6.4 Change of Variables in Double Integrals

- Let's look at how we do change of variables (i.e. u-substitution) in 1-D first:

<u>Ex</u>: $\int_1^2 e^{5x}\,dx$ $\quad u = 5x \rightarrow x = \frac{1}{5}u$

$\quad du = 5dx \rightarrow dx = \frac{1}{5}du$

$$= \int_5^{10} e^{u}\,\frac{1}{5}\,du$$

- viewing $x = \frac{1}{5}u = x(u)$ as a function of u, we see:

$$\int_1^2 e^{5x}\,dx = \int_{x^{-1}(1)}^{x^{-1}(2)} e^{5x(u)}\,x'(u)\,du$$

- More generally,

$$\int_I f(x)\,dx = \int_{I^*} f(x(u)) \left|\frac{\partial x}{\partial u}\right| du$$

- Here, I is the interval of integration (i.e. $I = [a,b]$).
- $x(u)$ is a transformation which maps I^* to I:

$x : I^* \rightarrow I$

- The question: how do we extend this notion to functions of more than 1 variable?
- First, lets look at the transformations:
  - In 1-D, we had a transformation from one <u>interval</u> I^* to another <u>interval</u> I
  - In 2-D, we need a transformation from one <u>region</u> D^* to another <u>region</u> D
  - So, our transformation will be of the form:

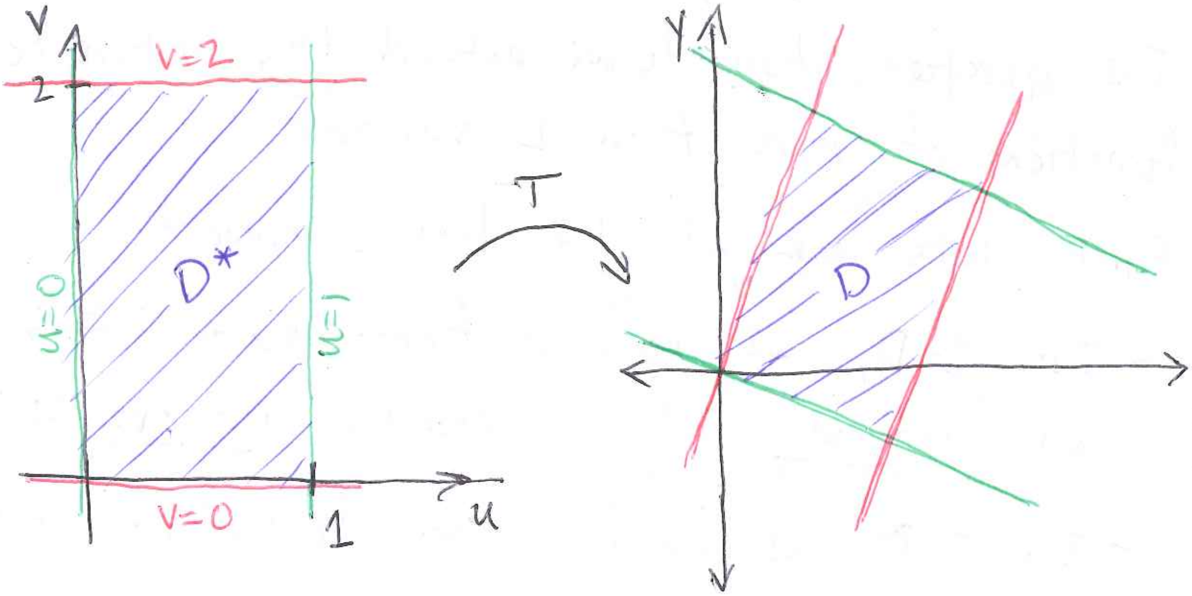
$$T(u,v) = (x(u,v),\ y(u,v))$$

- Let's look at some examples:

<u>Ex</u>: Consider the map $T(u,v) = (u+2v, 3u-v)$, with domain $D^* = [0,1] \times [0,2]$. What is the image (call it D) of T?

