$, where S is the part of the surface $x + y^2 + z = 4$ in the first octant, using a projection of S onto one of the coordinate planes.

24. Let S be the part of the plane $x + y + 2z = 4$ in the first octant, oriented by the upward-pointing normal. Compute $\iint_S (xy^2 + z^2)\,dS$

(a) Using a parametrization of S,

(b) By viewing S as the graph of the function $z = 2 - x/2 - y/2$ and using the formula of Example 7.30, and

(c) By using any of the three projections of S onto the coordinate planes.

25. Find the area of the hemisphere S defined by $x^2 + y^2 + z^2 = a^2, a > 0, y \geq 0$, using a projection of S onto a coordinate plane.

26. Find the surface area of the strip on the sphere $x^2 + y^2 + z^2 = a^2$ $(a > 0)$, defined by the angles ϕ_1 and ϕ_2, where $\phi_1 < \phi_2$ (ϕ_1 and ϕ_2 are defined in the same way as the angle ϕ in spherical coordinates).

▶ 7.4 SURFACE INTEGRALS OF VECTOR FIELDS

The aim of this section is to give a generalization of integrals of scalar functions to integrals of vector functions over surfaces in $\mathbb{R}^3$. An important application of this concept is the flux of a vector field.

DEFINITION 7.12 Surface Integral of a Vector Field

Let S be a smooth surface in $\mathbb{R}^3$ parametrized by a C^1 map $\mathbf{r} = \mathbf{r}(u, v)\colon D \to \mathbb{R}^3$ (where D is an elementary region in $\mathbb{R}^2$) and let $\mathbf{F}\colon S \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector field on S. The *surface integral* $\iint_S \mathbf{F} \cdot d\mathbf{S}$ *of* $\mathbf{F}$ *over* S is defined by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v)\,dA,$$

where $\mathbf{N}(u, v) = \mathbf{T}_u(u, v) \times \mathbf{T}_v(u, v)$. ◀

The surface integral of a vector field depends only on the values of the vector field at points on the surface. According to the definition, it is reduced to a double integral of the *real-valued* function $\mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v)$ over an elementary region D. (Note the analogy with the path integrals of vector fields that we studied in Section 5.3.)

Insight: assume that $\mathbf{F}$ represents the velocity of a fluid (and that it is constant, which implies that $\|\mathbf{F}\| = C$). Take a surface S to be a subset of the xy-plane (thus, its normal is $\mathbf{N} = \mathbf{k}$), as shown in Figure 7.54.

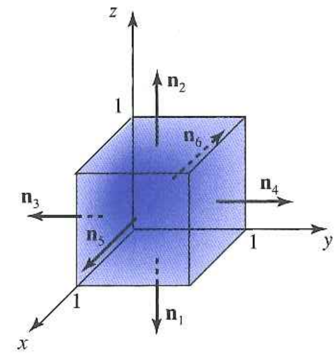


Figure 7.56 Surface of the cube oriented by outward-pointing normals.

Finally, parametrize the front and back sides of S by $\mathbf{r}_5(u, v) = (1, u, v)$, and $\mathbf{r}_6(u, v) = (0, u, v)$, $0 \le u, v \le 1$. Then $\mathbf{n}_5 = \mathbf{i}$ and $\mathbf{n}_6 = -\mathbf{i}$, and

$$\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_5} \mathbf{F} \cdot \mathbf{n}_5 \, dS = \iint_{[0,1]\times[0,1]} (\mathbf{i} + v\mathbf{k}) \cdot \mathbf{i} \, dA = 1$$

and

$$\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_6} \mathbf{F} \cdot \mathbf{n}_6 \, dS = \iint_{[0,1]\times[0,1]} (v\mathbf{k}) \cdot (-\mathbf{i}) \, dA = 0.$$

Therefore, $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0 + 1 + 0 + 0 + 1 + 0 = 2$ is the flux out across S. ◀

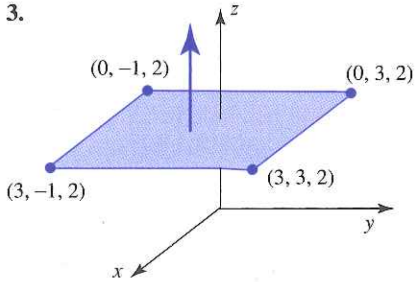
▶ EXERCISES 7.4

1. The surface S is defined by $x^2 + y^2 = 1$, $1 \le z \le 2$; assume that it is oriented by the outward-pointing normal vector. Among the vector fields $\mathbf{F}_1(x, y, z) = \mathbf{i}$, $\mathbf{F}_2(x, y, z) = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{F}_3(x, y, z) = \mathbf{k}$, identify those whose (outward) flux is zero.

