- From our previous example:

$$\vec{a}_N = \vec{a} - \vec{a}_T$$

$$= (2,0,2) - \left(\frac{16t^2}{8t^2+1}, \frac{8t}{8t^2+1}, \frac{16t^2}{8t^2+1}\right)$$

$$= \left(\frac{2}{8t^2+1}, -\frac{8t}{8t^2+1}, \frac{2}{8t^2+1}\right) \checkmark$$

- How "curvy" is a curve? Given a path $\vec{c}$, the curvature of $\vec{c}$ at a point $\vec{c}(s)$ is

$$\boxed{K(s) = \left\| \frac{d\vec{T}(s)}{ds} \right\|}$$

  - this is for curves parametrized by arclength. In general:

$$\frac{d\vec{T}(s)}{ds} = \frac{d\vec{T}(t)}{dt} \cdot \frac{dt}{ds} = \frac{T'(t)}{\|\vec{c}'(t)\|}$$

$$\Rightarrow \boxed{K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|}}$$

Ex: Consider $\vec{c}(t) = (r\cos t, r\sin t)$, the circle of radius $r$.

$$\Rightarrow \vec{c}'(t) = (-r\sin t, r\cos t)$$

$$\Rightarrow \|\vec{c}'(t)\| = \sqrt{(-r\sin t)^2 + (r\cos t)^2} = r$$

$$\Rightarrow \vec{T}(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} = (-\sin t, \cos t)$$

$$\Rightarrow \vec{T}\,'(t) = (-\cos t, -\sin t)$$

$$\Rightarrow K(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{c}\,'(t)\|} = \frac{1}{r}$$

- this tells us that the smaller the radius is, the larger the curvature, i.e. small circles are more "curvy"

<u>Ex:</u> Compute the curvature of the graph of $y = x^2$. Where is the curvature largest?

- First, we parametrize the graph:

$$\vec{c}(t) = (t, t^2), \quad t \in (-\infty, \infty)$$

- Computing the curvature:

$$\vec{c}\,'(t) = (1, 2t) \Rightarrow \|\vec{c}\,'(t)\| = \sqrt{1+4t^2}$$

$$\Rightarrow \vec{T}(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} = \left((1+4t^2)^{-1/2}, 2t(1+4t^2)^{-1/2}\right)$$

$$\Rightarrow \vec{T}\,'(t) = \left(-\tfrac{1}{2}(1+4t^2)^{-3/2} \cdot 8t, 2(1+4t^2)^{-3/2}\right)$$

$$= \left(-\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}\right)$$

$$\Rightarrow \|\vec{T}'(t)\| = \sqrt{\left(-\frac{4t}{(1+4t^2)^{3/2}}\right)^2 + \left(\frac{2}{(1+4t^2)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{16t^2+4}{(4t^2+1)^3}}$$

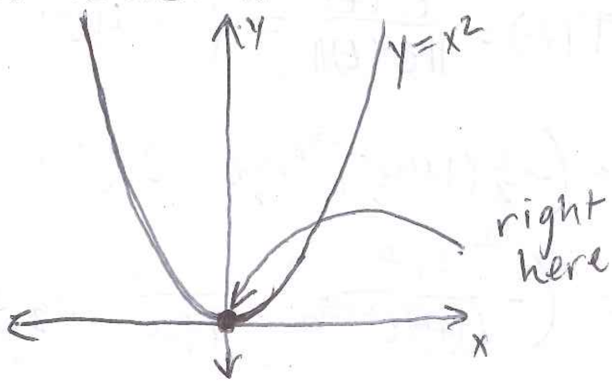
$$= \sqrt{\frac{4}{(4t^2+1)^2}} = \frac{2}{4t^2+1}$$

$$\Rightarrow K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{2/(4t^2+1)}{\sqrt{4t^2+1}}$$

$$= \frac{2}{(4t^2+1)^{3/2}}$$

- we see that $K(t)$ is largest when the denominator is smallest $\rightarrow$ $\boxed{t=0}$. In this case $\boxed{K(0)=2}$, and the corresponding pos. on the curve is $\boxed{\vec{c}(0)=(0,0)}$

- This makes sense: where is the curvature largest?



