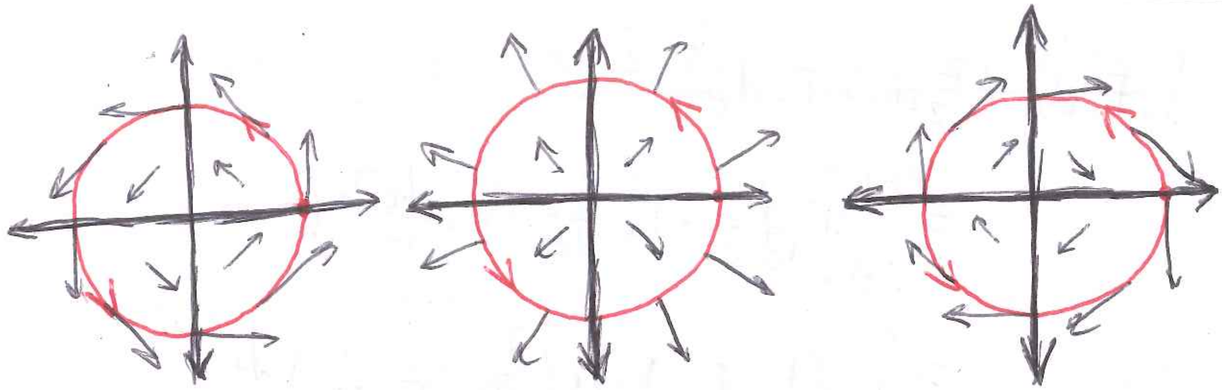


$\vec{F}_1$ $\qquad\qquad$ $\vec{F}_2$ $\qquad\qquad$ $\vec{F}_3$

$$\int_{\vec{c}} \vec{F}_1 \cdot d\vec{s} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} dt = 2\pi$$

$$\int_{\vec{c}} \vec{F}_2 \cdot d\vec{s} = \int_0^{2\pi} (\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} 0\, dt = 0$$

$$\int_{\vec{c}} \vec{F}_3 \cdot d\vec{s} = \int_0^{2\pi} (\sin t, -\cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} -1\, dt = -2\pi$$

- This is why it's called circulation: viewing $\vec{F}_i$ as the velocity field of a fluid, $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ measures the tendency of the fluid particles to "circulate" along $\vec{c}$.

- Sometimes, we use different notation for line integrals: instead of $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$, we write

$$\int_{\vec{c}} F_1\, dx + F_2\, dy + F_3\, dz$$

$$\int_{\vec{c}} F_1\,dx + F_2\,dy + F_3\,dz$$

$$= \int_a^b \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right] dt$$

$$= \int_a^b (F_1, F_2, F_3) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt = \int_{\vec{c}} \vec{F} \cdot d\vec{s}$$

Ex: Compute $\int_{\vec{c}} x^2\,dx + y\,dy + 2yz\,dz$ along the path $\vec{c}(t) = (1, t, -t^2)$, $t \in [0,1]$

$$\int_{\vec{c}} x^2\,dx + y\,dy + 2yz\,dz$$

$$= \int_0^1 \left[x^2 \frac{dx}{dt} + y \frac{dy}{dt} + 2yz \frac{dz}{dt}\right] dt$$

$$= \int_0^1 \left[(1)^2 \cdot 0 + (t) \cdot (1) + 2(t)(-t^2) \cdot (-2t)\right] dt$$

$$= \int_0^1 t + 4t^4\,dt = \boxed{\frac{13}{10}}$$

5.4 Path Integrals Independent of Path

- A function $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ (or $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$) is called a gradient vector field if there is some function $f: \mathbb{R}^2 \to \mathbb{R}$ (or $f: \mathbb{R}^3 \to \mathbb{R}$) such that $\vec{F} = \nabla f$

  - Almost the same as a conservative vector field, but without the $-$ sign: $\vec{F} = -\nabla V$, where V is the potential function.

Ex: Imagine an object of mass M at the origin $(0,0,0)$. By Newton's law of gravitation, the force exerted on an object of mass m at the position $\vec{r} = (x, y, z)$ is

$$\vec{F}(\vec{r}) = -\frac{GMm\,\vec{r}}{\|\vec{r}\|^3}$$

$\vec{F}$ is a gradient vector field, since

$$\vec{F}(\vec{r}) = \nabla\left[GMm\frac{1}{\|\vec{r}\|}\right]$$

-Check:

$$\nabla\left[\frac{GMm}{\|\vec{r}\|}\right] = \nabla\left[\frac{GMm}{\sqrt{x^2+y^2+z^2}}\right]$$

$$\cancel{= \nabla\left(GMm \frac{1}{\sqrt{x^2+y^2+z^2}}\right)}$$

$$= \nabla\left[GMm\,(x^2+y^2+z^2)^{-1/2}\right]$$

$$= GMm\Big(-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x,$$
$$-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y,$$
$$-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2z\Big)$$

