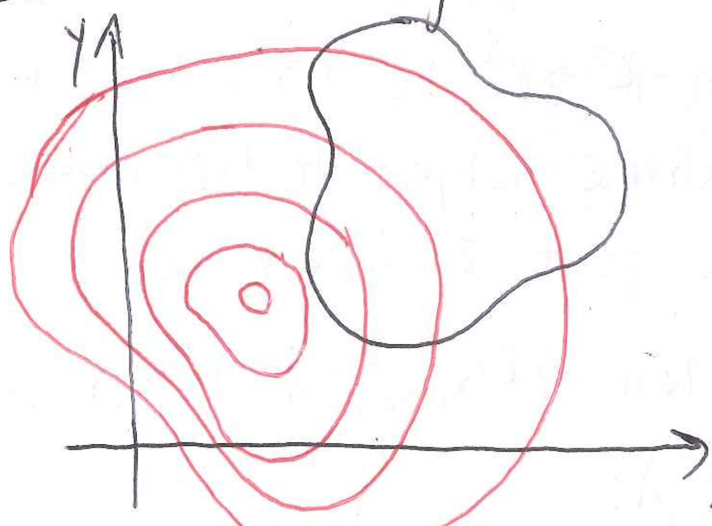
4.4 Optimization w/ Constraints and Lagrange Multipliers

- We want to optimize functions (i.e. find maxes and mins) subject to contraints.

<u>Ex:</u> Let $T(x,y) = x^2 - y + 200$ be the temperature of a metal disk $D = \{(x,y) \mid x^2+y^2 \leq 1\}$. Where is the maximum temperature along the edge of the disk?

- What this is asking: maximize T <u>subject to the constraint</u> $x^2+y^2=1$.

- Let's look at the general 2D case: Consider $f: \mathbb{R}^2 \to \mathbb{R}$, subject to some constraint $g(x,y) = k$. We can parametrize the constraint w/ a path $\vec{c}(t)$.



contour curves of f (in red)

$g(x,y) = k$ (in black)

- In order to maximize (or minimize) f along $\vec{c}$, we set $D(f(\vec{c}(t))) = 0 \Rightarrow \nabla f(\vec{c}) \cdot \vec{c}\,'(t) = 0$
- Since $\vec{c}$ parametrizes a level curve of g (i.e. $\vec{c}$ describes $g(x,y) = k$), we know that ∇g and $\vec{c}$ are perpendicular:

$$\nabla g(\vec{c}) \cdot \vec{c}\,'(t) = 0$$

- So: both ∇f and ∇g are perpendicular to the same vector. Thus:

$$\boxed{\nabla f = \lambda \nabla g}$$

for some constant λ.

- Theorem: Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be functions. If f has a local extreme subject to the constraint $g(x,y) = k$ at the point $\vec{x}_0 = (x_0, y_0)$ and if $\nabla g(x_0, y_0) \neq \vec{0}$, then $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some real number λ.

Ex: Find the maximum of $T(x,y) = x^2 - y + 200$ subject to the constraint $\underbrace{x^2+y^2}_{g(x,y)} = 1$.

$\nabla T = (2x, -1)$ $\qquad$ $\nabla g = (2x, 2y)$

$$\nabla T = \lambda \nabla g \Rightarrow (2x, -1) = (\lambda 2x, \lambda 2y)$$

$$\Rightarrow 2x = \lambda 2x, \quad -1 = \lambda 2y$$

$$\rightarrow 2x(1-\lambda) = 0 \Rightarrow x = 0 \text{ or } \lambda = 1$$

If $x = 0 \rightarrow (0)^2 + y^2 = 1 \rightarrow y = \pm 1 \Rightarrow \boxed{(0, \pm 1)}$

If $\lambda = 1 \rightarrow -1 = (1)2y \rightarrow y = \frac{1}{2} \rightarrow x^2 + (\frac{1}{2})^2 = 1$

$\rightarrow x = \pm \frac{\sqrt{3}}{2} \Rightarrow \boxed{(\pm\frac{\sqrt{3}}{2}, \frac{1}{2})}$

