Ex: Consider the partial differential equation

$$x^2 u_{xx} + y^2 u_{yy} + x u_x + y u_y = 0$$

Rewrite this equation using the change of variables $x = e^s$, $y = e^t$.

- Let's find all second-order derivs w.r.t. s and t:

$$\frac{\partial u}{\partial s} = u_s = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= u_x \cdot e^s + u_y \cdot 0 = u_x e^s$$

$$\Rightarrow \frac{\partial^2 u}{\partial t \partial s} = u_{st} = \frac{\partial}{\partial t}\left[u_x e^s\right]$$

$$= \frac{\partial u_x}{\partial t} \cdot e^s + u_x \cancelto{0}{\frac{\partial e^s}{\partial t}}$$

$$= \left[\frac{\partial u_x}{\partial x}\cancelto{0}{\frac{\partial x}{\partial t}} + \frac{\partial u_x}{\partial y}\cancel{\frac{\partial y}{\partial t}}\right] e^s$$

$$= \cancel{\quad} = u_{xy} e^t e^s$$

$$= u_{xy}\, x y$$

$$\Rightarrow \frac{\partial^2 u}{\partial s^2} = u_{ss} = \frac{\partial}{\partial s}\left[u_x e^s\right]$$

$$= \frac{\partial u_x}{\partial s} e^s + u_x \frac{\partial e^s}{\partial s}$$

$$= \left[\frac{\partial u_x}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u_x}{\partial y}\cancelto{0}{\frac{\partial y}{\partial s}}\right]e^s + u_x \cdot e^s$$

$$= u_{xx}\cdot e^{2s} + u_x e^s$$

$$= \boxed{u_{xx}\cdot x^2 + u_x \cdot x}$$

Similarly, we find that

$$u_{ts} = u_{xy}\cdot xy,$$

$$u_{tt} = \boxed{u_{yy}\cdot y^2 + u_y \cdot y}$$

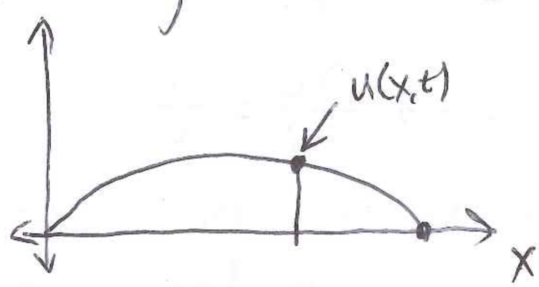
- so the original equation:

$$x^2 u_{xx} + y^2 u_{yy} + x u_x + y u_y = 0$$

simplifies to

$$u_{ss} + u_{tt} = 0$$

Ex: Consider the motion of a vibrating string. Let $u(x,t)$ be the vertical displacement of the string at location x at time t:



snapshot of string at time t.

- in an ideal setting, u ~~[illegible]~~ satisfies the wave equation:

$$u_{tt} = c^2 u_{xx}$$

- u_{tt} describes the acceleration of a particular point of the string, and u_{xx} gives the concavity at the point.
  - → the acceleration of a particular point on the string is proportional to the concavity of the string

- Let $v = x + ct$, $w = x - ct$. Show that $u_{vw} = 0$

Using the chain rule, we get that

$$\frac{\partial u}{\partial v} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial v}$$

since $v = x + ct$, $w = x - ct$ ~~[illegible]~~

$$\Rightarrow \frac{\partial v}{\partial x} = 1 \rightarrow \frac{\partial x}{\partial v} = 1$$

$$\Rightarrow \frac{\partial v}{\partial t} = c \rightarrow \frac{\partial t}{\partial v} = \frac{1}{c}$$

so:

$$\frac{\partial u}{\partial v} = u_x + u_t \cdot \frac{1}{c}$$

Then

$$\frac{\partial^2 u}{\partial w \partial v} = u_{vw} = \frac{\partial}{\partial w}\left[u_x + \frac{1}{c} u_t\right]$$

$$= \frac{\partial u_x}{\partial w} + \frac{1}{c}\frac{\partial u_t}{\partial w}$$

$$= \frac{\partial u_x}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial u_x}{\partial t}\frac{\partial t}{\partial w}$$

$$+ \frac{1}{c}\left[\frac{\partial u_t}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial u_t}{\partial t}\frac{\partial t}{\partial w}\right]$$

