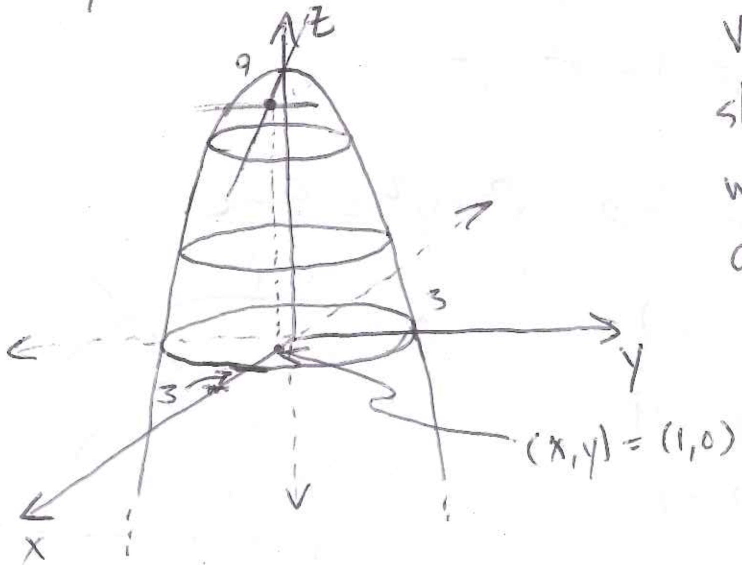
2.4 Derivatives

- Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- Consider the function $f(x,y) = 9 - x^2 - y^2$. How would we describe the "derivative" at the point $(x,y) = (1,0)$?



We can compute the slope in one direction, but we also need a slope in another direction

- We can compute the slope (i.e. derivative) in the x-direction: define $g(x) = f(x,b)$ (take b to be a constant). Then

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h, b) - f(x, b)}{h}$$

- this is called the partial derivative of f w.r.t. x denoted $\frac{\partial f}{\partial x}$. Similarly:

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(a, y+h) - f(a, y)}{h}$$

Ex: Let $f(x,y) = (x-3)^2 e^{3y}$. Then

$$\frac{\partial f}{\partial x} = 2(x-3)e^{3y} \qquad \frac{\partial f}{\partial y} = 3(x-3)^2 e^{3y}$$

Ex: Let $f(x,y) = 3xy^2 + \ln(x^2+y^2)$. Then

$$\frac{\partial f}{\partial x} = 3y^2 + \frac{1}{x^2+y^2} \cdot 2x$$

$$\frac{\partial f}{\partial y} = 6xy + \frac{1}{x^2+y^2} \cdot 2y$$

- More general definition: Let $f : \mathbb{R}^m \to \mathbb{R}$, with $f(\vec{x}) = f(x_1, x_2, \ldots, x_m)$. The partial derivative of f w.r.t. x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_m) - f(x_1, \ldots, x_i, \ldots, x_m)}{h}$$

- sometimes denoted

$$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial f}{\partial x_i} = D_i f$$

<u>Ex</u>: Let $f(x,y,z,w) = e^{xyzw} \sin(2x+z^2)$. Then

$$\frac{\partial f}{\partial x} = yzwe^{xyzw} \sin(2x+z^2) + 2e^{xyzw} \cos(2x+z^2)$$

$$\frac{\partial f}{\partial y} = xzwe^{xyzw} \sin(2x+z^2)$$

$$\frac{\partial f}{\partial z} = xywe^{xyzw} \sin(2x+z^2) + 2ze^{xyzw} \cos(2x+z^2)$$

$$\frac{\partial f}{\partial w} = xyze^{xyzw} \sin(2x+z^2)$$

- How about derivatives of vector-valued functions? We need information about <u>all</u> partial derivatives of <u>all</u> component functions. Let $f: \mathbb{R}^m \to \mathbb{R}^n$, and define

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

(here $f(x_1, \ldots, x_m) = (f_1, f_2, \ldots, f_n)$)

- Under certain technical conditions, Df is called **the derivative of f**

  - In the case $n=1$ (i.e. $f: \mathbb{R}^m \to \mathbb{R}$), the derivative has a special name: the **gradient of f**, denoted ∇f.

