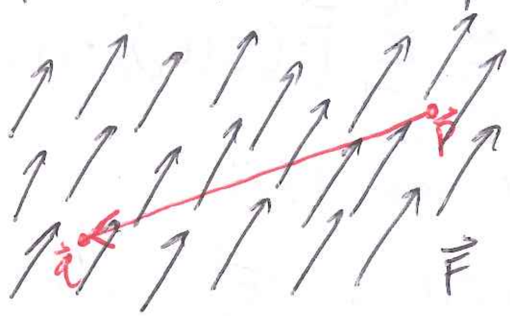
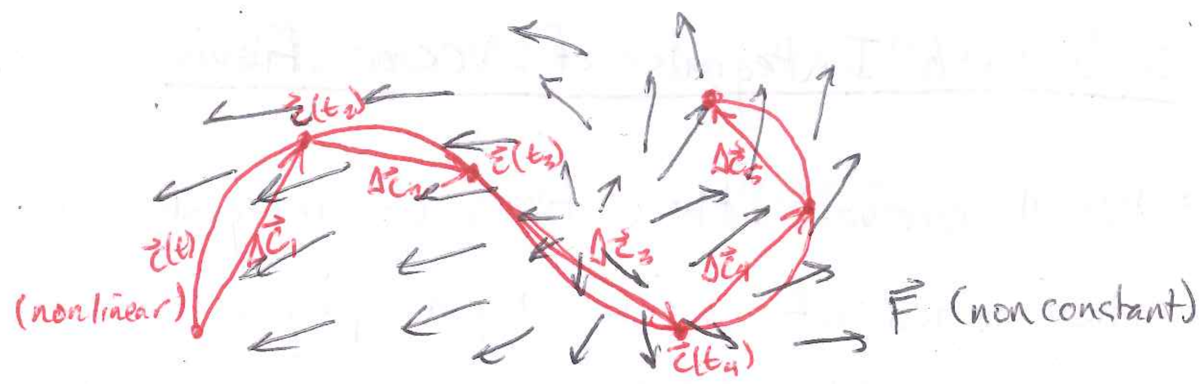
5.3 Path Integrals of Vector Fields

- We'll motivate these types of integrals using the notion of work from physics
- Suppose we have a constant vector field $\vec{F}$, and a particle in this field moves from a point $\vec{p}$ to a point $\vec{q}$:



  - the work that the field $\vec{F}$ did on the particle is given by $\boxed{W = \vec{F} \cdot \vec{d}}$, where $\vec{d}$ is the displacement vector $\vec{q} - \vec{p}$.

- What we'd like to compute is the total work done by a nonconstant field $\vec{F}$, along a nonlinear particle path.
  - like before, we approximate the path with line segments:



- If we take the vector field to be approximately constant along each segment, then the work done is about

$$W \approx \sum_{i=1}^{n} \vec{F}(\vec{c}(t_i)) \cdot \Delta \vec{c}_i$$

- Last time, we said that

$$\Delta c_i = \vec{c}(t_{i+1}) - \vec{c}(t_i) \approx \vec{c}\,'(t_i)\,\Delta t_i$$

so that gives us

$$W \approx \sum_{i=1}^{n} \vec{F}(\vec{c}(t_i)) \cdot \vec{c}\,'(t_i)\,\Delta t_i$$

- This is a Riemann sum, so taking the limit as $n \to \infty$ gives the actual integral: the path integral (or line integral) of $\vec{F}$ along $\vec{c}$ is

$$\boxed{\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt}$$

Ex: Compute the line integral $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$, where $\vec{F}(x,y) = (-e^{x+y}, 3x)$ and $\vec{c}(t) = (t^2, 3-2t^2), t \in [-1,1]$

① $\vec{F}(\vec{c}(t)) = \vec{F}(t^2, 3-2t^2)$

$$= (-e^{t^2+3-2t^2}, 3t^2)$$

$$= (-e^{3-t^2}, 3t^2)$$

② $\vec{c}\,'(t) = (2t, -4t)$

③ $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{-1}^{1} (-e^{3-t^2}, 3t^2) \cdot (2t, -4t)\, dt$

$$= \int_{-1}^{1} -2te^{3-t^2} - 12t^3\, dt$$

$$= (e^{3-t^2} - 3t^4)\Big|_{-1}^{1} = \boxed{0}$$

- We're integrating a dot product: $\vec{F}(\vec{c}) \cdot \vec{c}\,'$, which says something about the <u>angle</u> between the two vectors:
  - If $\vec{F}(\vec{c}) \cdot \vec{c}\,' > 0$, then $\vec{F}(\vec{c})$ and $\vec{c}\,'$ are going roughly the same direction ($\vec{c}$ is "moving with" $\vec{F}$)

