Math 5B - Midterm - 8/23

Name: ______________________________ Perm: ____________

- Read all directions carefully.
- Show all your work. Problems without work shown will receive no credit.
- No calculators.
- When you finish, staple your notecard to the back of your test.
- Good luck!

Problem 1 *(12 points total))*

(a) *(6 pts)* Compute the gradient of $f(x,y) = 4x^2 - 2xy + y^2$ at the point $\mathbf{p} = (2,1)$.

The gradient of f is

$$\nabla f(x,y) = (8x - 2y, -2x + 2y),$$

so $\nabla f(2,1) = (8 \cdot 2 - 2 \cdot 1, -2 \cdot 2 + 2 \cdot 1) = (14, -2)$.

(b) *(4 pts)* Compute the directional derivative of f at $\mathbf{p} = (2,1)$ in the direction $\mathbf{v} = (-1,1)$.

Since $\mathbf{v}$ is not a unit vector, we use

$$\mathbf{u} = \mathbf{v}/\|\mathbf{v}\| = (-1/\sqrt{2}, 1/\sqrt{2}).$$

Then the directional derivative of f at $\mathbf{p}$ in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}} f(2,1) = (14, -2) \cdot (-1/\sqrt{2}, 1/\sqrt{2}) = -\frac{16}{\sqrt{2}}$$

(c) *(2 pts)* Describe in words in what direction the gradient of a function "points".

The gradient of a function always points "uphill," or in the direction of greatest increase.

Problem 2 *(16 points total)* Alfred the ant is walking on a table along the curve parametrized by the path $\mathbf{c}(t) = (e^t \cos(t), e^t \sin(t))$, where t is measured in seconds and each component of $\mathbf{c}$ is measured in centimeters.

(a) *(6 pts)* In what direction was Alfred traveling at $t = \frac{\pi}{2}$, and how fast was he going?

The direction Alfred was traveling is given by the velocity:

$$\mathbf{c}'(t) = e^t(\cos(t) - \sin(t), \sin(t) + \cos(t))$$

so he was traveling in the direction $\mathbf{c}'(\frac{\pi}{2}) = (-e^{\pi/2}, e^{\pi/2})$. The speed of Alfred is given by the norm of the derivative:

$$\|\mathbf{c}'(t)\| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = \sqrt{2}e^t,$$

so his speed was $\|\mathbf{c}'(\frac{\pi}{2})\| = \sqrt{2}e^{\pi/2}$.

(b) *(5 pts)* How far did Alfred travel along his path in the first 2 seconds (i.e. from $t = 0$ to $t = 2$)?

The arclength of Alfred's path is

$$\int_0^2 \|\mathbf{c}'(t)\| dt = \int_0^2 \sqrt{2}e^t dt = \sqrt{2}(e^2 - 1).$$

(c) *(5 pts)* Compute the curvature of Alfred's path at $t = 1$.

The unit tangent vector is given by

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|} = \frac{1}{\sqrt{2}}(\cos t - \sin t, \sin t + \cos t).$$

In order to calculate the curvature, we need the norm of the *derivative* of $\mathbf{T}$:

$$\|\mathbf{T}'(t)\| = \|\frac{1}{\sqrt{2}}(-\sin t - \cos t, \cos t - \sin t)\| = 1.$$

Then the curvature of Alfred's path is $\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{c}'(t)\|} = \frac{1}{\sqrt{2}e^t}$, so at $t = 1$, his curvature was $\kappa(1) = \frac{1}{\sqrt{2}e}$.

Problem 3 *(16 points total)*

(a) *(5 pts)* Compute the Laplacian Δu for $u(x, y) = e^x \sin(y)$. Is u harmonic?

$$\Delta u(x, y) = u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0.$$

Since $\Delta u = 0$, u is harmonic.