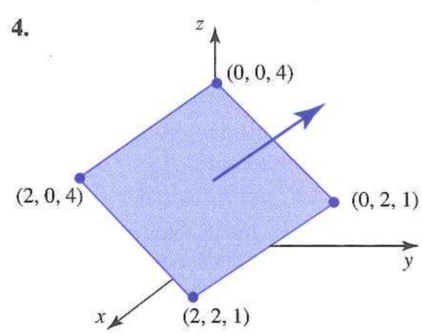
2. Consider the graph of the function $g(x, y) = 1 - x^2 - y^2$, $-1 \le x, y \le 1$, oriented by the upward-pointing normal. Determine the sign of the outward flux of the following fields: $\mathbf{F}_1(x, y, z) = \mathbf{k}$, $\mathbf{F}_2(x, y, z) = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{F}_3(x, y, z) = -x\mathbf{i} - y\mathbf{j}$.

Exercises 3 to 6: Compute the flux of the vector field $\mathbf{F} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ across the given region (assumed subset of a plane), with the indicated orientation.

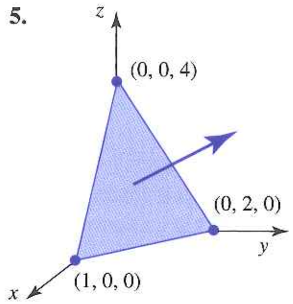
3.



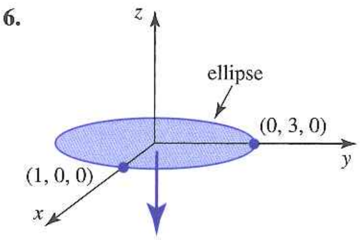
4.



5.



6.



Exercises 7 to 14: Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

7. $\mathbf{F} = 4y\mathbf{i} + (3x - 1)\mathbf{j} + z\mathbf{k}$, S is the part of the plane $3x + y - z = 1$ (with the upward normal) inside the vertical cylinder of radius 2 whose axis of symmetry is the z-axis

8. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, S is the part of the cone $z = \sqrt{x^2 + y^2}$ (oriented with the inward normal) inside the vertical cylinder $x^2 + y^2 = 9$

9. $\mathbf{F} = x^2\mathbf{i} + 2z\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, oriented with an outward normal

10. $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$, S is the surface parametrized by $\mathbf{r}(u, v) = (u \cos v, u \sin v, v)$, $0 \leq u \leq 1$, $0 \leq v \leq 4\pi$, and oriented with an upward-pointing normal

11. $\mathbf{F} = z\mathbf{k}$, S is the paraboloid $z = x^2 + y^2$ (oriented with the normal pointing away from it) between the planes $z = 1$ and $z = 2$

12. $\mathbf{F} = 2\mathbf{i} - xy\mathbf{j}$, S is the graph of the function $z = f(x, y) = x^2y^3 - 1$, where $0 \leq x, y \leq 1$, oriented by the upward-pointing normal

13. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, S is the surface parametrized by $\mathbf{r}(u, v) = (e^u \cos v, e^u \sin v, v)$, $0 \leq u \leq \ln 2$, $0 \leq v \leq \pi$, and oriented with an upward-pointing normal

14. $\mathbf{F} = x^2y\mathbf{i} - (y + x)\mathbf{j} - z^2x\mathbf{k}$, S is the part of the plane $x + 2y + 8z = 8$ in the first octant with the normal vector pointing upward

15. Let $T(x, y, z) = x^2 + y^2 + 3z^2$ be the temperature at a point (x, y, z) in $\mathbb{R}^3$. Compute the heat flux outward across the surface $x^2 + y^2 = 1$, $-1 \leq z \leq 1$.

16. Let $T(x, y, z) = e^{-x^2-y^2-z^2}$ be the temperature at a point (x, y, z) in $\mathbb{R}^3$. Compute the heat flux outward across the sphere $x^2 + y^2 + z^2 = 1$.

17. Consider the vector field $\mathbf{F} = c\mathbf{k}$, where c is a constant.

(a) Compute the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, with the outward-pointing normal.

(b) Compute the flux of $\mathbf{F}$ across the disk $x^2 + y^2 \leq a^2$ in the xy-plane, with the upward-pointing normal.

(c) Why are the answers in (a) and (b) the same?

Exercises 18 to 23: Find the flux of $\mathbf{F}$ across the surface S.

18. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$, out of the closed region bounded by the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 4$

19. $\mathbf{F} = x\mathbf{i}$, out of the closed region bounded by the paraboloids $z = x^2 + y^2$ and $z = 12 - x^2 - y^2$

20. $\mathbf{F} = x\mathbf{i}$, out of the closed region bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, $a > b$

21. $\mathbf{F} = y^3(\mathbf{j} - \mathbf{k})$, across the part of the plane $2x + y + z = 16$ in the first octant in the direction away from the origin

22. $\mathbf{F} = \mathbf{i} + xy\mathbf{j}$, across the closed cylinder (and in the direction away from it) $x^2 + y^2 = 1$, with the top disk at $z = 2$ and the bottom disk in the xy-plane

23. $\mathbf{F} = x^2y\mathbf{i} + xy^3\mathbf{j} + 2xyz\mathbf{k}$, upward across the surface $z = 2x^2y$, $0 \le x \le 1$, $0 \le y \le 2$

24. Compute the flux through the surface of the plane $z = 2$, $0 \le x, y \le a$ $(a > 0)$ of the constant unit vector field $\mathbf{F}$ that makes an angle of α rad $(0 \le \alpha \le \pi/2)$ with respect to the plane.