Ex: Find Df, where $f(x,y) = (x^2+2y, 3y^2, \cos(y))$.

Here, $f_1(x,y) = x^2+2y$; $f_2(x,y) = 3y^2$; and $f_3(x,y) = \cos(y)$. Then

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2 \\ 0 & 6y \\ 0 & -\sin(y) \end{bmatrix}$$

Ex: Find ∇f, where $f(x,y) = 9 - x^2 - y^2$.

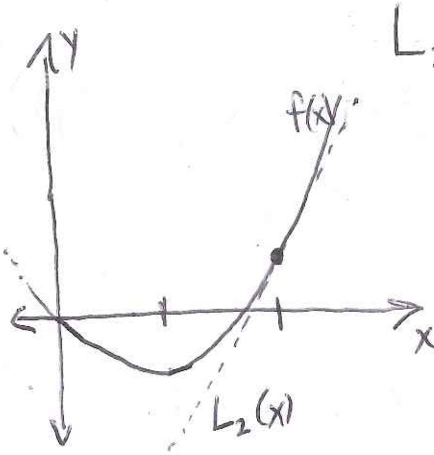
$$\Rightarrow \nabla f = Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -2x & -2y \end{bmatrix}$$

- Recall tangent line approximations: if f is a scalar function of 1 variable (i.e. $f: \mathbb{R} \to \mathbb{R}$), then

$$L_a(x) = f'(a)(x-a) + f(a)$$

is the tangent line approximation of f at $x=a$.

Ex: Let $f(x) = x^3 - 3x$. At $x = 2$,



$$\begin{aligned} L_2(x) &= f'(2)(x-2) + f(2) \\ &= [3(2)-3](x-2) + [(2)^3 - 3(2)] \\ &= 3(x-2) + 2 \\ &= 3x - 4 \end{aligned}$$

- How do we generalize this to higher dimensions?

Let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a function. Then

$$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

is the linear approximation of f at $\vec{a}$.

Ex: What is the linear approximation of $f(x,y) = 9 - x^2 - y^2$ at the point $\vec{a} = (1,0)$?

$$\begin{aligned} L_{(1,0)}(x,y) &= f(1,0) + Df(1,0)((x,y)-(1,0)) \\ &= [9-(1)^2-(0)^2] + \begin{bmatrix} -2(1) & -2(0) \end{bmatrix} \begin{bmatrix} x-1 \\ y-0 \end{bmatrix} \\ &= 8 + (-2)(x-1) + (0)(y-0) \end{aligned}$$

$= -2x + 10$

<u>Ex</u>: Find the linear approximation of

$f(x,y,z) = \ln(x^2 - y^2 + z)$

at the point $\vec{p} = (3,3,1)$.

First: $Df(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$

$$= \begin{bmatrix} \frac{2x}{x^2-y^2+z} & \frac{-2y}{x^2-y^2+z} & \frac{1}{x^2+y^2+z} \end{bmatrix}$$

$\Rightarrow Df(3,3,1) = \begin{bmatrix} 6 & -6 & 1 \end{bmatrix}$

So: $L_{(3,3,1)}(x,y,z) = f(3,3,1) + Df(3,3,1) \cdot \begin{bmatrix} x-3 \\ y-3 \\ z-1 \end{bmatrix}$

$$= \ln(3^2 - 3^2 + 1) + \begin{bmatrix} 6 & -6 & 1 \end{bmatrix} \begin{bmatrix} x-3 \\ y-3 \\ z-1 \end{bmatrix}$$