- The domain is bounded by lines:

$$u=0 \longrightarrow T(0,v) = (2v, -v) \longrightarrow y = -\frac{x}{2}$$
$$u=1 \longrightarrow T(1,v) = (2v+1, -v+3) \rightarrow y = -\frac{x}{2} + \frac{7}{2}$$
$$v=0 \longrightarrow T(u,0) = (u, 3u) \longrightarrow y = 3x$$
$$v=2 \longrightarrow T(u,2) = (u+4, 3u-2) \rightarrow y = 3x - 14$$



- Here we see the point of changing variables: D would be difficult to integrate over, but D^* is much much easier

Ex: Consider the transformation

$$T(r,\theta) = (r\cos\theta, r\sin\theta),$$

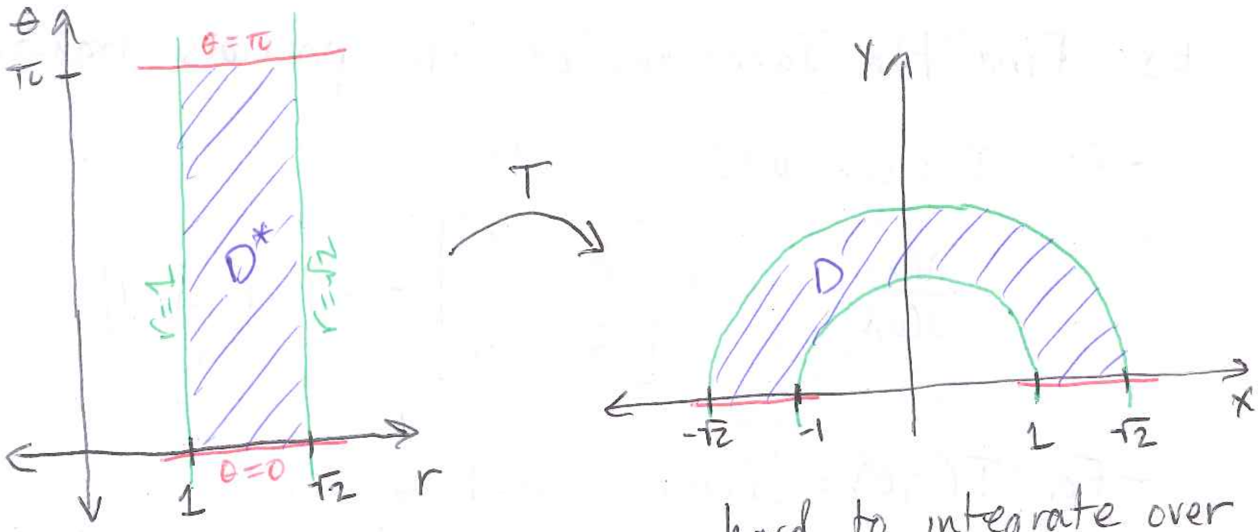
with domain $D^* = [1,\sqrt{2}] \times [0,\pi]$. What is the image of T?

$r=1 \longrightarrow T(1,\theta) = (\cos\theta, \sin\theta) \longrightarrow$ radius 1 circle

$r=\sqrt{2} \longrightarrow T(\sqrt{2},\theta) = (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta) \longrightarrow$ radius $\sqrt{2}$ circle

$\theta = 0 \longrightarrow T(r,0) = (r,0) \longrightarrow y=0,\ x \geq 0$

$\theta = \pi \longrightarrow T(r,0) = (-r,0) \longrightarrow y=0,\ x \leq 0$



hard to integrate over

easy to integrate over

- Back to the issue at hand: how to translate

$$\int_I f(x)dx = \int_{I^*} f(x(u)) \left|\frac{\partial x}{\partial u}\right| du$$

into functions of more than 1 variable

- We see that $x(u)$; $x: I^* \to I$ is equivalent to $T(u,v)$; $T: D^* \to D$
- What about $\left|\frac{\partial x}{\partial u}\right|$? Derivatives! In the case of $T(u,v) = (x(u,v), y(u,v))$:

$$DT = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

- To make it similar to the 1-D case, sometimes DT is denoted $\frac{\partial(x,y)}{\partial(u,v)}$
- DT (or $\frac{\partial(x,y)}{\partial(u,v)}$) is called the Jacobian of T

