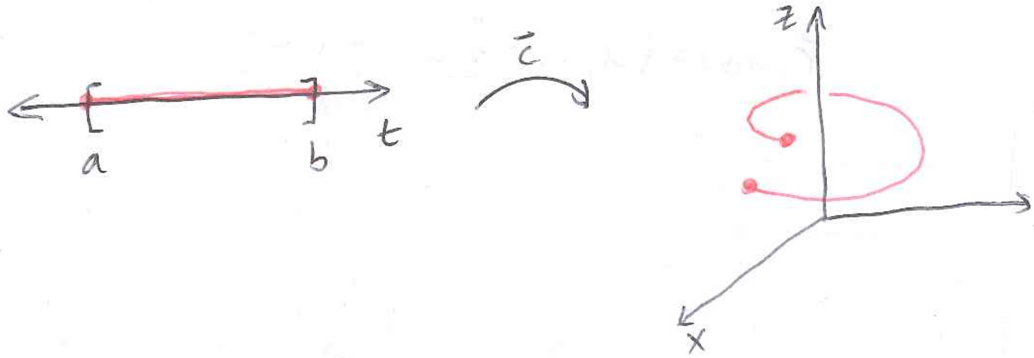
7.1 Parametrized Surfaces

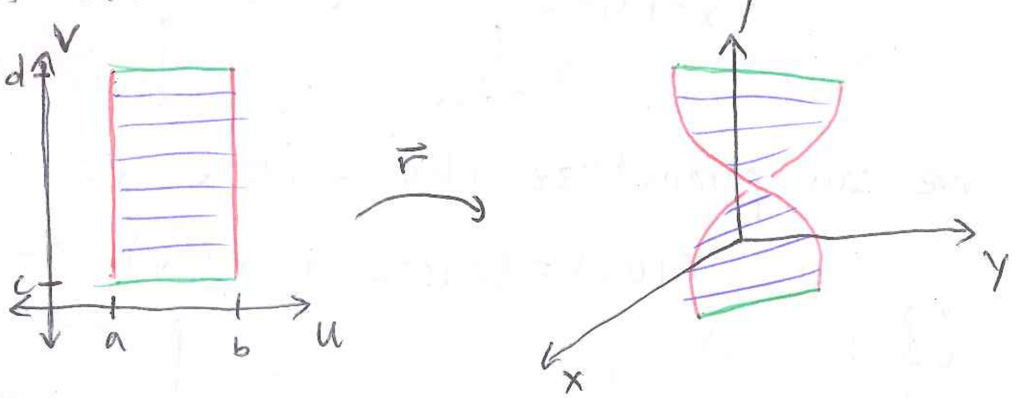
- Up to now, the only thing we have been parametrizing curves: $\vec{r}(t) = (x(t), y(t), z(t))$
  - One way to view this is "bending a wire"



- Now we want to parametrize <u>surfaces</u> via fns of the form:

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

  - This can be viewed as "deforming sheets":



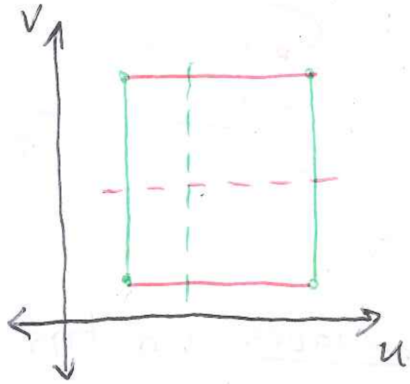
- <u>Defn</u>: A <u>parametrized surface</u> is a map $\vec{r}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ where D is a region in $\mathbb{R}^2$. The image of $\vec{r}$ $S = \vec{r}(D)$ is called a <u>surface</u>.