25. Let $\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi$ be the representation of the vector field $\mathbf{F}$ in spherical coordinates. Show that the flux of $\mathbf{F}$ out of the sphere $x^2 + y^2 + z^2 = a^2$, $a > 0$, satisfies $\iint_S \mathbf{F} \cdot \mathbf{S} = a^2 \int_0^{2\pi} \left(\int_0^{\pi} F_\rho \sin\phi d\phi \right) d\theta$.

26. Let S be the (closed) surface consisting of the part of the cone $z^2 = x^2 + y^2$, $1 \le z \le 2$ together with the top and bottom disks (in the planes $z = 2$ and $z = 1$). Show that the vector fields $\mathbf{F}_1 = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$ and $\mathbf{F}_2 = (y^2 + z)\mathbf{i} + (6y + x)\mathbf{j}$ have the same outward flux.

27. Let $\mathbf{F} = (x + y)\mathbf{i} + \mathbf{j} + z\mathbf{k}$ and assume that S is the part of the plane $x + 2y + 8z = 8$ in the first octant, oriented by the downward-pointing normal. Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using a projection of S onto a coordinate plane.

28. Let S be the part of the plane $z = 2$ defined by $0 \le x, y \le a$, $a > 0$. Let c be a positive constant.

(a) Compute the flux of a vertical field $\mathbf{F} = c\mathbf{k}$ across S.

(b) Compute the flux of $\mathbf{F} = cz\mathbf{k}$ across S.

(c) Compute the flux of $\mathbf{F} = cz^2\mathbf{k}$ across S.

(d) Compute the flux of $\mathbf{F} = c(\mathbf{j} + \mathbf{k})/\sqrt{2}$ across S.

(e) Compute the flux of $\mathbf{F} = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j}$ across S.

(f) Interpret the results of (a)–(e).

29. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}$ and S are as in Exercise 9, using the projection of S onto the xy-plane.

30. Let S be the graph of a C^1 function $z = f(x, y)$, $(x, y) \in D$, oriented as required by Definition 7.9. Show that, for a continuous vector field $\mathbf{F} = (F_1, F_2, F_3)$, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (-F_1(\partial f/\partial x) - F_2(\partial f/\partial y) + F_3)\, dA$.

▶ 7.5 INTEGRALS: PROPERTIES AND APPLICATIONS

Although we have defined all kinds of integrals in the last three chapters, we have to admit that we were in a way repeating the same things over and over again (such as, for example, the construction of Riemann sums). As a conclusion, we will present a unified view of the integrations we have discussed, list their properties, and show several applications.

Notation

Throughout this section, we will use $\mathcal{M}$ to denote either of the following:

(a) A curve $\mathbf{c}$ (that will be viewed in applications as an approximation of a thin wire);

(b) A plane region D (that will represent a thin flat plate in $\mathbb{R}^2$);

(c) A surface S (that will represent a thin sheet (possibly curved) in $\mathbb{R}^3$); or

(d) A solid W (that will represent a three-dimensional solid in $\mathbb{R}^3$).

Alternatively (the Laplacian is computed to be $\Delta f = 2 + 2 = 4$),

$$\int_{\mathbf{c}} \frac{\partial f}{\partial n} ds = \iint_{\{x^2+y^2\le 4\}} \Delta f \, dA = 4 \text{ area}(\{x^2 + y^2 \le 4\}) = 16\pi.$$

▶ EXERCISES 8.1

1. Consider a constant vector field $\mathbf{F}(x, y) = a\mathbf{i} + b\mathbf{j}$ in $\mathbb{R}^2$ $(a, b \in \mathbb{R})$, and let $\mathbf{c}$ be any simple closed curve in $\mathbb{R}^2$.

(a) Without using Green's Theorem, find the circulation $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ of $\mathbf{F}$ around $\mathbf{c}$.

(b) Use Green's Theorem to confirm your answer to (a).

2. Let $\mathbf{F}$ be a C^1 vector field in $\mathbb{R}^2$ whose scalar curl at $(3, -2)$ is equal to 7. Approximate the counterclockwise circulation $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ of $\mathbf{F}$ around the circle $\mathbf{c}$ of radius 0.02 centered at $(3, -2)$.

Exercises 3 to 7: Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ using Green's Theorem.

3. $\mathbf{F} = -2y\mathbf{i} + x\mathbf{j}$, $\mathbf{c}(t) = (2\cos t, \sin t)$, $t \in [0, 2\pi]$

4. $\mathbf{F} = (x^2 + 1)^{-1}\mathbf{j}$, $\mathbf{c}$ is the boundary of the rectangle $[0, 2] \times [0, 3]$, oriented counterclockwise

5. $\mathbf{F} = e^{x+y}\mathbf{j} - e^{x-y}\mathbf{i}$, $\mathbf{c}$ is the boundary of the triangle defined by the lines $y = 0$, $x = 1$, and $y = x$, oriented counterclockwise

6. $\mathbf{F} = (2 - y^3)\mathbf{i} + (y + x^3 + 2)\mathbf{j}$, $\mathbf{c}$ is the circle of radius 5 centered at the origin and oriented counterclockwise

7. $\mathbf{F} = 2x^2y^2\mathbf{i} - x\mathbf{j}$, $\mathbf{c}$ consists of the curve $y = 2x^3$ from $(0, 0)$ to $(1, 2)$ followed by the straight-line segment from $(1, 2)$ back to $(0, 0)$, oriented counterclockwise

8. Let D be a type-2 region given by $c \le y \le d$ and $\phi(y) \le x \le \psi(y)$, let $\mathbf{c}$ be its positively oriented boundary, and let $\mathbf{F}_2(x, y) = (0, Q(x, y))$.