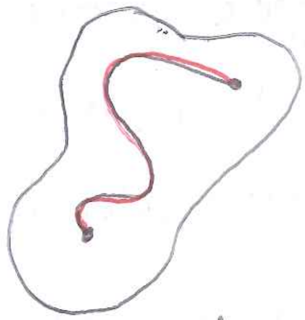
$$\cancel{= GMm}$$

$$= \frac{GMm}{(x^2+y^2+z^2)^{3/2}}\,(-x,-y,-z)$$

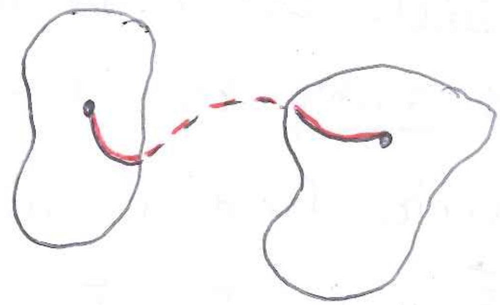
$$= -GMm\,\frac{1}{\|\vec{r}\|^3}\,\vec{r} = \vec{F}(\vec{r}) \checkmark$$

- Gradient vector fields are <u>super</u> nice as we'll see, so we want to know how to determine if a given vector field is a gradient vec. field.
- A set $U \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ is called <u>connected</u> if any two points in U can be connected by a curve that is completely contained in U

Ex:



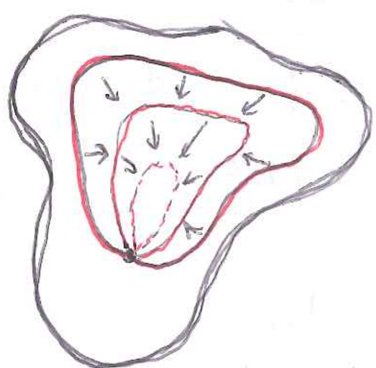
connected



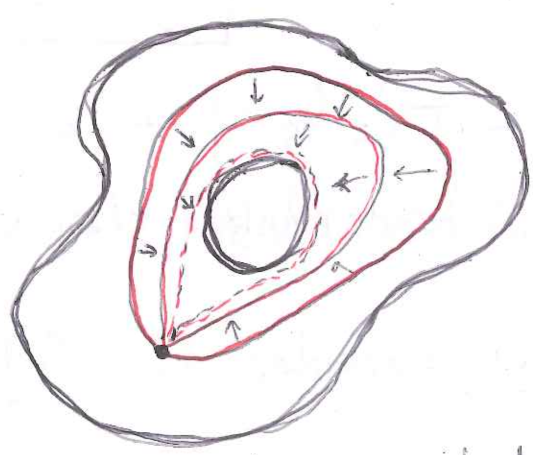
not connected

- A set $\mathcal{U}$ is called simply-connected if it is connected and if every simple closed curve can be shrunk to a point without leaving $\mathcal{U}$

Ex



simply-connected



not simply-connected

– intuitively, simply-connected (in $\mathbb{R}^2$) means there are no "holes"

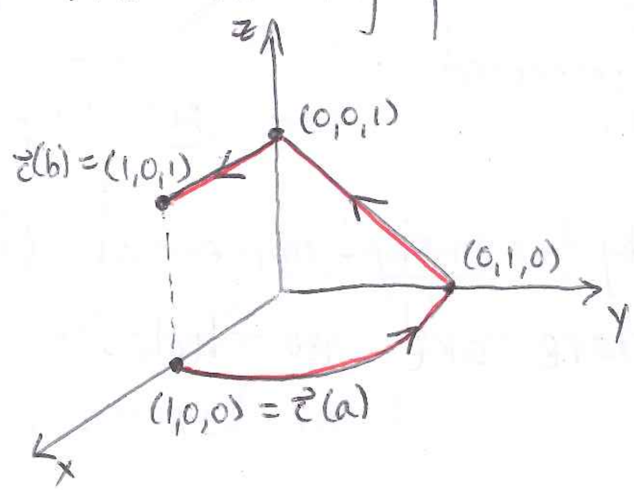
Theorem: Let $\vec{F}$ be a vector field defined on a simply-connected set $U \subseteq \mathbb{R}^2$. $\vec{F}$ is a gradient vector field if and only if the scalar curl of $\vec{F}$ is 0. If $\vec{F}$ is defined on a simply-connected set $U \subseteq \mathbb{R}^3$, then $\vec{F}$ is a gradient vector field if and only if $\nabla \times \vec{F} = \vec{0}$.