Ex: Find the Jacobians of the previous transformations

- For $T(u,v) = (u+2v, 3u-v)$:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = -1-6 = \boxed{-7}$$

- For $T(r,\theta) = (r\cos\theta, r\sin\theta)$:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

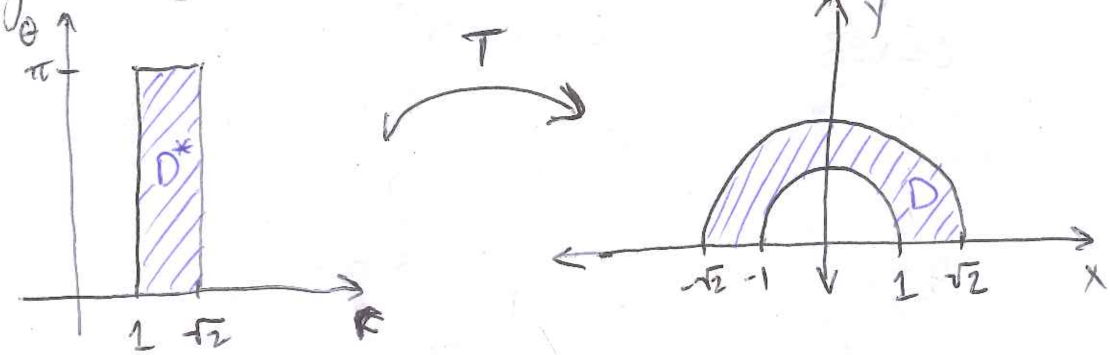
- Note: $T(r,\theta) = (r\cos\theta, r\sin\theta)$ is called a change to polar coordinates, where position is described via radius (r) and angle (θ).

• Now we're ready for the actual formula: Let D and D^* be regions in $\mathbb{R}^2$, with $T: D^* \to D$ a map such that $T(D^*) = D$. Then

$$\cancel{\iint_D f\,dA = \iint_{D^*}}$$

$$\iint_D f(x,y)\,dA = \iint_{D^*} f(T(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA^*$$

Ex: Evaluate $\iint_D e^{x^2+y^2}\,dA$, where D is the region $\{(x,y) \mid 1 \le x^2+y^2 \le 2,\ y \ge 0\}$.



- Using the transformation from the second example:

$$T(r,\theta) = (r\cos\theta, r\sin\theta) \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

our new integral is

$$\iint_{D^*} e^{(r\cos\theta)^2+(r\sin\theta)^2} |r|\,dA^*$$

$$= \int_0^{\pi}\int_1^{\sqrt{2}} e^{r^2} |r|\,dr\,d\theta \qquad \begin{matrix} u=r^2 \\ du=2r\,dr \end{matrix}$$

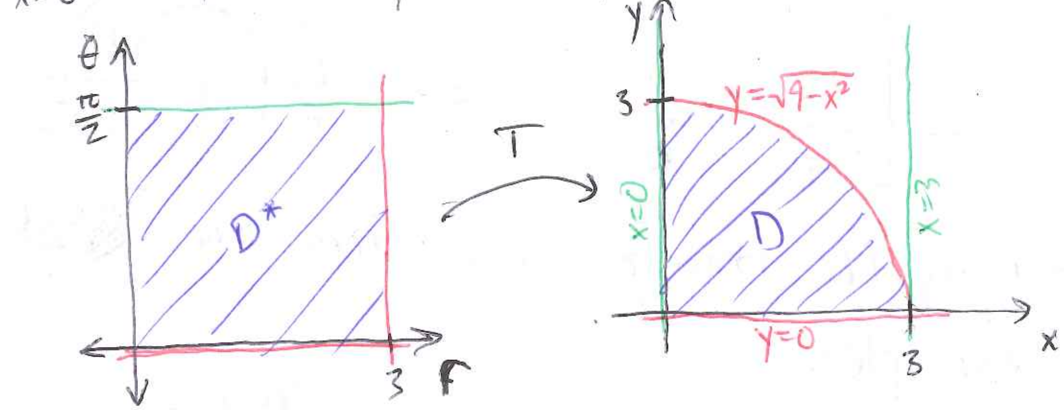
$$= \int_0^{\pi} \left(\frac{1}{2} e^{r^2}\Big|_1^{\sqrt{2}}\right) d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(e^2 - e^1)\,d\theta = \boxed{\frac{\pi}{2}(e^2 - e)}$$

Ex: Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} e^{x^2+y^2} \, dy\, dx$.