# 4.1 Higher order Partial Derivatives

- Recall some notation:

$$\frac{\partial f}{\partial x} = f_x, \quad \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y \partial z} = f_{zy}, \text{ etc}$$

Ex: Compute all second-order derivatives for $f(x,y) = x^2 e^{2y}$.

$$f_x = 2xe^{2y} \qquad\qquad f_y = 2x^2 e^{2y}$$

$$f_{xx} = 2e^{2y} \quad f_{xy} = 4xe^{2y} \quad f_{yx} = 4xe^{2y} \quad f_{yy} = 4x^2e^{2y}$$

- Note that $f_{xy} = f_{yx}$. Is this always the case?

- Theorem: Let $f$ be a real-valued function of $m$ variables $x_1, \dots, x_m$. If all second-order partial derivatives are continuous, then

$$\boxed{\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}}$$

  - This says that as long as everything is continous, the order of diffentiation doesn't matter.

- This theorem can save us a lot of work:

<u>Ex</u>: Find $f_{xzzy}$ for

$$f(x,y,z) = x^2 e^{y^2-x^2} + x^2 y z^2 - \cos(x^2+y^2)$$

- All components of this function has as many continuous derivatives as you want, so the order of partials doesn't matter
- The first and last terms don't have any $z$'s, so let's do the $\frac{\partial}{\partial z}$'s first:

$$f_z = 2x^2yz \Rightarrow f_{zz} = 2x^2y$$

$$\Rightarrow f_{zzx} = 4xy$$

$$\Rightarrow f_{zzxy} = 4x = f_{xzzy}$$

- As a check, let's try a different order:

$$f_x = 2xe^{y^2-x^2} - 2x^3e^{y^2-x^2} + 2xyz^2$$
$$+ \sin(x^2+y^2)\cdot 2x$$

$$\Rightarrow f_{xz} = 4xyz \Rightarrow f_{xzz} = 4xy$$

$$\Rightarrow f_{xzzy} = 4x \quad \checkmark$$

- <u>Change of variables:</u> Suppose we have a real-valued function $u(x,y)$, and we want to change variables to $s,t$ via some functions $x = x(s,t)$, $y = y(s,t)$.
  - how do we compute $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$?

  The chain rule:

<u>Ex</u>: Let $u(x,y) = e^x \sin y$ and suppose $x = s^2$, $y = st$. Compute $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$.

- We can view the change of variables as just a function:

$$\vec{F}(s,t) = (s^2, st)$$

$x = x(s,t) = s^2$

$y = y(s,t) = st$

- Now all we're doing is finding

$$D\left[u(\vec{F}(s,t))\right] = \left[\frac{\partial u}{\partial s} \;\; \frac{\partial u}{\partial t}\right]$$

- Using the chain rule:

$$D\left[u(\vec{F})\right] = [Du](\vec{F}) \cdot D\vec{F}$$

We know that $Du = [e^x \sin y \;\; e^x \cos y]$

and that

$$D\vec{F} = \begin{bmatrix} 2s & 0 \\ t & s \end{bmatrix}$$

Then

$$[Du](\vec{F}) \cdot D\vec{F} = \begin{bmatrix} e^{s^2}\sin(st) & e^{s^2}\cos(st) \end{bmatrix} \begin{bmatrix} 2s & 0 \\ t & s \end{bmatrix}$$

$$= \begin{bmatrix} 2se^{s^2}\sin(st) + te^{s^2}\cos(st), \\ se^{s^2}\cos(st) \end{bmatrix}$$