(b) *(5 pts)* Compute the first order Taylor polynomial of u centered at the point $\mathbf{x}_0 = (\ln(2), \frac{\pi}{2})$.

$$\begin{aligned} T_1(x, y) &= u(\ln 2, \pi/2) + \nabla u(\ln 2, \pi/2) \cdot (x - \ln 2, y - \pi/2) \\ &= e^{\ln 2} \sin(\pi/2) + (e^{\ln 2} \sin(\pi/2), e^{\ln 2} \cos(\pi/2)) \cdot (x - \ln 2, y - \pi/2) \\ &= 2 + 2(x - \ln 2). \end{aligned}$$

(c) *(6 pts)* Compute the second order Taylor polynomial of u centered at the point $\mathbf{x}_0 = (\ln(2), \frac{\pi}{2})$.

$$\begin{aligned} T_2(x, y) &= T_1(x, y) + \frac{1}{2} [Hu(\ln 2, \pi/2)(x - \ln 2, y - \pi/2)] \cdot (x - \ln 2, y - \pi/2) \\ &= T_1(x, y) + \frac{1}{2} \left[\begin{bmatrix} e^{\ln 2} \sin(\pi/2) & e^{\ln 2} \cos(\pi/2) \\ e^{\ln 2} \cos(\pi/2) & -e^{\ln 2} \sin(\pi/2) \end{bmatrix} (x - \ln 2, y - \pi/2) \right] \cdot (x - \ln 2, y - \pi/2) \\ &= T_1(x, y) + \frac{1}{2} \left[\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} (x - \ln 2, y - \pi/2) \right] \cdot (x - \ln 2, y - \pi/2) \\ &= T_1(x, y) + (x - \ln 2, -y + \pi/2) \cdot (x - \ln 2, y - \pi/2) \\ &= T_1(x, y) + (x - \ln 2)^2 - (y - \pi/2)^2 \\ &= 2 + 2(x - \ln 2) + (x - \ln 2)^2 - (y - \pi/2)^2. \end{aligned}$$

Problem 4 *(8 points total)* Suppose that you are building a cylindrical silo with a hemispherical top. Denote the height of the cylindrical portion of the silo by h, and the radius of the silo by r (measured in meters). **The bottom of the silo must also be covered.**

Hint: The volume of a sphere is $\frac{4}{3}\pi r^3$*, and the surface area of a sphere is* $4\pi r^2$*.*

(a) *(2 pts)* Express the volume $V(r,h)$ and surface area $S(r,h)$ in terms of r and h.

The volume and surface area of the silo are given by

$$V(r,h) = \frac{2}{3}\pi r^3 + \pi r^2 h, \qquad S(r,h) = 3\pi r^2 + 2\pi r h.$$

(b) *(6 pts)* You only have enough funds for $S = 125\pi$ m^2 of material. Find r and h that maximizes the volume of the silo, i.e. maximize V subject to the constraint $S = 125\pi$.

Using the method of Lagrange multipliers, we find

$$\nabla V(r,h) = (2\pi r^2 + 2\pi r h, \pi r^2), \qquad \nabla S(r,h) = (6\pi r + 2\pi h, 2\pi r),$$

and set $\nabla V = \lambda \nabla S$. Then

$$2\pi r^2 + 2\pi r h = \lambda(6\pi r + 2\pi h), \qquad \pi r^2 = 2\pi\lambda r.$$

Solving for λ in the second equation gives $\lambda = \frac{r}{2}$ (we discount the possibility $r = 0$). Substituting into the first equation gives

$$2\pi r^2 + 2\pi r h = \frac{r}{2}(6\pi r + 2\pi h) \quad \Rightarrow \quad 2r^2 + 2rh = 3r^2 + rh \quad \Rightarrow \quad r = h.$$

Subsituting $r = h$ into the constraint equation $S = 125\pi$ gives

$$3\pi r^2 + 2\pi r \cdot r = 125\pi \quad \Rightarrow \quad 5r^2 = 125 \quad \Rightarrow \quad r = 5.$$

So, the volume is maximized when $r = 5, h = 5$.

Problem 5 *(8 points total)* Consider the following partial differential equation:

$$u_x u_y = 0,$$

and the following change of variables:

$$v = x + y, \qquad w = x - y.$$

(a) *(4 pts)* Write u_x and u_y in terms of u_v and u_w.

$u_x = \frac{\partial u}{\partial x} = u_v v_x + u_w w_x = u_v + u_w$

$u_y = \frac{\partial u}{\partial y} = u_v v_y + u_w w_y = u_v - u_w$

(b) *(4 pts)* Substitute your answers from part (a) into the original partial differential equation to verify that

$$(u_v)^2 = (u_w)^2.$$

$$\begin{aligned} u_x u_y = 0 &\Rightarrow (u_v + u_w)(u_v - u_w) = 0 \\ &\Rightarrow u_v^2 - u_w^2 = 0 \\ &\Rightarrow u_v^2 = u_w^2. \end{aligned}$$