- Why are gradient vector fields super nice? Because of this fact:

$$\boxed{\int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))}$$

  - This is just a generalization of the fundamental theorem of calculus.

Ex: Consider the following path:



Compute $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$, for $\vec{F}(x,y,z) = (y+z, x, x)$.

- Before, we would have to parametrize each segment of the curve, and then calculate $\underline{\underline{3}}$ integrals – ugh.

- Is $\vec{F}$ a gradient vector field?

  ① $\vec{F}$ is defined for all $(x,y,z) \in \mathbb{R}^3$, so it is defined on a simply-connected set

  ② Is $\nabla \times \vec{F} = \vec{0}$?

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial y} x - \frac{\partial}{\partial z} x, \frac{\partial}{\partial z}(y+z) - \frac{\partial}{\partial x} x, \right.$$
$$\left. \frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(y+z) \right)$$

$$= (0-0, 1-1, 1-1) = (0,0,0)$$

So $\vec{F}$ $\underline{\text{is}}$ a gradient vector field. Let's find f such that $\vec{F} = \nabla f$.

$$\Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (y+z, x, x)$$

$$\Rightarrow \frac{\partial f}{\partial x} \overset{①}{=} y+z, \quad \frac{\partial f}{\partial y} \overset{②}{=} x, \quad \frac{\partial f}{\partial z} \overset{③}{=} x$$

$$① \Rightarrow f = xy + xz + C(y,z)$$

$$\Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial C}{\partial y}(y,z) \overset{②}{=} x \longrightarrow \frac{\partial C}{\partial y}(y,z) = 0$$

$$\rightarrow C(y,z) = C(z)$$

Then $f = xy + xz + C(z)$

$$\Rightarrow \frac{\partial f}{\partial z} = x + \frac{\partial C}{\partial z}(z) \overset{(3)}{=} x \rightarrow \frac{\partial C}{\partial z}(z) = 0$$

$$\rightarrow C(z) = C$$

Then $f(x,y,z) = xy + xz + C$, for some constant C. Now the magic happens:

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} \nabla f \cdot d\vec{s}$$

$$= f(\vec{c}(b)) - f(\vec{c}(a))$$

$$= f(1,0,1) - f(1,0,0)$$

$$= (1)(0) - (1)(1) - [(1)(0) - (1)(0)] = 1$$

- We would have gotten the same answer for <u>any</u> path that started at $(1,0,0)$ and ended at $(1,0,1)$

☆☆ Path integrals of gradient vector fields depend only on the endpoints of the path, not the path itself! ☆☆

Ex: Compute the work of the gravitational force $\vec{F}(\vec{r}) = GMm\frac{\vec{r}}{\|\vec{r}\|^3}$ acting on an object moving from $(2,2,1)$ to $(3,-1,2)$.

- We already showed that

$$\vec{F}(\vec{r}) = \nabla\left[GMm\frac{1}{\|\vec{r}\|}\right].$$

Then

$$\int_{(2,2,1)}^{(3,-1,2)} \vec{F}\cdot d\vec{s} = \int_{(2,2,1)}^{(3,-1,2)} \nabla\left[GMm\frac{1}{\|\vec{r}\|}\right]\cdot d\vec{s}$$

$$= GMm\left[\frac{1}{\|(3,-1,2)\|} - \frac{1}{\|(2,2,1)\|}\right]$$

$$= GMm\left[\frac{1}{\sqrt{14}} - \frac{1}{3}\right]$$

- If $\vec{c}$ is a simple <u>closed</u> curve, then $\vec{c}(a) = \vec{c}(b)$ and so

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_{\vec{c}} \nabla f\cdot d\vec{s}$$

$$= f(\vec{c}(b)) - f(\vec{c}(a)) = 0$$

- Probably the trippiest fact yet though: given a vector field $\vec{F}$ defined on a connected set U, if $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ is independent of the path $\vec{c}$, then $\vec{F}$ is a gradient vector field.

- Let's summarize: Given a vector field $\vec{F}$ defined on a connected set $U \subseteq \mathbb{R}^2$ (or $\mathbb{R}^3$)

  (a) $\vec{F}$ is a gradient vector field

  <u>is equivalent to</u>

  (b) $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = 0$ for all simple closed curves

  <u>is equivalent to</u>

  (c) $\int_{\vec{c}_1} \vec{F} \cdot d\vec{s} = \int_{\vec{c}_2} \vec{F} \cdot d\vec{s}$ for any 2 simple curves $\vec{c}_1$ and $\vec{c}_2$ having the same endpoints and orientation

  - (a), (b), and (c) all imply that $\boxed{\overset{(d)}{\nabla \times \vec{F} = \vec{0}}}$
  - If U is <u>simply-connected</u>, then (d) also implies (a), (b), and (c)