- This tells us that

$$\frac{\partial u}{\partial s} = 2se^{s^2}\sin(st) + te^{s^2}\cos(st)$$

$$\frac{\partial u}{\partial t} = se^{s^2}\cos(st)$$

- More generally:

$$[Du](\vec{F}) \cdot D\vec{F} = \begin{bmatrix} \frac{\partial u}{\partial x}(x(s,t), y(s,t)) & \frac{\partial u}{\partial y}(x(s,t), y(s,t)) \end{bmatrix}$$

$$\cdot \begin{bmatrix} \frac{\partial x}{\partial s}(s,t) & \frac{\partial x}{\partial t}(s,t) \\ \frac{\partial y}{\partial s}(s,t) & \frac{\partial y}{\partial t}(s,t) \end{bmatrix}$$

$$= \left[ \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}, \quad \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} \right]$$

so really, it all simplifies down to

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}, \\ \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \end{aligned}}$$

- We can verify that this is true for the particular example by changing variables <u>first</u>:

$$u(s,t) = e^{s^2} \sin(st)$$

and <u>then</u> doing the partials:

$$\frac{\partial u}{\partial s} = 2se^{s^2} \sin(st) + t e^{s^2} \cos(s,t) \checkmark$$

$$\frac{\partial u}{\partial t} = s e^{s^2} \cos(st) \checkmark$$

- Why bother with changes of variables? For partial differential equations, it often simplifies things:

Ex: Consider the partial differential equation

$$x^2 u_{xx} + y^2 u_{yy} + x u_x + y u_y = 0$$

Rewrite this equation using the change of variables $x = e^s$, $y = e^t$.

- Let's find all second-order derivs w.r.t. $s$ and $t$:

$$\frac{\partial u}{\partial s} = u_s = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= u_x \cdot e^s + u_y \cdot 0 = u_x e^s$$

$$\Rightarrow \frac{\partial^2 u}{\partial t \partial s} = u_{st} = \frac{\partial}{\partial t}\left[u_x e^s\right]$$

$$= \frac{\partial u_x}{\partial t} \cdot e^s + u_x \cancelto{0}{\frac{\partial e^s}{\partial t}}$$

$$= \left[\frac{\partial u_x}{\partial x}\cancelto{0}{\frac{\partial x}{\partial t}} + \frac{\partial u_x}{\partial y}\cancel{\frac{\partial y}{\partial t}}\right] e^s$$

$$= u_{xy} e^t e^s$$

$$= u_{xy}\, x y$$

$$\Rightarrow \frac{\partial^2 u}{\partial s^2} = u_{ss} = \frac{\partial}{\partial s}\left[u_x e^s\right]$$

$$= \frac{\partial u_x}{\partial s} e^s + u_x \frac{\partial e^s}{\partial s}$$

$$= \left[\frac{\partial u_x}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u_x}{\partial y}\cancelto{0}{\frac{\partial y}{\partial s}}\right]e^s + u_x \cdot e^s$$

$$= u_{xx}\cdot e^{2s} + u_x e^s$$

$$= \boxed{u_{xx}\cdot x^2 + u_x \cdot x}$$

Similarly, we find that

$$u_{ts} = u_{xy}\cdot xy,$$

$$u_{tt} = \boxed{u_{yy}\cdot y^2 + u_y \cdot y}$$

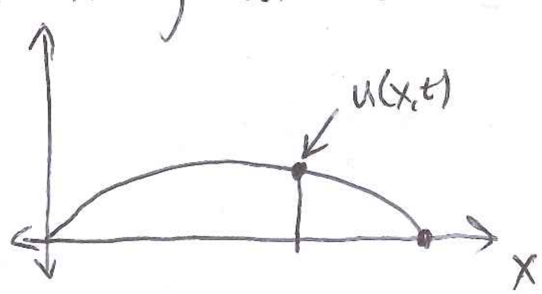
- so the original equation:

$$x^2 u_{xx} + y^2 u_{yy} + x u_x + y u_y = 0$$

simplifies to

$$u_{ss} + u_{tt} = 0$$

Ex: Consider the motion of a vibrating string. Let $u(x,t)$ be the vertical displacement of the string at location $x$ at time $t$:




snapshot of string at time $t$.