(a) Show that the integral of $\mathbf{F}_2$ along the line segments $y = c$ and $y = d$ is zero.

(b) Prove that $\int_{\mathbf{c}} \mathbf{F}_2 \cdot d\mathbf{s} = \int_c^d Q(\psi(y), y) - Q(\phi(y), y)\, dy$.

(c) Show that $Q(\psi(y), y) - Q(\phi(y), y) = \int_{\phi(y)}^{\psi(y)} (\partial Q(x, y)/\partial x)\, dx$.

(d) Conclude that $\int_{\mathbf{c}} \mathbf{F}_2 \cdot d\mathbf{s} = \iint_D (\partial Q/\partial x)\, dA$.

Exercises 9 to 13: Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ directly, or using Green's Theorem.

9. $\mathbf{F} = x^2y^2\mathbf{i} + y^4\mathbf{j}$, $\mathbf{c}$ is the curve $x^2 + y^2 = 1$, oriented counterclockwise

10. $\mathbf{F} = (2x + 3y + 2)\mathbf{i} - (x - 4y + 3)\mathbf{j}$, $\mathbf{c}$ is the ellipse $x^2 + 4y^2 = 4$, oriented clockwise

11. $\mathbf{F} = \cosh y\mathbf{i} + x \sinh y\mathbf{j}$, $\mathbf{c}$ is the boundary of the triangle defined by the lines $y = 4x$, $y = 2x$, and $x = 1$, oriented counterclockwise [recall that $\cosh y = (e^y + e^{-y})/2$ and $\sinh y = (e^y - e^{-y})/2$]

12. $\mathbf{F} = e^x(\mathbf{i} + \mathbf{j})$, $\mathbf{c}$ is the boundary of the triangle with vertices $(0, 0)$, $(1, 2)$, and $(0, 2)$, oriented counterclockwise

13. $\mathbf{F} = \arctan(y/x)\mathbf{i} + \arctan(x/y)\mathbf{j}$, $\mathbf{c}$ is the circle $x^2 + y^2 = 2$, oriented counterclockwise

14. Assume that the curves involved are oriented counterclockwise.

(a) Compute $\displaystyle\int_{\mathbf{c}} \frac{x\,dy - y\,dx}{x^2 + y^2}$, where $\mathbf{c}$ is the circle $x^2 + y^2 = 1$.

(b) Compute $\displaystyle\int_{\mathbf{c}} \frac{x\,dy - y\,dx}{x^2 + y^2}$, where $\mathbf{c}$ is the circle $(x - 1)^2 + (y - 1)^2 = 1$.

15. Using a path integral, compute the area of the region D bounded by the curves $y = 2x^2$ and $y = 4x$.

16. Using a path integral, compute the area of the region D bounded by the curves $x = y^2$, $x = 2$, and $x = 3$.

17. Using a path integral, compute the area of the region D in the first quadrant bounded by the astroid $x^{2/3} + y^{2/3} = 1$.

18. Using a path integral, compute the area of the region bounded by the x-axis and the cycloid $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$, where $0 \le t \le 2\pi$.

19. Compute the work of the force $\mathbf{F} = x\mathbf{i} + (x^2 + 3y^2)\mathbf{j}$ done on a particle that moves along the straight-line segments from $(0, 0)$ to $(3, 0)$, then from $(3, 0)$ to $(1, 2)$, and then from $(1, 2)$ back to $(0, 0)$.

20. Let D be a region that satisfies Assumption 8.1, with a positively oriented boundary $\partial D = \mathbf{c}$. Assume that D is of constant density ρ. Express its mass m and moments M_x and M_y (with respect to the y-axis and x-axis) in terms of path integrals.

21. Let D be a region that satisfies Assumption 8.1, with a positively oriented boundary $\partial D = \mathbf{c}$. Assume that D is of constant density ρ. Express its moments of inertia about the x-axis and y-axis in terms of path integrals.

22. Let D be the disk $x^2 + y^2 \le 1$, let $\mathbf{c}$ be its positively oriented boundary, and let $f(x, y) = x^2 + 3y^2$. By computing both sides, check that $\iint_D \Delta f \, dA = \int_{\mathbf{c}} D_{\mathbf{n}} f \, ds$, where $\mathbf{n}$ is the outward normal to $\mathbf{c}$ and $D_{\mathbf{n}} f$ is the directional derivative in the direction of the normal. Δf denotes the Laplacian of f, defined by $\Delta f = f_{xx} + f_{yy}$.