$T(0,1) = 199$

$T(0,-1) = 201$

$T(\frac{\sqrt{3}}{2}, \frac{1}{2}) = 201.25$

$T(-\frac{\sqrt{3}}{2}, \frac{1}{2}) = 201.25$

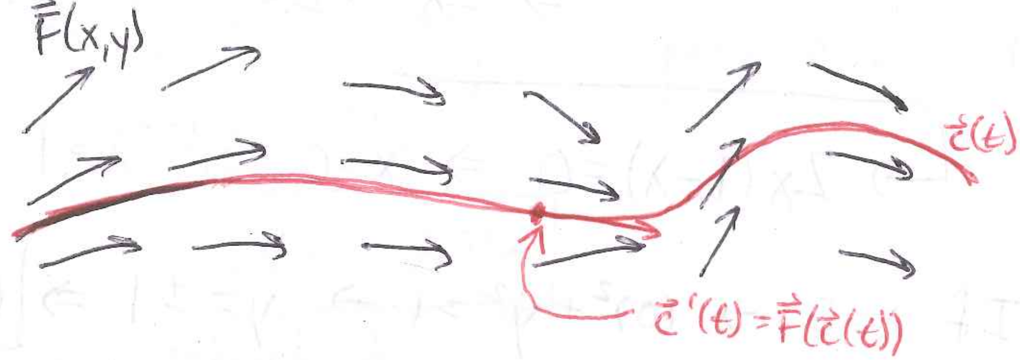
T reaches a max of 201.25 at the points $(\pm\frac{\sqrt{3}}{2}, \frac{1}{2})$ subject to $x^2+y^2=1$.

4.5 Flow lines

- Suppose $\vec{F}(x,y)$ is a vector field. A flow line of $\vec{F}$ is a path $\vec{c}(t)$ such that

$$\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$$

  - Intuitively, a flow line is a path whose derivative coincides exactly with the vector field



<u>Ex</u>: Consider the vector field $\vec{F}(x,y) = (-y, x)$. Let's check that $\vec{c}_1(t) = (\cos t, \sin t)$ and $\vec{c}_2(t) = (3\cos t + \sin t, 3\sin t - \cos t)$ are both flow lines.

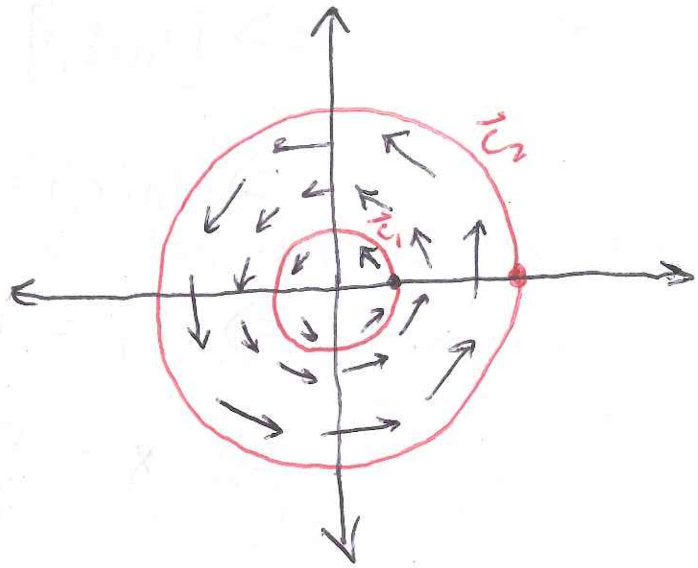
$$\vec{c}_1\,'(t) = (-\sin t, \cos t)$$

$$\vec{F}(\vec{c}_1(t)) = \vec{F}(\cos t, \sin t) = (-\sin t, \cos t) \checkmark$$

$$\vec{c}_2\,'(t) = (-3\sin t + \cos t, 3\cos t + \sin t)$$

$$\vec{F}(\vec{c}_2(t)) = \vec{F}(3\cos t + \sin t, 3\sin t - \cos t) = (-3\sin t + \cos t, 3\cos t + \sin t) \checkmark$$

The picture:



<u>Ex</u>: Consider an electrostatic field:

$$\vec{F}(x,y) = \frac{Qq}{4\pi\left(\sqrt{x^2+y^2}\right)^3}(x,y)$$

$$= \frac{Qq}{4\pi}\frac{\vec{x}}{\|\vec{x}\|^3} \qquad K = \frac{Qq}{4\pi}$$

- To find the flow lines, we need to solve $\vec{c}\,' = \vec{F}(\vec{c})$:

$$c_1'(t) = K\frac{c_1(t)}{\|\vec{c}(t)\|^3}$$

$$c_2'(t) = K\frac{c_2(t)}{\|\vec{c}(t)\|^3}$$

- Dividing the first by the second:

$$\frac{c_1'}{c_2'} = \frac{c_1}{c_2} \Rightarrow \frac{c_1'}{c_1} = \frac{c_2'}{c_2}$$