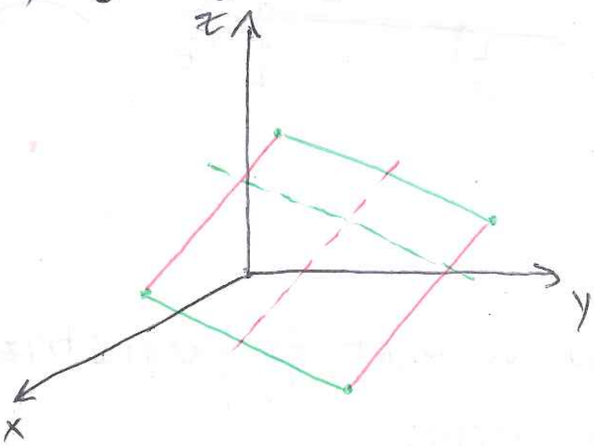
<u>Ex</u>: A plane in $\mathbb{R}^3$ can be represented by the equation $ax+by+cz+d=0$. If $c\neq 0$, then

$$z=-\frac{a}{c}x-\frac{b}{c}y-\frac{d}{c}$$

and we can parametrize the plane via

$$\vec{r}(u,v)=\left(u,v,-\frac{a}{c}u-\frac{b}{c}v-\frac{d}{c}\right)$$



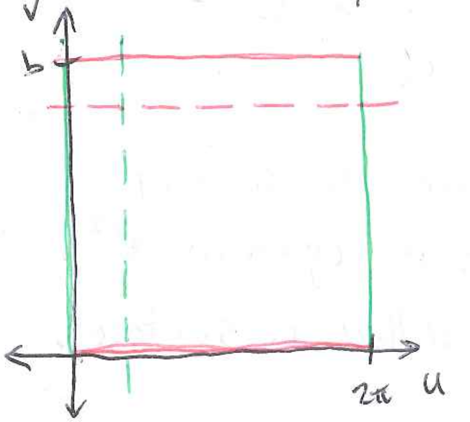


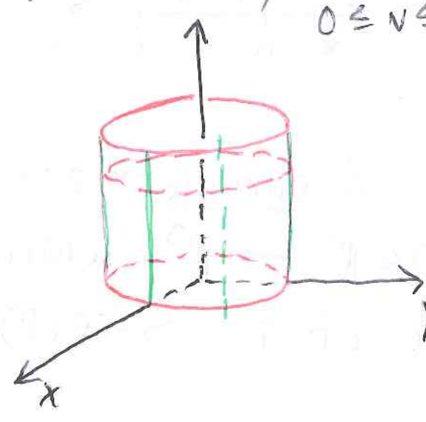
<u>Ex</u>: A cylinder in $\mathbb{R}^3$ is described via

$$\underset{\substack{\uparrow\\ \text{radius}}}{x^2+y^2=a^2}, \quad \underset{\substack{\uparrow\\ \text{height}}}{0\le z\le b}$$

