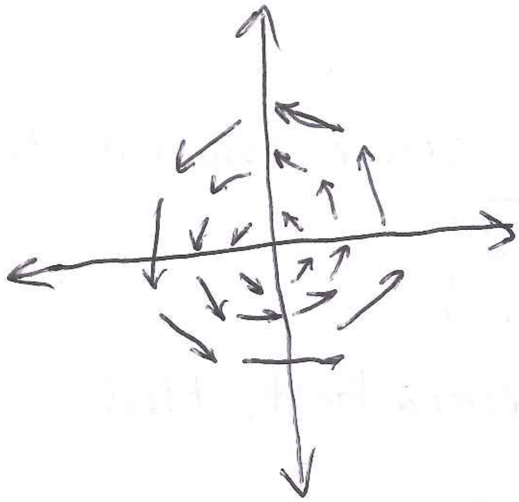
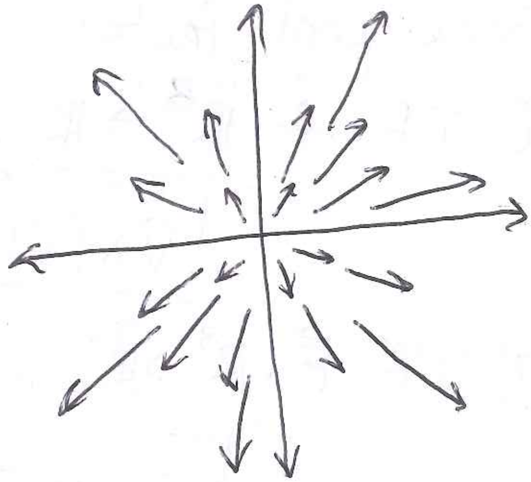


$\nabla \cdot \vec{F}_1 = 0$

$\nabla \times \vec{F}_1 \neq 0$



$\nabla \cdot \vec{F}_2 \neq 0$

$\nabla \times \vec{F}_2 = 0$

- Interesting... It turns out that if you interpret $\vec{F}$ as the velocity of a fluid, $\nabla \cdot \vec{F}$ measures the rate of expansion of $\vec{F}$. Likewise, $\nabla \times \vec{F}$ measures the tendency of the fluid to rotate around some axis.
  - A vector field with $\nabla \cdot \vec{F} = 0$ is called incompressible.
  - A vector field with $\nabla \times \vec{F} = \vec{0}$ is called irrotational.

- Some cool facts:

① If $f: \mathbb{R}^3 \to \mathbb{R}$ is a scalar function, then

$$\boxed{\nabla \times (\nabla f) = \vec{0}}$$

② If $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, then

$$\boxed{\nabla \cdot (\nabla \times \vec{F}) = 0}$$

Ex: Calculate $\nabla \times (\nabla f)$ for $f(x,y,z) = x^2+y^2+z^2$.

$$\nabla f(x,y,z) = (2x, 2y, 2z)$$

$$\Rightarrow \nabla \times \nabla f = \left(\frac{\partial}{\partial y}2z - \frac{\partial}{\partial z}2y, \frac{\partial}{\partial z}2x - \frac{\partial}{\partial x}2z, \frac{\partial}{\partial x}2y - \frac{\partial}{\partial y}2x\right)$$

$$= (0,0,0) \checkmark$$

Ex: Calculate $\nabla \cdot (\nabla \times \vec{F})$ for $\vec{F}(x,y,z)$

$= (2xy, 2yz, 2xy)$

$$\nabla \times \vec{F} = (2x - 2y, 0 - 2y, 0 - 2x)$$

$$= (2x-2y, -2y, -2x)$$

$$\Rightarrow \nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot (2x-2y, -2y, -2x)$$
$$= 2 - 2 + 0 = 0$$

- <u>Recall</u>: A vector field $\vec{F}$ is conservative if there is some scalar function V such that $\vec{F} = -\nabla V$.
  - The previous facts (① in particular), let's us check if a field is conservatie: if $\vec{F}$ is conservative, then

$$\nabla \times \vec{F} = \nabla \times (-\nabla V)$$
$$= -\nabla \times (\nabla V)$$
$$= \vec{0}$$

  - So, if $\nabla \times \vec{F} \neq \vec{0}$, then $\vec{F}$ is <u>not</u> conservative.
- We can also express the Laplacian operator using $\nabla \cdot$: for a scalar function f,

$$\Delta f = \nabla \cdot (\nabla f)$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \cdot (\nabla f) = \nabla \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \checkmark$$

- These differential operators are fundamental to physics, chemistry, biology, etc. One important concept: Fick's Law
  - Fick's Law says that if f is a function that describes the concentration of something, then the flux (call it $\vec{F}$) of concentration (i.e. the direction the "something" moves) is proportional to the gradient of f:

$$\vec{F} = -k \nabla f$$

  - Conservation of mass says that the change in mass at some location is equal to the negative divergence of the flux:

$$\frac{\partial f}{\partial t} = -\nabla \cdot \vec{F}$$

- Putting these together gives the <u>heat equation</u>:

$$\frac{\partial f}{\partial t} = -\nabla \cdot \vec{F}$$

$$= -\nabla \cdot (-k \nabla f)$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial t} = k \Delta f}$$

- f can represent many things: temperature, concentration of bacteria in air, concentration of air molecules in a box, etc.

- The heat equation describes how all these systems evolve in time.