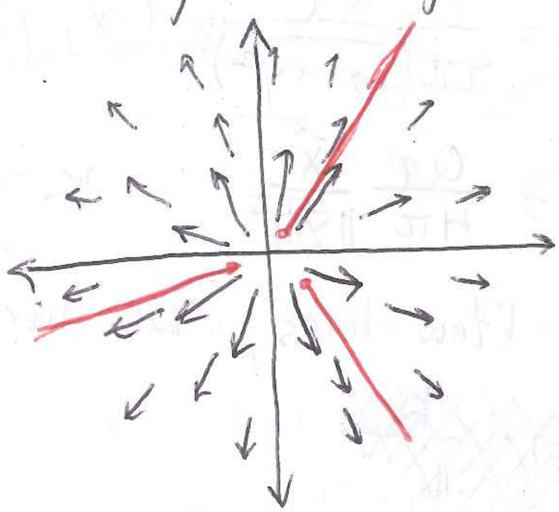
$$\Rightarrow [\ln(c_1)]' = [\ln(c_2)]'$$

$$\Rightarrow \ln(c_1) = \ln(c_2) + C$$

$$\Rightarrow c_1 = e^{C} c_2$$

$$\Rightarrow x = \underbrace{D}_{\text{some constant}} y$$

- this says that flow lines are straight lines which go through the origin:



Flow lines in red
electrostatic field in black.

4.6 Divergence and Curl of a Vector Field

- First, some definitions:
  - Let $\vec{F}$ be a vector field. The <u>divergence</u> of $\vec{F}$ is the real-valued function

$$\boxed{\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

  - Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. The <u>curl</u> of $\vec{F}$ is the vector field

$$\boxed{\operatorname{curl} \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)}$$

  - Let $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field. The <u>scalar curl</u> of $\vec{F}$ is

$$\boxed{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}$$

- Usually, divergence and curl are denoted

$$\boxed{\begin{aligned} \operatorname{div} \vec{F} &= \nabla \cdot \vec{F} \\ \operatorname{curl} \vec{F} &= \nabla \times \vec{F} \end{aligned}}$$

- Why are they denoted like that? If we define ∇ to be the "vector"

$$\nabla = \left[\frac{\partial}{\partial x} \ \frac{\partial}{\partial y} \ \frac{\partial}{\partial z}\right],$$

then

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3,$$

"dot product"

$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial}{\partial z} F_2, \frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3, \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1\right)$$

"cross product"

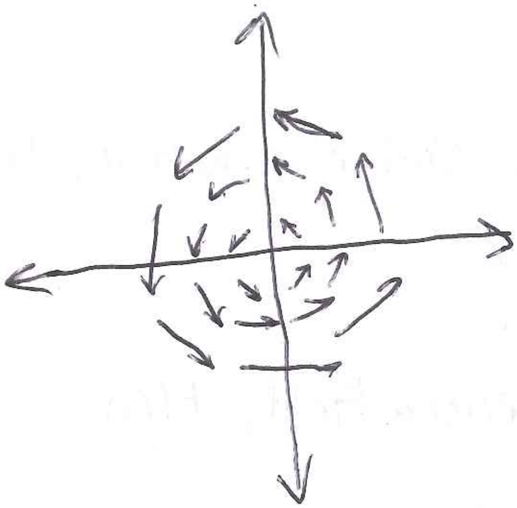
Ex: Compute the divergence and scalar curls of $\vec{F}_1(x,y) = (-y, x)$. and $\vec{F}_2(x,y) = (x,y)$

$$\nabla \cdot \vec{F}_1 = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0$$

$$\nabla \times \vec{F}_1 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2$$

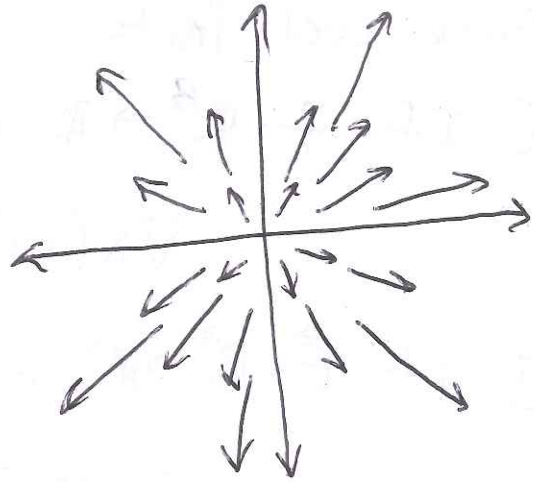
$$\nabla \cdot \vec{F}_2 = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2$$

$$\nabla \times \vec{F}_2 = \frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial y} = 0$$



$\nabla \cdot \vec{F}_1 = 0$

$\nabla \times \vec{F}_1 \neq 0$



$\nabla \cdot \vec{F}_2 \neq 0$

$\nabla \times \vec{F}_2 = 0$

- Interesting... It turns out that if you interpret $\vec{F}$ as the <u>velocity</u> of a fluid, $\nabla \cdot \vec{F}$ measures the <u>rate of expansion</u> of $\vec{F}$. Likewise, $\nabla \times \vec{F}$ measures the tendency of the fluid to <u>rotate</u> around some axis.
  - A vector field with $\nabla \cdot \vec{F} = 0$ is called <u>incompressible</u>.
  - A vector field with $\nabla \times \vec{F} = \vec{0}$ is called <u>irrotational</u>.