23. Check that (see Exercise 22 for the notation) $\iint_D \Delta f \, dA = \int_{\mathbf{c}} D_{\mathbf{n}} f \, ds$ for the function $f(x, y) = e^x \cos y$, where D is the rectangle $[0, 1] \times [0, 2]$ and $\mathbf{c}$ is its positively oriented boundary.

24. Check that (see Exercise 22 for the notation) $\iint_D \Delta f \, dA = \int_{\mathbf{c}} D_{\mathbf{n}} f \, ds$ for the function $f(x, y) = e^{x+y}$, where D is the rectangle $[0, 1] \times [0, 1]$ and $\mathbf{c}$ is its positively oriented boundary.

▶ 8.2 THE DIVERGENCE THEOREM

The Divergence Theorem (or Gauss' Divergence Theorem) is similar to Green's Theorem: it relates an integral over a closed geometric object (a closed surface) to an integral over the region (in this case, a three-dimensional solid region) enclosed by it.

Elementary regions in $\mathbb{R}^3$ are regions in $\mathbb{R}^3$ bounded by surfaces that are graphs of real-valued functions of two variables. Depending on which of the variables are involved, the regions are called type 1, type 2, or type 3. A region is of type 1 if its "bottom" and "top" sides are graphs of continuous functions $\kappa_1(x, y)$ and $\kappa_2(x, y)$. A region is of type 2 if its "back" and "front" sides are graphs of continuous functions $\kappa_1(y, z)$ and $\kappa_2(y, z)$, and of type 3 if its "left" and "right" sides are graphs of continuous functions $\kappa_1(x, z)$ and $\kappa_2(x, z)$ [of course, the names "top," "bottom," "left," etc., for sides depend on the point from which we look at the xyz-coordinate system; see Section 6.5 for precise definitions; here, we drop the notation (3D) since the context will clearly distinguish between two-dimensional and three-dimensional elementary regions].

A region is of type 4 if it is of type 1, type 2, and type 3. For example, a rectangular box whose sides are parallel to coordinate axes is of type 4. The ball $\{(x, y, z) | x^2 + y^2 + z^2 \le 1\}$ and upper half-ball $\{(x, y, z) | x^2 + y^2 + z^2 \le 1, z \ge 0\}$ are of type 4.

Before giving the statement of the theorem, we describe the regions that will be involved.

▶ EXERCISES 8.2

1. Consider the vector field $\mathbf{F} = \mathbf{r}/\|\mathbf{r}\|^3$, where $\mathbf{r} \neq \mathbf{0}$.

(a) Show that $div\, \mathbf{F} = 0$.

(b) Find $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$, where S_1 is the sphere of radius 1 centered at the origin, oriented by the outward normal. Can the Divergence Theorem be used to compute this integral?

(c) If possible, use the Divergence Theorem to compute $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, where S_2 is the sphere of radius 1 centered at the point $(0, 0, 2)$, oriented by the outward normal.

2. Let $\mathbf{F}$ be the velocity vector of a fluid, and assume that the only information known about it is that $div\, \mathbf{F}(3, 0, -1) = 4$. Approximate the flux out of a sphere of radius 0.1 centered at $(3, 0, -1)$. Give a reason why your answer is an approximation and not the actual value of the flux.

3. Assume that $\mathbf{F}$ is a vector field such that $div\, \mathbf{F}(x, y, z) = 3$ for all $(x, y, z) \in \mathbb{R}^3$. Find the flux of $\mathbf{F}$ out of the parallelepiped with sides 3, 2, and 5.

4. Find the flux of the vector field $\mathbf{F} = \mathbf{c} \times \mathbf{r}$, where $\mathbf{c}$ is a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, out of any sphere of radius 1.

Exercises 5 to 13: Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is a closed surface oriented by an outward normal.

5. $\mathbf{F}(x, y, z) = (y^2 + \sin z)\mathbf{i} + (e^{\sin z} + 2)\mathbf{j} + (xy + \ln x)\mathbf{k}$, S is the surface of the cube $0 \le x, y, z \le 1$

6. $\mathbf{F}(x, y, z) = (x^2 + z^2)\mathbf{i} + (y^2 + z^2)\mathbf{k}$, S is the surface of the parallelepiped $0 \le x, y \le 2$, $0 \le z \le 4$

7. $\mathbf{F}(x, y, z) = (x + y^2 + 1)\mathbf{i} + (y + xz)\mathbf{j}$, S consists of the part of the cone $z^2 = x^2 + y^2$ bounded by the disks $0 \le x^2 + y^2 \le 1$, $z = 1$, and $0 \le x^2 + y^2 \le 4$, $z = 2$

8. $\mathbf{F}(x, y, z) = (2x + 3y)\mathbf{i} - (4y + 3z)\mathbf{j} + 4z\mathbf{k}$, S consists of the paraboloid $z = x^2 + y^2$, $0 \le z \le 1$, and the disk $0 \le x^2 + y^2 \le 1$, $z = 1$

9. $\mathbf{F}(x, y, z) = -e^x \cos y\mathbf{i} + e^x \sin y\mathbf{j} + \mathbf{k}$, S is the surface of the sphere $x^2 + y^2 + z^2 = 1$

