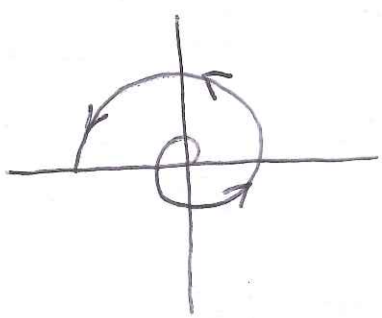
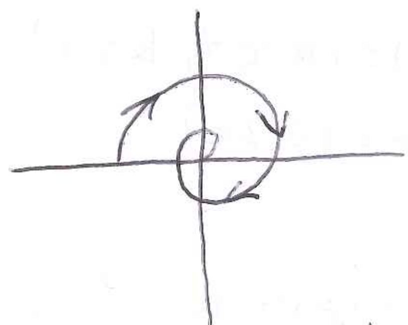
- If $\vec{c}$ is defined on the interval $[a,b]$, $\vec{c}(a)$ is called the initial point, and $\vec{c}(b)$ is called the terminal point. Together, they are the endpoints of $\vec{c}$.
- The direction corresponding to increasing t values gives the positive orientation of $\vec{c}$

Ex: $\vec{c}(t) = (t\cos t, t\sin t)$



positive orientation



negative orientation

- Some common parametrizations:
  - Lines: to parametrize the line segment from $\vec{u} = (u_1, u_2, u_3)$ to $\vec{v} = (v_1, v_2, v_3)$, we have

$$\vec{c}(t) = \vec{u} + t(\vec{v} - \vec{u}), \quad t \in [0,1]$$

$$= (u_1 + t(v_1 - u_1), u_2 + t(v_2 - u_2), u_3 + t(v_3 - u_3))$$

- **Circles:** to parametrize a circle of radius r centered at the origin, we have

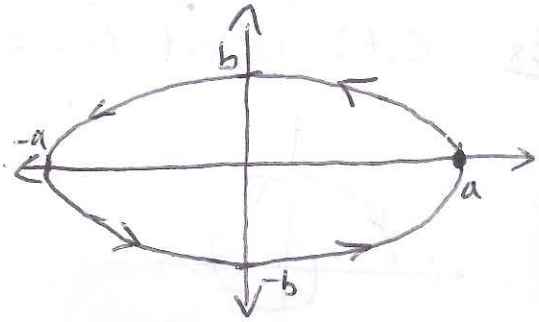
$$\vec{c}(t) = (r\cos t, r\sin t),\ t \in [0, 2\pi]$$

For a circle centered at $\vec{d} = (d_1, d_2)$, we have

$$\vec{c}(t) = (r\cos t + d_1, r\sin t + d_2),\ t \in [0, 2\pi]$$

- **Ellipses:** to parametrize the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$\vec{c}(t) = (a \cdot \cos t, b \cdot \sin t)$$
$$t \in [0, 2\pi]$$



- Parametrizations of a curve are not unique:

$$\vec{c}(t) = (2\cos t, 2\sin t),\ t \in [0, 2\pi]$$
$$\vec{c}(t) = (-2\cos t, 2\sin t),\ t \in [0, 2\pi]$$
$$\vec{c}(t) = \left(\tfrac{2}{\sqrt{17}}(4\cos t + \sin t), \tfrac{2}{\sqrt{17}}(4\sin t - \cos t)\right),\ t \in [0, 2\pi]$$

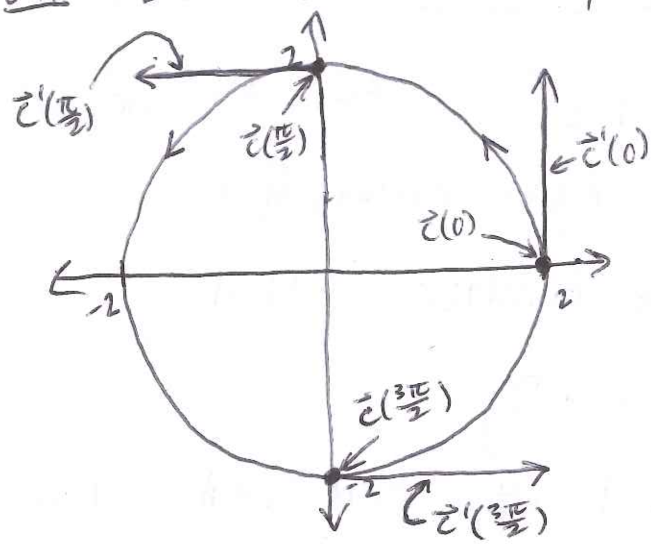
all parametrize the circle of radius 2

- Let's talk about derivatives:

$$D\vec{c}(t) = (x'(t), y'(t), z'(t)) = \vec{c}\,'(t)$$

- $\vec{c}\,'(t)$ is a tangent vector to the path at the point $\vec{c}(t)$.