we can parametrize this surface via

$$\vec{r}(u,v)=(a\cos u, a\sin u, v), \quad 0\le u\le 2\pi, \quad 0\le v\le b.$$



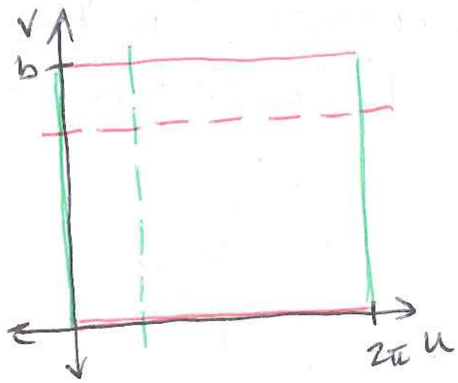


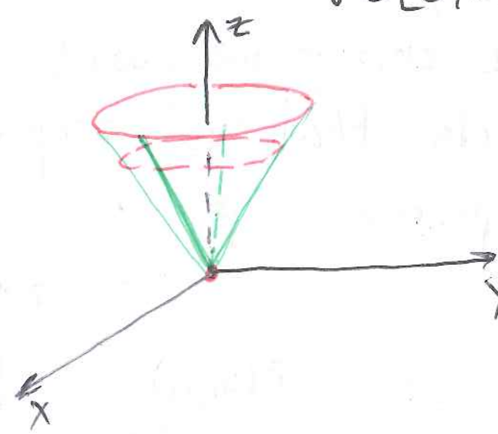
Ex: A cone is described via

$$x^2+y^2=z^2, \quad 0\le z\le b$$

which can be parametrized via

$$\vec{r}(u,v)=(v\cos u,\ v\sin u,\ v), \quad u\in[0,2\pi], \quad v\in[0,b]$$





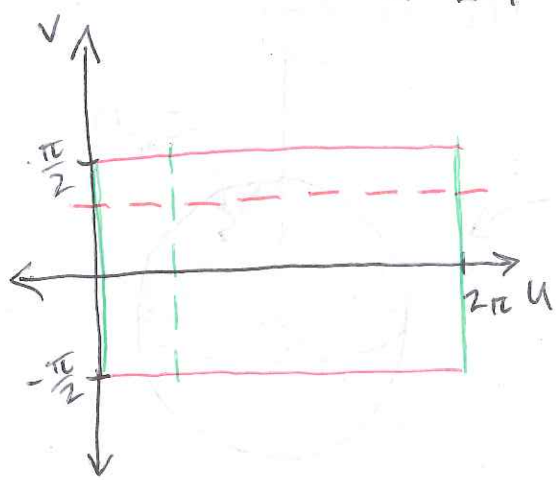
Ex: A sphere of radius a is described via

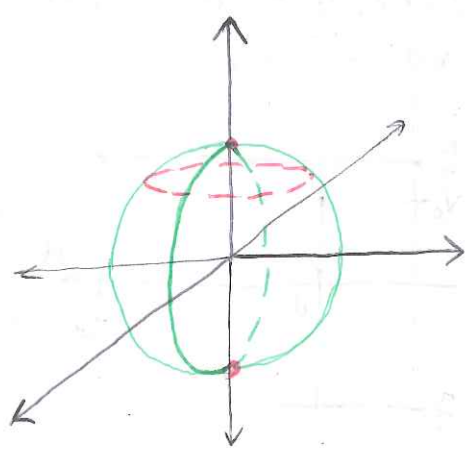
$$x^2+y^2+z^2=a^2$$

which can be parametrized by

$$\vec{r}(u,v)=(a\cos u\cos v,\ a\sin u\cos v,\ a\sin v),$$

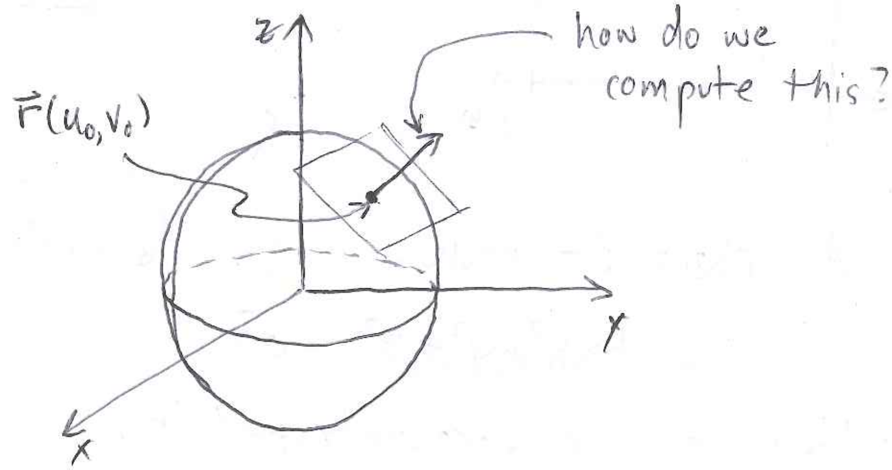
$$u\in[0,2\pi], \quad v\in\left[-\tfrac{\pi}{2},\tfrac{\pi}{2}\right]$$