10. $\mathbf{F}(x, y, z) = x^{-1}\mathbf{i} + z^{-1}\mathbf{j} - yz^{-1}\mathbf{k}$, S is the surface of the parallelepiped $1 \le x \le 2$, $2 \le y, z \le 4$

11. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$, S consists of the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$, and the disk $0 \le x^2 + y^2 \le 1$ in the xy-plane

12. $\mathbf{F}(x, y, z) = 2x\mathbf{i} + xy^2\mathbf{j} + xyz\mathbf{k}$, S is the boundary of the three-dimensional solid inside $x^2 + y^2 = 2$, outside $x^2 + y^2 = 1$, and between the planes $z = 0$ and $z = 4$

13. $\mathbf{F}(x, y, z) = ye^z\mathbf{i} + yz\mathbf{k}$, S is the surface of the tetrahedron in the first octant, bounded by the plane $x + 2y + z = 4$

14. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Prove that $\iint_S \mathbf{r} \cdot d\mathbf{S} = 3v(W)$, where $v(W)$ is the volume of the three-dimensional region W, bounded by S (i.e., $\partial W = S$).

15. Let W be a solid three-dimensional region that satisfies Assumption 8.2, and denote by S its positively oriented boundary. Prove that $\iiint_W \|\mathbf{r}\|^{-2}\, dV = \iint_S \|\mathbf{r}\|^{-2}\mathbf{r} \cdot d\mathbf{S}$.

16. Use the Divergence Theorem to compute $\iint_S (x + y)\, dS$, where S consists of the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$ (oriented positively), and the disk $0 \le x^2 + y^2 \le 1$ in the xy-plane.

17. Use the Divergence Theorem to compute the surface integral $\iint_S xyz\, dS$, where S is the sphere $x^2 + y^2 + z^2 = 1$, oriented by the outward-pointing normal.

18. Compute $\iint_S (x^2 + y^2)\, dS$, where S consists of the part of the paraboloid $z = 2(x^2 + y^2)$ between $z = 0$ and $z = 4$, together with the top disk $0 \le x^2 + y^2 \le 2$, $z = 4$, oriented by the outward normal.

19. Let W be a solid three-dimensional region that satisfies Assumption 8.2, bounded by a closed, positively oriented surface S. Show that, for C^2 functions f and g, $\iint_S f\nabla g\, d\mathbf{S} = \iiint_W (f\Delta g + \nabla f \cdot \nabla g)\, dV$.

20. Compute $\iint_S \mathbf{c} \cdot d\mathbf{S}$, if $\mathbf{c}$ is a constant vector field and S is a closed surface.

21. Let W be a solid three-dimensional region that satisfies Assumption 8.2, bounded by a closed, positively oriented surface S. Show that $\iint_S D_{\mathbf{n}} f\, dS = \iiint_W \Delta f\, dV$, where f is of class C^2 and $D_{\mathbf{n}} f$ denotes the directional derivative of f in the direction of the outward unit normal to S.

22. Let $\mathbf{F}$ be a C^1 vector field in $\mathbb{R}^2$. Assume that $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{n}\, ds = 0$ for any closed curve $\mathbf{c}$ in $\mathbb{R}^2$ (with the outward normal $\mathbf{n}$). What (if anything) can be said about the divergence of $\mathbf{F}$?

23. Let $\mathbf{F} = y\mathbf{i}/(x^2 + y^2) - x\mathbf{j}/(x^2 + y^2)$. Compute the outward flux $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{n}\, ds$ of $\mathbf{F}$ across the rectangle $R = [-1, 1] \times [-1, 2]$.

Exercises 24 to 28: Find the outward flux $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{n}\, ds$ and the counterclockwise circulation $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ of the vector field $\mathbf{F}$ along the curve $\mathbf{c}$.

24. $\mathbf{F}(x, y) = (2x - 1 + y)\mathbf{i} - (x - 3y)\mathbf{j}$, $\mathbf{c}$ is the square with the vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$, and $(-1, 1)$

25. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} - xy\mathbf{j}$, $\mathbf{c}$ is the triangle defined by $y = x$, $y = 2x$, and $x = 1$

26. $\mathbf{F}(x, y) = 2xy\mathbf{i} + 3y^2\mathbf{j}$, $\mathbf{c}$ is the boundary of the region in the first quadrant defined by $y = x^2$ and $y = 1$

27. $\mathbf{F}(x, y) = e^x e^y\mathbf{i} + 2e^y\mathbf{j}$, $\mathbf{c}$ is the boundary of the rectangle $R = [0, 3] \times [0, 4]$ oriented counterclockwise

28. $\mathbf{F}(x, y) = e^x \cos y\mathbf{i} + (xy + e^x \sin y)\mathbf{j}$, $\mathbf{c}$ is the boundary of the region defined by the curves $y = \ln x$, $y = 0$, and $x = e$

29. Consider the vector field $\mathbf{F}(x, y, z) = e^{-x^2}\mathbf{k}$, and let W be the cube $[0, 1] \times [0, 1] \times [0, 1]$. The boundary $\partial W = S$ consists of six squares.

(a) Try to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ directly, that is, by computing surface integrals.

