2.1 Real-valued and Vector-valued Functions of Several Variables

- A function $f: \mathbb{R}^m \to \mathbb{R}^n$ is called a real-valued function if $n=1$, and a vector-valued function if $n>1$
  - Sometimes real-valued functions are called scalar ~~valued~~ functions.

- Some examples:
  - $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$: a real-valued (or scalar) function of one variable (one input, x)
  - $f: \mathbb{R} \to \mathbb{R}^2$, $f(t) = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$, a vector-valued function of one variable (one input, t)

    ★ sometimes instead of vector notation $\begin{bmatrix} \sim \\ \sim \end{bmatrix}$, you'll see component notation:

    $$f(t) = (2e^{2t}, e^{2t})$$

    These are exactly the same. The book uses component notation more frequently.

- $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 2x^3 - 3\ln(xy)$; a real-valued function of <u>2 variables</u> (2 inputs, x and y)

- $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (2e^{x+y}, -3z+2)$: a vector-valued function of 3 variables

- For a given vector-valued function $f: \mathbb{R}^m \to \mathbb{R}^n$, the <u>component functions</u> are the individual coordinates of the outputs. We usually put numbered subscripts on them:

$$f(t) = (2e^{2t}, e^{2t}) \Rightarrow f_1(t) = 2e^{2t},\ f_2(t) = e^{2t}$$

$$f(x,y,z) = (2e^{x+y}, -3z+2)$$

$$\Rightarrow f_1(x,y,z) = 2e^{x+y},\ f_2(x,y,z) = -3z+2$$

- One important function that we'll be using very frequently is the <u>distance function</u>:

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}$$

This describes the distance from the point (x,y,z) to the origin $(0,0,0)$.

- We use it so often that we give it its own notation:

$$\sqrt{x^2+y^2+z^2} = \|(x,y,z)\|$$

<u>or</u>

$$\vec{x} = (x_1, x_2, x_3) \Rightarrow \sqrt{x_1^2+x_2^2+x_3^2} = \|\vec{x}\|$$

- Sometimes it's also called the <u>Euclidean distance</u> or the <u>Euclidean norm</u>.

• The <u>domain</u> of a real- or vector-valued function is the set of all input values such that the output values are defined (i.e. no div. by 0, sqrt of negatives, ln's of #'s ≤ 0, etc.)

<u>Ex</u>: Find the domain of

$$f(x,y) = \frac{x-3y^2}{x^2-y}$$

The only time f is undefined is when the denominator $=0$: i.e. the curve $y=x^2$. Thus, the domain is

$$\{(x,y) \mid y \neq x^2\}$$

"the set of all (x,y) pairs"

"such that"

"the second coord. is not equal to the first coord. squared"

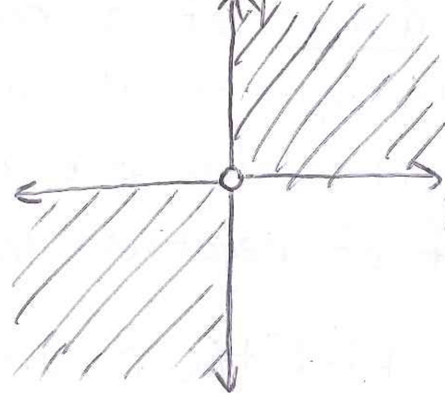
Ex: Find the domain of

$$f(x,y) = \left(\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, \sqrt{xy} \right)$$

From the first two components, we see that $(x,y) \neq (0,0)$. The last component requires that $xy \geq 0$. Then the domain is

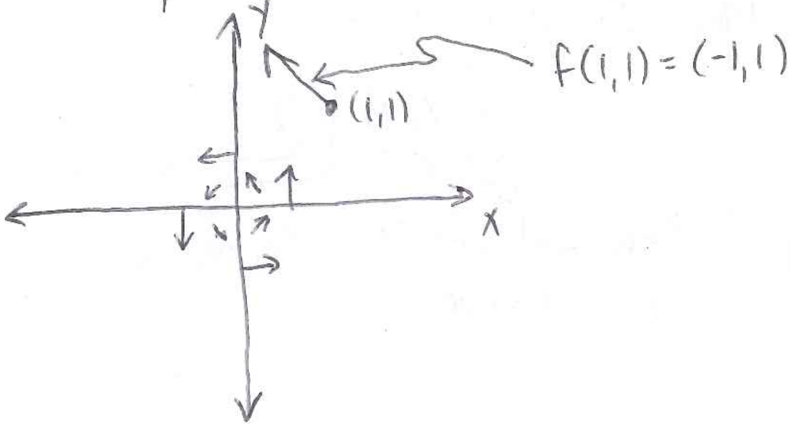
$$\{ (x,y) \mid x \neq 0 \text{ and } y \neq 0 \text{ and } xy \geq 0 \}$$

In terms of a region:



- A vector-field is a vector-valued function $f: \mathbb{R}^2 \to \mathbb{R}^2$ (in the plane) or $f: \mathbb{R}^3 \to \mathbb{R}^3$ (in space)