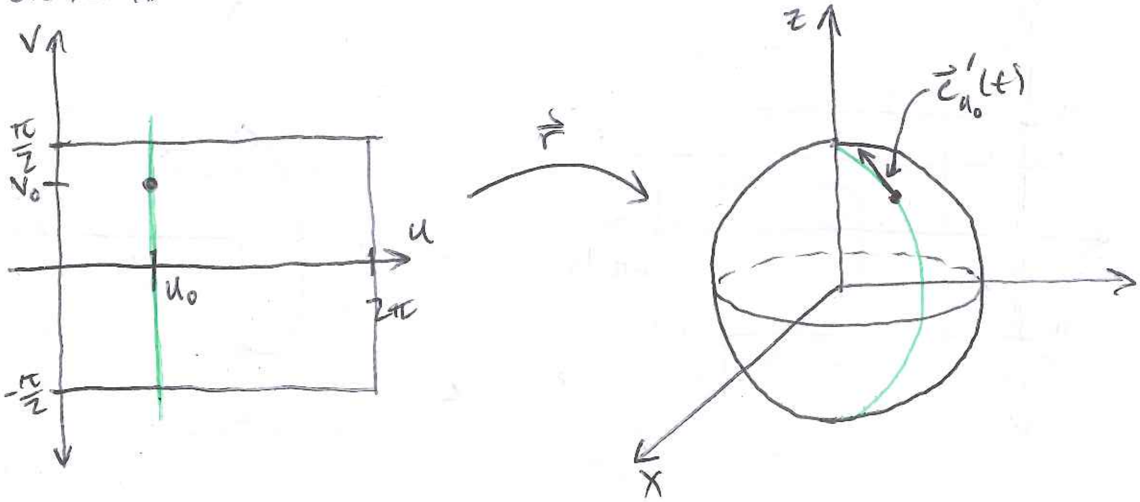


- You can view this as a parametrization of the earth's surface: horizontal lines are mapped to latitudes, and vertical lines are mapped to longitudes.

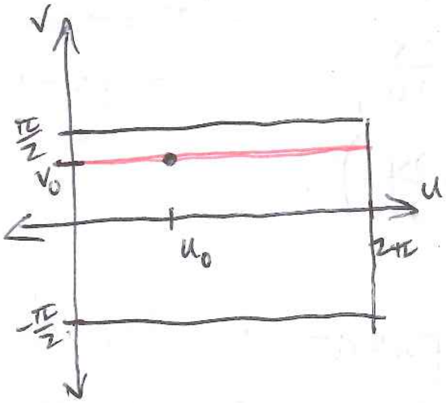
- One object we would like to determine is the vector that is perpendicular to a surface at a particular point:

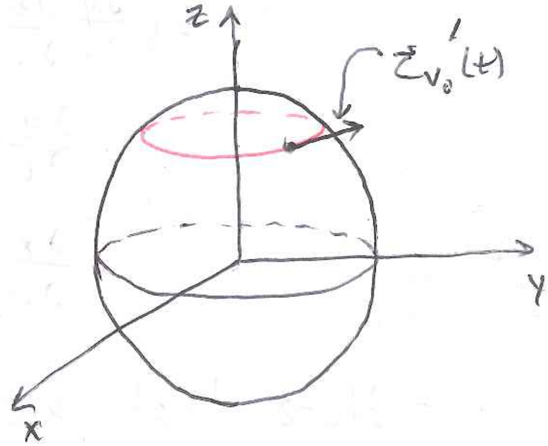


- Consider the path $\vec{c}_{u_0}(t) = \vec{r}(u_0, t)$. This describes a curve on the surface:



- Then $\vec{c}_{u_0}'(t)$ gives a tangent vector to the surface. We can also consider the curve $\vec{c}_{v_0}(t) = \vec{r}(t, v_0)$:





- $\vec{c}_{v_0}'(t)$ gives <u>another</u> tangent vector to the surface.

- We're looking for a vector that is perpendicular to both vectors, which gives the definition:

- The vector $\vec{N} = \vec{c}_{v_0}' \times \vec{c}_{u_0}'$ is called the <u>normal vector</u> (or <u>surface normal</u>) to S.