(b) Use the Divergence Theorem to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

▶ 8.3 STOKES' THEOREM

Stokes' Theorem is similar in spirit to Green's Theorem: it relates the path integral of a vector field around a closed curve $\mathbf{c}$ in $\mathbb{R}^3$ to an integral over a surface S whose boundary is $\mathbf{c}$. As usual, we have to make precise the assumptions on the curves and surfaces involved. We will do it in two stages: first for a surface that is the graph of a function $z = f(x, y)$ and then for a general parametrized surface.

Let S be a surface defined as the graph of a function $z = f(x, y)$, where $(x, y) \in D$. Assume that the domain $D \subseteq \mathbb{R}^2$ is a region to which Green's Theorem applies (see Assumption 8.1 in Section 8.1). The boundary ∂D of D is a simple closed curve (i.e., a closed curve that does not intersect itself), or several such curves, oriented positively (as we walk along the boundary, the region D is on our left). Parametrize S by (for convenience, we depart from using the standard parameters u and v and use x and y instead)

$$\mathbf{r}(x, y) = (x, y, f(x, y)), \qquad (x, y) \in D,$$

Combining the above, we get

$$\int_{\mathbf{c}} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

and that is the statement of Green's Theorem; see formula (8.4). ◀

When talking about gradient vector fields in Section 5.4, we stated the fact that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$ for any oriented, simple closed curve $\mathbf{c}$ is equivalent to $curl\,\mathbf{F} = \mathbf{0}$ if the domain U of $\mathbf{F}$ is simply-connected [read Theorem 5.8, following the equivalences (b) $\Leftrightarrow$ (a) $\Leftrightarrow$ (d)]. However, we gave the proof only in the case where U is a star-shaped set. We will now outline the proof in a general case.

Let $\mathbf{F}$ be a vector field that is defined and is C^1 on a simply-connected set $U \subseteq \mathbb{R}^3$, and assume that $curl\,\mathbf{F} = \mathbf{0}$. Find a surface S that does not go through the points where $\mathbf{F}$ is not defined or not C^1 and whose boundary is a closed curve $\mathbf{c}$ (this can always be done; however, the proof is beyond the scope of this book). By Stokes' Theorem,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \iint_S curl\,\mathbf{F} \cdot d\mathbf{S} = 0,$$

and we are done.

▶ EXERCISES 8.3

1. Let $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$. Use Stokes' Theorem to find the path integral $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where:

(a) $\mathbf{c}$ is the boundary of the square with vertices $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$, oriented counterclockwise, as seen from above.

(b) $\mathbf{c}$ is the boundary of the square with vertices $(1, 1, 0)$, $(0, 1, 0)$, $(0, 1, 1)$, and $(1, 1, 1)$, oriented by the normal $\mathbf{n} = \mathbf{j}$.

2. Take a closed surface S oriented by the outward normal, and break it into two parts S_1 and S_2 that share a boundary curve $\mathbf{c}$. Assume that S_1 and S_2 are oriented by the same normal as S, and that they satisfy the assumptions of Stokes' Theorem. Show that $\iint_{S_1} curl\,\mathbf{F} \cdot d\mathbf{S} = -\iint_{S_2} curl\,\mathbf{F} \cdot d\mathbf{S}$, for a C^1 vector field $\mathbf{F}$ defined on S. Conclude that $\iint_S curl\,\mathbf{F} \cdot d\mathbf{S} = 0$.

3. Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ directly, and then use Stokes' Theorem: let $\mathbf{F} = (x+1)^2\mathbf{i} - x^2\mathbf{k}$, and let $\mathbf{c}$ be the intersection of the cylinder $x^2 + 2x + y^2 = 3$ and the plane $z = x$, oriented counterclockwise, as seen from above.

Exercises 4 to 10: Find the circulation $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ of the vector field $\mathbf{F}$ along the curve $\mathbf{c}$ in the given direction.

4. $\mathbf{F}(x, y, z) = y^2\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}$, $\mathbf{c}$ is the ellipse $x^2 + 4y^2 = 4$, $z = 0$, oriented counterclockwise

5. $\mathbf{F}(x, y, z) = (2x + y)\mathbf{i} - (3x - y - z)\mathbf{k}$, $\mathbf{c}$ is the boundary of the triangle cut out from the plane $x + 4y + 3z = 1$ by the first octant, oriented clockwise as seen from the origin

6. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, $\mathbf{c}$ is the boundary of the circle $x^2 + y^2 = 4$ in the plane $z = 4$, oriented counterclockwise as seen from the origin

7. $\mathbf{F}(x, y, z) = (x^2 + z^2)\mathbf{i} + y^2z^2\mathbf{j}$, $\mathbf{c}$ is the boundary of the rectangle cut out from the plane $y = z$ by the planes $x = 1$, $x = 2$, $y = 0$, and $y = 4$, oriented counterclockwise as seen from above

8. $\mathbf{F}(x, y, z) = -2y\mathbf{i} + z\mathbf{j} - z\mathbf{k}$, $\mathbf{c}$ is the intersection of the cylinder $z^2 + x^2 = 1$ and the plane $y = x + 1$, oriented counterclockwise as seen from the origin