- If $\vec{F}(\vec{c})\cdot\vec{c}\,'<0$, then $\vec{F}(\vec{c})$ and $\vec{c}\,'$ are pointing roughly away from each other ($\vec{c}$ is "moving against" $\vec{F}$)

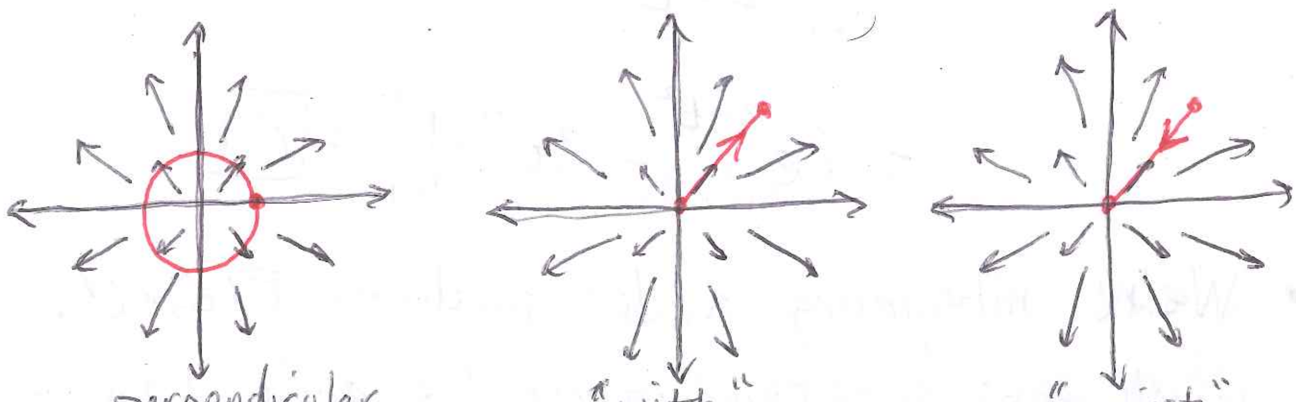
- If $\vec{F}(\vec{c})\cdot\vec{c}\,'=0$, then $\vec{F}(\vec{c})$ and $\vec{c}\,'$ are perpendicular

<u>Ex</u>: Let $\vec{F}(x,y)=(x,y)$ and consider 3 paths:

$c_1(t)=(\cos t, \sin t)$, $t\in[0,2\pi]$

$c_2(t)=(t,t)$, $t\in[0,1]$

$c_3(t)=(1-t,1-t)$, $t\in[0,1]$



perpendicular   "with"   "against"

- For $\vec{c}_1$: $\vec{c}_1\,'(t)=(-\sin t, \cos t)$, $\vec{F}(\vec{c}_1(t))=(\cos t, \sin t)$

$$\Rightarrow \int_{\vec{c}_1}\vec{F}\cdot d\vec{s}=\int_0^{2\pi}(\cos t,\sin t)\cdot(-\sin t,\cos t)\,dt$$

$$=\int_0^{2\pi}0\,dt=0$$

- For $\vec{c}_2$: $\vec{c}_2\,'(t) = (1,1)$, $\vec{F}(\vec{c}(t)) = (t,t)$

$$\Rightarrow \int_{\vec{c}_2} \vec{F} \cdot d\vec{s} = \int_0^1 (t,t)\cdot(1,1)\,dt$$

$$= \int_0^1 2t\,dt = 1 \underline{\underline{\geq 0}}$$

- For $\vec{c}_3$: $\vec{c}_3\,'(t) = (-1,-1)$, $\vec{F}(\vec{c}(t)) = (1-t, 1-t)$

$$\Rightarrow \int_{\vec{c}_3} \vec{F} \cdot d\vec{s} = \int_0^1 (1-t, 1-t)\cdot(-1,-1)\,dt$$