Ex: Let $\vec{c}(t) = (2\cos t, 2\sin t)$, $t \in [0, 2\pi]$



$\vec{c}\,'(t) = (-2\sin t, 2\cos t)$

$\Rightarrow \vec{c}\,'(0) = (0, 2)$

$\Rightarrow \vec{c}\,'(\frac{\pi}{2}) = (-2, 0)$

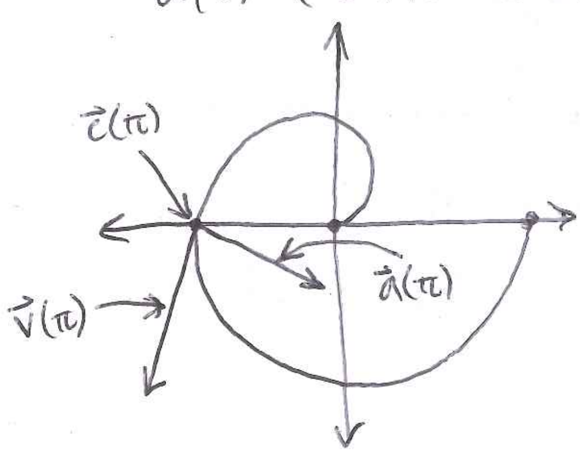
$\Rightarrow \vec{c}\,'(\frac{3\pi}{2}) = (2, 0)$

- If you view a parametrization $\vec{c}(t)$ as the position of a particle at time t, then

$$\vec{v}(t) = \vec{c}\,'(t) = \text{velocity}$$
$$\|\vec{v}(t)\| = \text{speed}$$
$$\vec{a}(t) = \vec{v}\,'(t) = \text{acceleration}$$

Ex: Let $\vec{c}(t) = (t\cos t, t\sin t)$, $t \in [0, 2\pi]$. Then

$$\vec{v}(t) = (\cos t - t\sin t, \sin t + t\cos t)$$
$$\vec{a}(t) = (-2\sin t - t\cos t, 2\cos t - t\sin t)$$



$\Rightarrow \vec{v}(\pi) = (-1, \pi)$

$\Rightarrow \vec{a}(\pi) = (\pi, -2)$

2.6 Properties of Derivatives

- For the most part, all the properties of normal derivatives carry over to higher dimensions:

linearity

a) If $f, g: \mathbb{R}^m \to \mathbb{R}^n$ are functions, then

$$D(f+g) = Df + Dg$$

b) If $f: \mathbb{R}^m \to \mathbb{R}^n$ is a function and $c \in \mathbb{R}$, then

$$D(cf) = cDf$$

product rule

c) If $f, g: \mathbb{R}^m \to \mathbb{R}$ are scalar functions, then

$$D(fg) = (Df)g + f(Dg)$$

quotient rule

d) If $f, g: \mathbb{R}^m \to \mathbb{R}$ are scalar functions, then

$$D\left(\frac{f}{g}\right) = \frac{(Df)g - f(Dg)}{g^2}$$

product rule

e) If $f, g: \mathbb{R} \to \mathbb{R}^n$ are vector-valued ~~functions,~~ curves,

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

f) If $f, g: \mathbb{R} \to \mathbb{R}^3$ are curves in space, then

$$(f \times g)' = f' \times g + f \times g'$$

- The chain rule also carries over: If $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are functions, then

$$D(g \circ f) = [(Dg) \circ f] \cdot Df$$

<u>Ex</u>: Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $\vec{c} : \mathbb{R} \to \mathbb{R}^3$ be defined by

$$f(x,y,z) = x^2 + y^2 + z^2$$
$$\vec{c}(t) = (\cos t, \sin t, t)$$

Then

$$D(f \circ \vec{c}) = [(Df) \circ \vec{c}] \cdot D\vec{c}$$

$$= \left[\frac{\partial f}{\partial x}(\vec{c}) \ \ \frac{\partial f}{\partial y}(\vec{c}) \ \ \frac{\partial f}{\partial z}(\vec{c}) \right] \cdot \vec{c}\,'$$

$$= \left[2\cos t \ \ 2\sin t \ \ 2t \right] \cdot \left[-\sin t \ \ \cos t \ \ 1 \right]$$

$$= -2\cos t \sin t + 2\cos t \sin t + 2t$$

$$= 2t$$

If you look carefully in this case, you can see that

$$D(f \circ \vec{c}) = [(Df) \circ \vec{c}] \cdot D\vec{c} = \nabla f(\vec{c}) \cdot \vec{c}\,'$$