9. $\mathbf{F}(x, y, z) = y^2(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\mathbf{c}$ is the circle on the sphere $x^2 + y^2 + z^2 = 1$ defined by $z = 1/2$, oriented clockwise as seen from the origin

10. $\mathbf{F}(x, y, z) = 2x\mathbf{i} + y^2\mathbf{k}$, $\mathbf{c}$ is the boundary of the paraboloid $z = 4 - x^2 - y^2$ in the first octant, oriented clockwise as seen from the origin

11. Let $\mathbf{F}$ be a constant vector field. A surface S in $\mathbb{R}^3$ and its boundary curve $\mathbf{c}$ are assumed to satisfy the assumptions of Stokes' Theorem. Show that $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{2}\int_{\mathbf{c}} (\mathbf{F} \times \mathbf{r}) \cdot d\mathbf{s}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

12. Compute $\int_{\mathbf{c}} (2\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}) \cdot d\mathbf{s}$, where $\mathbf{c}$ is the circle $x^2 + y^2 = 1$, $z = 1$, oriented counterclockwise (as seen from above), by using the fact that $\mathbf{c}$ is the boundary of the cone $z^2 = x^2 + y^2$, $z = 1$.

13. Consider the vector field $\mathbf{F} = -2y\mathbf{i}/(x^2 + y^2) + 2x\mathbf{j}/(x^2 + y^2)$. Compute the counterclockwise circulation of $\mathbf{F}$ along the circle $x^2 + y^2 = 1$, $z = 0$, directly. Can you compute $\iint_S \mathit{curl}\,\mathbf{F} \cdot d\mathbf{S}$, over the disk S in the xy-plane enclosed by $\mathbf{c}$? Explain why your answers do not violate Stokes' Theorem.

Exercises 14 to 21: Compute the circulation $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ of the vector field $\mathbf{F}$ along the curve $\mathbf{c}$ by direct computation, using the Fundamental Theorem of Calculus or using Stokes' Theorem.

14. $\mathbf{F}(x, y) = 3xe^{-y}\mathbf{i}$, $\mathbf{c}$ consists of the path $y = x^2$ from $(0, 0)$ to $(2, 4)$, followed by the straight line from $(2, 4)$ back to $(0, 0)$

15. $\mathbf{F}(x, y) = 2x\mathbf{i}/(x^2 + y) + \mathbf{j}/(x^2 + y)$, $\mathbf{c}$ is the boundary of the rectangle $[1, 2] \times [0, 1]$, oriented counterclockwise

16. $\mathbf{F}(x, y) = x \sin y\mathbf{i} + y \sin x\mathbf{j}$, $\mathbf{c}$ is the boundary of the triangle defined by the lines $y = x$, $y = y = \pi x/2$, and $x = 1$, oriented counterclockwise

17. $\mathbf{F}(x, y, z) = y\mathbf{i} + 2z\mathbf{j} + 3x\mathbf{k}$, $\mathbf{c}$ is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y$, oriented counterclockwise as seen from above

18. $\mathbf{F}(x, y) = (2xy\mathbf{i} + \mathbf{j})e^{x^2}$, $\mathbf{c}$ consists of the straight-line segments from $(0, 0)$ to $(1, 1)$, then from $(1, 1)$ to $(0, 2)$, and then from $(0, 2)$ back to $(0, 0)$

19. $\mathbf{F}(x, y, z) = x\mathbf{i} - yz\mathbf{j} + \mathbf{k}$, $\mathbf{c}$ is the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$, oriented counterclockwise as seen from above

20. $\mathbf{F}(x, y, z) = 5\mathbf{i} + 2\mathbf{j} + z\mathbf{k}$, $\mathbf{c}$ is the ellipse $y^2 + 4z^2 = 4$ in the plane $x = 2$, oriented clockwise as seen from the origin

21. $\mathbf{F}(x, y, z) = (2x + y)\mathbf{i} + (2y - x)\mathbf{j}$, $\mathbf{c}$ is the helix $\mathbf{c}(t) = (\cos t, \sin t, t)$, $t \in [0, 3\pi]$, followed by the line segment from $(-1, 0, 3\pi)$ back to $(1, 0, 0)$

22. Show that if the curve $\mathbf{c} = \partial S$ and the surface S satisfy the assumptions of Stokes' Theorem, then $\int_{\mathbf{c}} f\nabla g \cdot d\mathbf{s} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$.

23. Show that if the curve $\mathbf{c} = \partial S$ and the surface S satisfy the assumptions of Stokes' Theorem, then $\int_{\mathbf{c}} f\nabla f \cdot d\mathbf{s} = 0$

24. Set up the integral for the counterclockwise circulation of the vector field $\mathbf{F} = e^x\mathbf{i}/(x^2 + 1)$ around the unit circle in the xy-plane. Then evaluate it using Stokes' Theorem.

▶ 8.4 DIFFERENTIAL FORMS AND CLASSICAL INTEGRATION THEOREMS

There are several reasons why we introduce and study differential forms. They provide a useful way of formalizing certain concepts, and also, they appear often in applications of vector calculus (see, for instance, Section 8.5). Moreover, we will be able to show that the