$$= \int_0^1 2(t-1)\,dt = -1 \underline{\underline{\leq 0}}$$

- This example brings up another issue: note that $\vec{c}_2$ and $\vec{c}_3$ parametrize the <u>same</u> curve, but the path integral is <u>different</u>.
  - not all is lost: the only thing that possibly changes between different parametrizations is a $-$ sign
  - let $\vec{c}$ be a parametrization of some curve, and let $\vec{\gamma} = \vec{c}(\varphi)$ be a reparam. of the curve. If $\vec{\gamma}$ traverses the curve in the <u>same direction as $\vec{c}$</u>, $\vec{\gamma}$ is called <u>orientation-preserving</u>

- If $\vec{\gamma}$ traverses the curve in the opposite direction as $\vec{c}$, then $\vec{\gamma}$ is called orientation-reversing.

- Path integrals of vector fields are independent of parametrization, up to a $-$ sign: If $\vec{c}$ and $\vec{\gamma}$ parametrize the same curve, then

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \begin{cases} \int_{\vec{\gamma}} \vec{F} \cdot d\vec{s} & \text{if } \vec{\gamma} \text{ is orientation-preserving} \\ -\int_{\vec{\gamma}} \vec{F} \cdot d\vec{s} & \text{if } \vec{\gamma} \text{ is orientation-reversing} \end{cases}$$

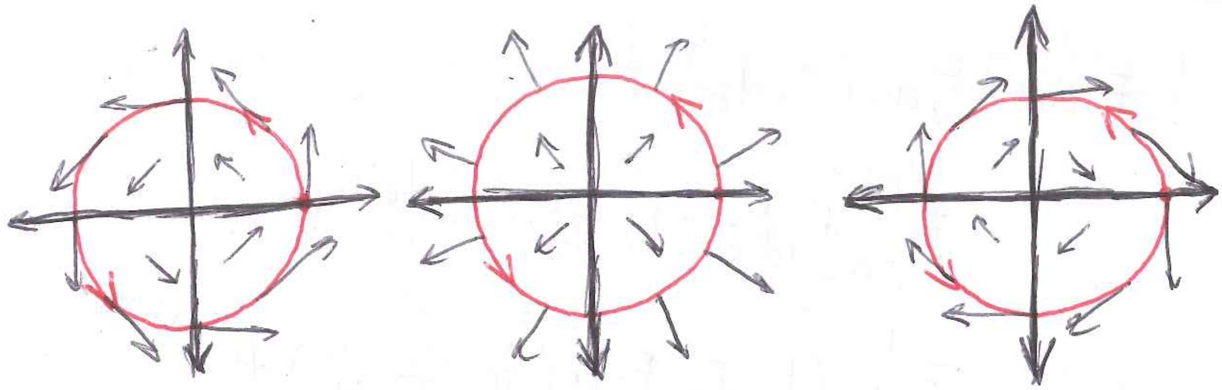
- If $\vec{c}$ is a simple closed curve, then $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ is called the circulation of $\vec{F}$ around $\vec{c}$.

Ex: Let $\vec{c}$ be the counterclockwise unit circle

$$\vec{c}(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]$$

and consider 3 vector fields:

$$\vec{F}_1(x,y) = (-y, x)$$
$$\vec{F}_2(x,y) = (x, y)$$
$$\vec{F}_3(x,y) = (y, -x)$$



$\vec{F}_1$ $\vec{F}_2$ $\vec{F}_3$

$$\int_{\vec{c}} \vec{F}_1 \cdot d\vec{s} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} dt = 2\pi$$

$$\int_{\vec{c}} \vec{F}_2 \cdot d\vec{s} = \int_0^{2\pi} (\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} 0\, dt = 0$$

$$\int_{\vec{c}} \vec{F}_3 \cdot d\vec{s} = \int_0^{2\pi} (\sin t, -\cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} -1\, dt = -2\pi$$

- This is why it's called circulation: viewing $\vec{F}_1$ as the velocity field of a fluid, $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ measures the tendency of the fluid particles to "circulate" along $\vec{c}$.

• Sometimes, we use different notation for line integrals: instead of $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$, we write

$$\int_{\vec{c}} F_1\, dx + F_2\, dy + F_3\, dz$$