$0 \leq x \leq 3$ → $x=0$, $x=3$

$0 \leq y \leq \sqrt{9-x^2}$ → $y=0$, $y=\sqrt{9-x^2} \Rightarrow x^2+y^2=9$



– Using $T(r,\theta) = (r\cos\theta, r\sin\theta)$:

$$\iint_{D^*} e^{(r\cos\theta)^2+(r\sin\theta)^2} \, dA^*$$

$$= \int_0^{\pi/2} \int_0^3 e^{r^2} \, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} e^{r^2} \Big|_0^3 \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(e^9 - e^1)\, d\theta = \frac{\pi}{4}(e^9 - e^1)$$

Ex: Use the change of variables $T(u,v) = (v, \frac{u}{v})$ to evaluate $\iint_D x^2y^2 \, dA$, where D is the region bdd. by the curves $y=x^2$, $y=2x^2$, $xy=1$, and $xy=2$.

- First we need to figure out what D^* looks like: Using the fact that $T(u,v) = (v, \frac{u}{v})$

$$\Rightarrow \quad x(u,v) = v, \quad y(u,v) = \frac{u}{v}$$

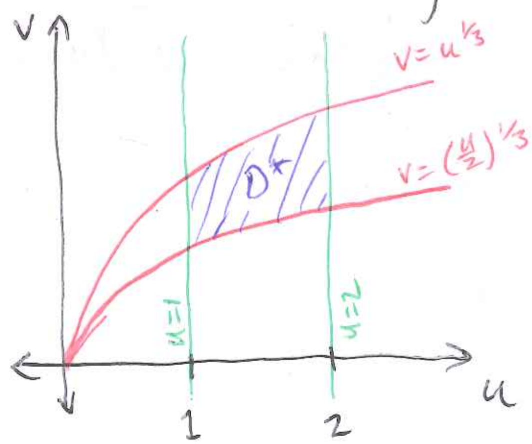
$$y = x^2 \rightarrow \frac{u}{v} = v^2 \rightarrow u = v^3 \text{ or } v = u^{1/3}$$

$$y = 2x^2 \rightarrow \frac{u}{v} = 2v^2 \rightarrow u = 2v^3 \text{ or } v = \left(\frac{u}{2}\right)^{1/3}$$

$$xy = 1 \rightarrow v \cdot \frac{u}{v} = 1 \longrightarrow u = 1$$

$$xy = 2 \rightarrow v \cdot \frac{u}{v} = 2 \longrightarrow u = 2$$

- So D^* should look something like



- Now our integral is

$$\iint_{D^*} (v)^2 \left(\frac{u}{v}\right)^2 \cancel{dA} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA^*,$$

where

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{bmatrix} 0 & 1 \\ 1/v & -\frac{u}{v^2} \end{bmatrix}\right| = \left|-\frac{1}{v}\right| = \left|\frac{1}{v}\right|$$

$$\Rightarrow \int_1^2 \int_{\left(\frac{u}{2}\right)^{1/3}}^{u^{1/3}} \frac{u^2}{v}\, dv\, du$$

$$= \int_1^2 \left(u^2 \ln v \Big|_{v=\left(\frac{u}{2}\right)^{1/3}}^{v=u^{1/3}} \right) du$$

$$= \int_1^2 u^2 \left[\ln\left(u^{1/3}\right) - \ln\left(\left(\frac{u}{2}\right)^{1/3}\right) \right] du$$

$$= \frac{1}{3} \int_1^2 u^2 \left(\ln(u) - \ln\left(\frac{u}{2}\right) \right) du$$

$$= \frac{1}{3} \int_1^2 u^2 \ln\left(\frac{u}{u/2}\right) du$$

$$= \frac{1}{3} \ln 2 \int_1^2 u^2 du = \boxed{\frac{7}{9} \ln 2}$$