  - Let's examine how to actually compute $\vec{N}$: if we denote $\vec{r} = (x, y, z)$ [x, y, z are all functions of u and v] then

$$\vec{c}_{v_0}' = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$\vec{c}_{u_0}' = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

$$\Rightarrow \vec{N} = \vec{c}\,'_{v_0} \times \vec{c}\,'_{u_0}$$

$$= \Big(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u},$$

$$\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u},$$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \Big)$$

Note: Sometimes we use $\vec{T}_u$ for $\vec{c}\,'_{v_0}$ and $\vec{T}_v$ for $\vec{c}\,'_{u_0}$ (T stands for tangent)

- Note that each component is the determinant of a 2x2 matrix. In fact, each component is the Jacobian of a function:

$$\boxed{\vec{N} = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right)}$$

- Personally, I think this is the easiest formula to remember

Ex: Find the normal vector to the cone, parametrized by

$$\vec{r}(u,v) = (v\cos u, v\sin u, v)$$

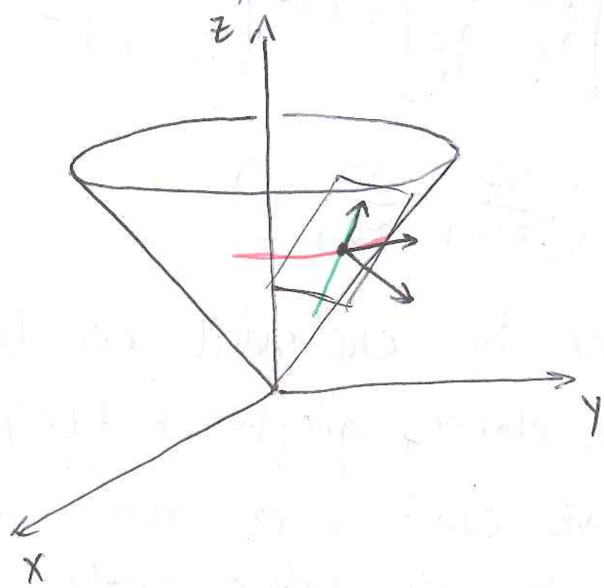
- Find all the Jacobians:

$$\frac{\partial(y,z)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \det\begin{bmatrix} v\cos u & \sin u \\ 0 & 1 \end{bmatrix} = v\cos u$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{bmatrix} = \det\begin{bmatrix} 0 & 1 \\ -v\sin u & \cos u \end{bmatrix} = v\sin u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \end{bmatrix} = -v$$

$$\Rightarrow \vec{N} = (v\cos u,\ v\sin u,\ -v)$$



- One type of surface that's easy to parametrize is the graph of a function $f(x,y)$:

$$\vec{r}(u,v) = (u, v, f(u,v))$$

- in this case, you can view $u=x$ and $v=y$

Ex: Find the normal vector for a surface

$$\vec{r}(u,v) = (u, v, f(u,v))$$

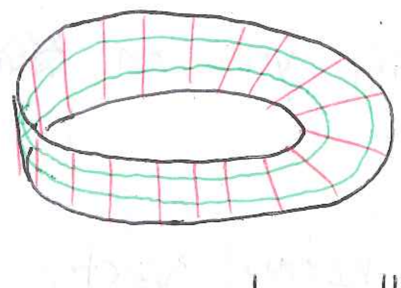
$$\frac{\partial(y,z)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \det\begin{bmatrix} 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial z} \end{bmatrix} = -\frac{\partial f}{\partial u}$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ 1 & 0 \end{bmatrix} = -\frac{\partial f}{\partial v}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$\Rightarrow \vec{N} = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)$$

- Surfaces can be one-sided or two-sided: spheres, cones, planes, graphs of $f(x,y)$ are <u>all</u> 2-sided — we can't move from one side of the surface to the other without crossing an edge or passing through the surface
  - The simplest one-sided surface is the Möbius strip:



  - Imagine you are an ant walking on this surface: you can get to the other "side" by following the green lines!