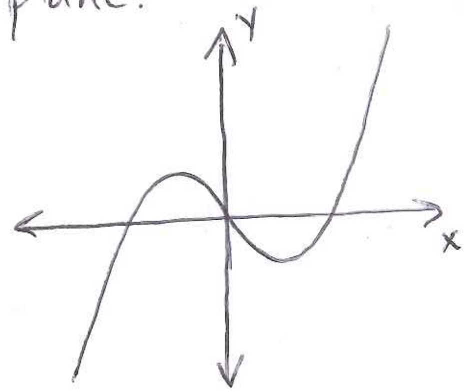
Ex: $f(x,y) = (-x, y)$



2.2 Graph of a Function of Several Variables

- The graph of a scalar function of on variable is a curve in the xy-plane:

$$f(x) = x^3 - 3x$$



- We can make this a little more precise:

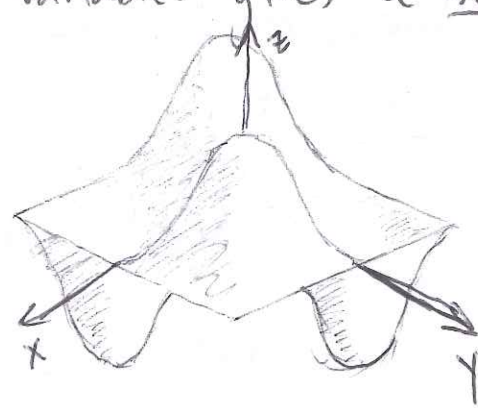
$$\text{Graph}(f) = \{(x,y) \mid y = f(x)\}$$

- We can generalize this to functions of 2 variables: given $f: \mathbb{R}^2 \to \mathbb{R}$, the graph of f is

$$\text{Graph}(f) = \{(x,y,z) \mid z = f(x,y)\}$$

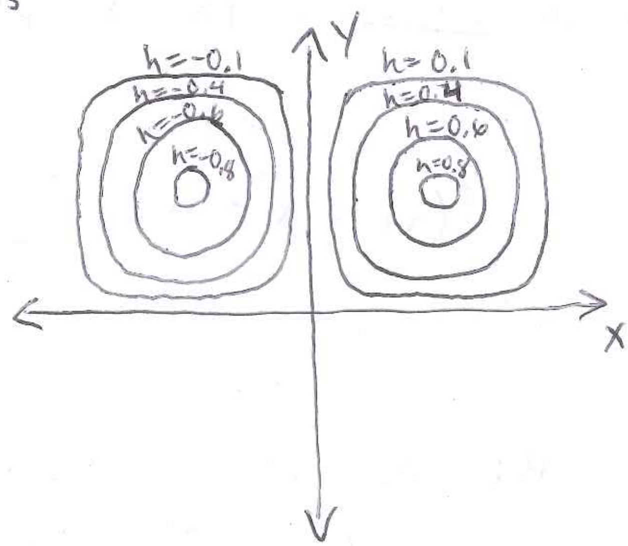
- instead of a curve (1 dimensional), the graph of a function of 2 variables gives a surface

$$f(x,y) = \sin(x)\sin(y)$$



- We aren't all trained artists though, so there's another way to describe surfaces: level curves or contour curves

$f(x,y) = \sin(x)\sin(y)$



- this idea is very similar to topographical maps

- The level set of value c of the function $f: \mathbb{R}^m \to \mathbb{R}$ is

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m \mid f(x_1, \dots, x_m) = c\}$$

- When $m=2$, we get a level curve or contour curve of level c:

$$\{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$$

Ex: Draw the contour diagram for $f(x,y) = 3x - 2y + 1$

We want to find a relationship between x and y for each particular c:

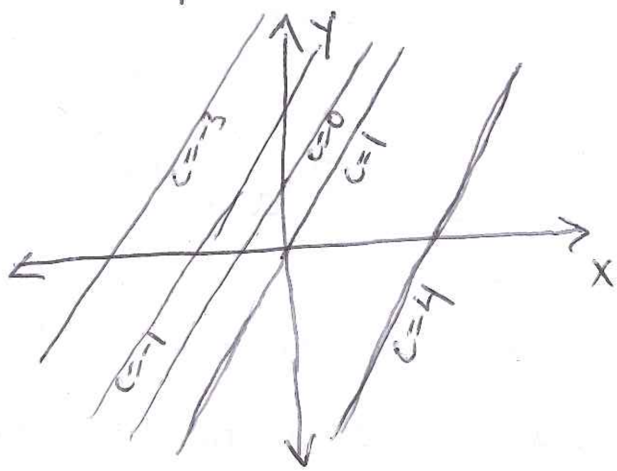
$$3x - 2y + 1 = c \rightarrow y = \frac{3}{2}x + \frac{(1-c)}{2}$$

$c=0 \rightarrow y = \frac{3}{2}x + \frac{1}{2}$

$c=1 \rightarrow y = \frac{3}{2}x$

$c=-1 \rightarrow y = \frac{3}{2}x + 1$

$\vdots$



<u>Ex</u>: Draw the contour diagram for

$$f(x,y) = 9 - x^2 - y^2$$

$$\Rightarrow 9 - x^2 - y^2 = c \rightarrow x^2 + y^2 = 9 - c$$

$c=0 \rightarrow x^2+y^2=9$

$c=5 \rightarrow x^2+y^2=4$

$c=9 \rightarrow x^2+y^2=0$

$c=10 \rightarrow x^2+y^2=-1$

<u>not</u> <u>possible</u>!

$c=-7 \rightarrow x^2+y^2=16$

$c=8 \rightarrow x^2+y^2=1$