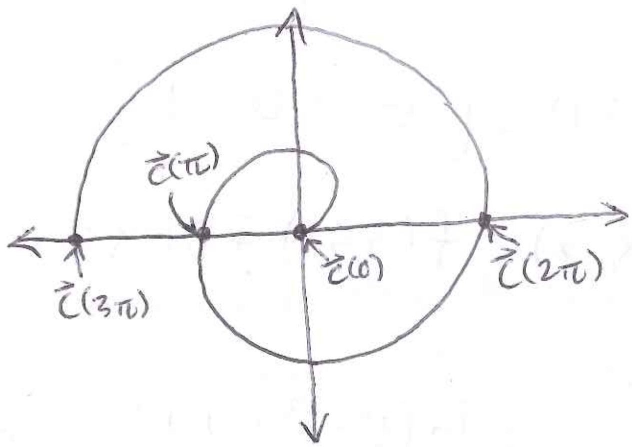
$$= 6(x-3) - 6(y-3) + (z-1)$$

$$= 6x - 6y + z - 1$$

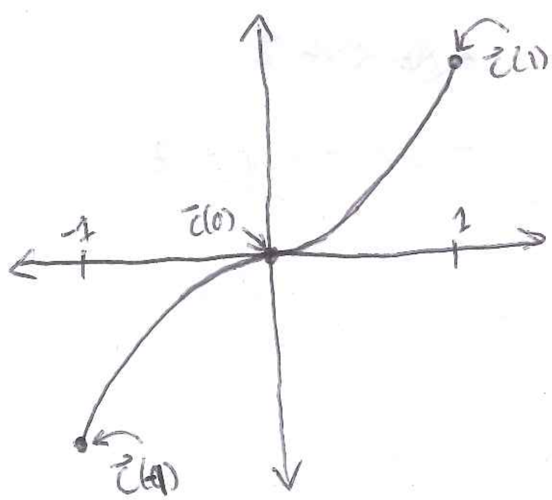
2.5 Paths and Curves in $\mathbb{R}^2$ and $\mathbb{R}^3$

- A path in $\mathbb{R}^2$ (~~and~~ or $\mathbb{R}^3$) is a function $\vec{c} : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ (or $\mathbb{R}^3$). The image of $\vec{c}$ is called a curve. $\vec{c}$ is also called a parametrization of the curve.

Ex: $\vec{c}(t) = (t\cos t,\ t\sin t)$, $t \in [0, 3\pi]$

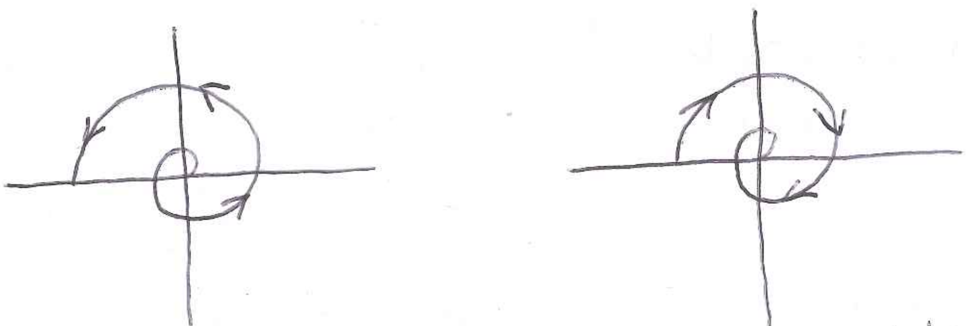


$\vec{c}(t) = (t, t^3)$, $t \in [-1, 1]$



- If $\vec{c}$ is defined on the interval $[a,b]$, $\vec{c}(a)$ is called the initial point, and $\vec{c}(b)$ is called the terminal point. Together, they are the endpoints of $\vec{c}$.
- The direction corresponding to increasing t values gives the positive orientation of $\vec{c}$

<u>Ex</u>: $\vec{c}(t) = (t\cos t, t\sin t)$



positive orientation   negative orientation

- Some common parametrizations:
  - <u>Lines</u>: to parametrize the line segment from $\vec{u} = (u_1, u_2, u_3)$ to $\vec{v} = (v_1, v_2, v_3)$, we have

$$\vec{c}(t) = \vec{u} + t(\vec{v} - \vec{u}), \quad t \in [0,1]$$

$$= (u_1 + t(v_1 - u_1), u_2 + t(v_2 - u_2), u_3 + t(v_3 - u_3))$$