Since $\frac{\partial w}{\partial x} = 1 \rightarrow \frac{\partial x}{\partial w} = 1$, and $\frac{\partial w}{\partial t} = -c \rightarrow \frac{\partial t}{\partial w} = \frac{1}{-c}$, we have that

$$\frac{\partial^2 u}{\partial w \partial v} = u_{xx} + u_{xt}\left(\frac{1}{-c}\right) + \frac{1}{c}\left[u_{tx} + u_{tt}\left(\frac{1}{-c}\right)\right]$$

$$= u_{xx} - \frac{1}{c^2} u_{tt}$$

$$= -c^2\left[u_{tt} - c^2 u_{xx}\right]$$

$$= 0 \quad \checkmark$$

Alternatively, we can write u_{tt} and u_{xx} in terms of v and w:

$$u_t = u_v v_t + u_w w_t$$

$$= u_v \cdot c + u_w(-c)$$

$$\Rightarrow \quad u_{tt} = c\left[u_{vv} v_t + u_{vw} w_t\right]$$

$$- c\left[u_{wv} v_t + u_{ww} w_t\right]$$

$$= c^2 u_{vv} - c^2 u_{vw}$$

$$- c^2 u_{wv} + c^2 u_{ww}$$

$$u_x = u_v v_x + u_w w_x$$

$$= u_v + u_w$$

$$\Rightarrow \quad u_{xx} = u_{vv} v_x + u_{vw} w_x$$

$$+ u_{wv} v_x + u_{ww} w_x$$

$$= u_{vv} + u_{vw} + u_{wv} + u_{ww}$$

Substituting into $u_{tt} = c^2 u_{xx}$:

$$\cancel{c^2 u_{vv}} - 2c^2 u_{vw} \cancel{+ c^2 u_{ww}} = (\cancel{u_{vv}} + 2u_{vw} + \cancel{u_{ww}})c^2$$

$$\Rightarrow -2c^2 u_{vw} = 2c^2 u_{vw}$$

$$\Rightarrow u_{vw} = 0 \quad \checkmark$$

- There is a very special differential operator called the Laplacian:

$$\boxed{\Delta u(x,y) = u_{xx} + u_{yy}}$$

  - functions that satisfy $\Delta u(x,y) = 0$ are called harmonic.
  - The equation $\boxed{\Delta u = 0}$ is so common, it has a name: Laplace's equation.

Ex: Is the function $u(x,y) = \ln(x^2+y^2)$ harmonic?

$$u_x = \frac{1}{x^2+y^2} \cdot 2x \Rightarrow u_{xx} = \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{1}{x^2+y^2} \cdot 2y \Rightarrow u_{yy} = \frac{2(x^2+y^2) - 4y^2}{(x^2+y^2)^2}$$

$$\Rightarrow u_{xx} + u_{yy} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2} + \frac{2x^2 - 2y^2}{(x^2+y^2)^2}$$

$$= 0$$

$\Rightarrow u$ is harmonic.

4.2 Taylor's Formula

- Recall: If $f: \mathbb{R} \to \mathbb{R}$ has derivatives of all orders at x_0, then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

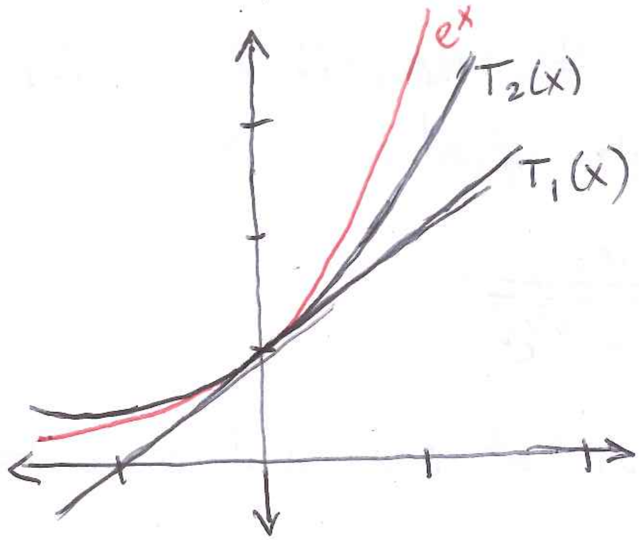
  - this is called the Taylor series for f centered at x_0

- The n^{th}-order Taylor polynomial for f at x_0:

$$\boxed{T_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}$$

Ex: Compute $T_1(x)$, $T_2(x)$, $T_3(x)$ for $f(x) = e^x$ at $x_0 = 0$.

$$\left.\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ f''(0) &= 1 \\ f'''(0) &= 1 \end{aligned}\right\} \quad \begin{aligned} T_1(x) &= 1 + 1(x-0) \\ T_2(x) &= 1 + 1(x-0) + \tfrac{1}{2}(x-0)^2 \\ T_3(x) &= 1 + 1(x-0) + \tfrac{1}{2}(x-0)^2 + \tfrac{1}{3!}(x-0)^3 \end{aligned}$$



- Taylor polynomials give good approximations to the function near x_0, and higher order Taylor polynomials are more accurate.

<u>Ex</u>: Use the 1^{st} order Taylor polynomial of $f(x) = x + e^{-2x}$ at $x_0 = 0$ to approximate $f(0.075)$.

$$\left.\begin{aligned} f(0) &= 1 \\ f'(0) &= -1 \end{aligned}\right\} \quad \begin{aligned} &T_1(x) = 1 - 1(x-0) = 1-x \\ &\Rightarrow f(0.075) \approx T_1(0.075) = 0.925 \end{aligned}$$

- The actual value is $f(0.075) = 0.93570798\ldots$ pretty close!

- Taylor polynomials are used all the time in numerical analysis, but the show up in physics too: