ON THE WELL POSEDNESS OF THE MODIFIED KORTEWEG-DE VRIES EQUATION IN WEIGHTED SOBOLEV SPACES

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Abstract. We study local and global well posedness of the k-generalized Korteweg-de Vries equation in weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$.

1. Introduction

This work is concerned with the initial value problems (IVP) associated to the k-generalized Korteweg-de Vries (k-gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad t,\, x \in \mathbb{R}, \;\; k \in \mathbb{Z}^+, \\ \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Our goal is to study well posedness of the IVP (1.1) in weighted Sobolev spaces

$$Z_{s,r} \equiv H^s(\mathbb{R}) \cap L^2(\,|x|^{2r}), \qquad s \in \mathbb{R}, \;\; r \geq 0. \tag{1.2}$$

We shall follow the notion of well posedness given in [10]: the IVP (1.1) is said to be locally well posed (LWP) in the function space X if for each $u_0 \in X$ there exist $T > 0$ and a unique solution $u \in C([-T,T] : X) \cap .... = Y_T$ of the equation in (1.1), with the map data $\rightarrow$ solution being locally continuous from X to Y_T.

This notion of LWP includes the "persistence" property, i.e. the solution describes a continuous curve on X. In particular, this implies that the solution flow of (1.1) defines a dynamical system in X. If T can be taken arbitrarily large, then the IVP (1.1) is said to be globally well posed (GWP).

We shall be mainly concerned with the modified Korteweg de Vries (mKdV) equation, i.e. $k = 2$ in (1.1). In [11] Kenig, Ponce and Vega showed that the IVP (1.1) with $k = 2$ is locally well posed in

$$\dot{H}^{1/4}(\mathbb{R}) = (-\partial_x^2)^{-1/8} L^2(\mathbb{R}) \supset H^{1/4}(\mathbb{R}) = J^{-1/4} L^2(\mathbb{R}) = (1-\partial_x^2)^{-1/8} L^2(\mathbb{R}).$$

More precisely, the following result was established in [11]:

Theorem A. ([11]) *For any $u_0 \in \dot{H}^{1/4}(\mathbb{R})$ there exist*

$$T = T(\|D_x^{1/4} u_0\|_2) \sim \|D_x^{1/4} u_0\|_2^{-4}, \tag{1.3}$$

and a unique solution $u(t)$ of the IVP (1.1) with $k = 2$ such that

$$\begin{aligned} & u \in C([-T,T] : \dot{H}^{1/4}(\mathbb{R})), \\ & \textit{and} \\ & \|D_x^{1/4}\, \partial_x u\|_{L_x^\infty L_T^2} + \|\partial_x u\|_{L_x^{20} L_T^{5/2}} + \|D_x^{1/4} u\|_{L_x^5 L_T^{10}} + \|u\|_{L_x^4 L_T^\infty} < \infty. \end{aligned} \tag{1.4}$$

For any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $\dot{H}^{1/4}(\mathbb{R})$ such that the map data $\to$ solution $\widetilde{u}_0 \to \widetilde{u}(t)$ from V into the class defined by (1.4) with T' instead of T is smooth.

Moreover, if in addition $u_0 \in H^s(\mathbb{R})$ with $s \geq 1/4$, then the solution

$$u \in C([-T,T] : H^s(\mathbb{R})),$$

and

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} + \|J_x^{s-1/4}\partial_x u\|_{L_x^{20} L_T^{5/2}} + \|J_x^s u\|_{L_x^5 L_T^{10}} < \infty.$$

Remarks: (a) The fact that the map data $\to$ solution is smooth is a direct consequence of the proof of Theorem A, based on the contraction principle, and the implicit function theorem. The estimate for the length of the time interval of existence (1.3) is inside the proof in [11] (which is partially reproduced in the proof of Theorem 1 below) or can also be obtained by a scaling argument.

(b) It was shown in [13], and [2] that in an appropriate sense the value $1/4$ in Theorem A is optimal.

(c) In [4] Colliander, Keel, Staffilani, Takaoka, and Tao showed that this LWP extends to a GWP if $s > 1/4$. The GWP for the limiting case $s = 1/4$ was established by Guo [9] and Kishimoto [14].

Theorem B. ([9], [14]) *Let $u_0 \in H^s(\mathbb{R})$ with $s \geq 1/4$. Then for any $T^* > 0$ the IVP (1.1) with $k = 2$ has a unique solution*

$$u \in C([-T^*,T^*] : H^s(\mathbb{R})) \cap .... \tag{1.5}$$

Remark: (a) The proof of Theorem B relies on the so called "I-method" introduced in [3], on the Miura transformation [16], and on sharp LWP for the Korteweg-de Vries (KdV) $k = 1$ in (1.1). This optimal LWP result for the KdV requires the use of the so called Bourgain spaces $X_{s,b}$, introduced in the context of non-linear dispersive equations in [1]. Consequently, the precise description of the class in (1.5) involves those spaces.

Concerning LWP in the weighted spaces $Z_{s,r}$ defined in (1.2) T. Kato [10] showed that persistent properties holds for solutions of the IVP (1.1) for any $m \in \mathbb{Z}^+$ in

$$Z_{s,m} = H^s(\mathbb{R}) \cap L^2(\,|x|^{2m}), \quad s \geq 2m, \quad m = 1, 2, ...$$

More precisely:

Theorem C. ([10]) *Let $m \in \mathbb{Z}^+$. Let $u \in C([-T,T] : H^s(\mathbb{R})) \cap .....$ with $s \geq 2m$ be the solution of the IVP (1.1). If $u(x,0) = u_0(x) \in L^2(|x|^{2m}dx)$, then*

$$u \in C([-T,T] : Z_{s,m}).$$

Remarks : (a) We recall the best known LWP and GWP results in $H^s(\mathbb{R})$ for the IVP (1.1) with $k \neq 2$:

- for $k = 1$ LWP is known for $s \geq -3/4$ (see [12] for the case $s > -3/4$ and [2], [9] and [14] for the limiting case $s = -3/4$), and GWP is known for $s \geq -3/4$ (see [4] for the case $s > -3/4$ and [9] and [14] for the limiting case $s = -3/4$),

- for $k = 3$ LWP is known for $s \geq -1/6$ (see [7] for the case $s > -1/6$ and [20] for the limiting case $s = -1/6$) and GWP is known for $s > -1/42$ (see [8]),

-for $k \geq 4$ LWP is known for $s \geq (k-4)/2k$ (see [11]). In [15] for the case $k = 4$ it is shown that there exist local smooth solutions which develop singularities in finite time.

(b) The proof of Theorem C in [10] is based on the commutative property of the operators

$$\Gamma = x - 3t\partial_x^2, \qquad \mathcal{L} = \partial_t + \partial_x^3, \qquad \text{so} \quad [\Gamma;\mathcal{L}] = 0. \tag{1.6}$$

In particular, if $\{U(t) : t \in \mathbb{R}\}$ denotes the unitary group of operators describing the solution of the linear IVP

$$\partial_t v + \partial_x^3 v = 0, \quad t,\, x \in \mathbb{R}, \qquad v(x,0) = v_0(x), \tag{1.7}$$

i.e.

$$U(t)v_0(x) = (e^{-it\xi^3}\widehat{v_0})^{\vee}(x), \tag{1.8}$$

then from (1.6) one has that

$$\begin{aligned} &x\,U(t)v_0(x) = U(t)(xv_0)(x) + 3tU(t)(\partial_x^2 v_0)(x),\\ &\quad\text{i.e.}\\ &\Gamma U(t)v_0(x) = U(t)(xv_0)(x). \end{aligned} \tag{1.9}$$

(c) The form of the operator Γ suggests that one should expect persistence in $Z_{s,r}$ only if $s \geq 2r$.

In [17] for the case of the mKdV, J. Nahas extended locally the result in Theorem C to the optimal range of the parameter s, r accordingly to Theorem A and (1.6), i.e. $s \geq 1/4$ and $s \geq 2r > 0$. Also in [17] for the case $k \geq 4$ in (1.1) Theorem C was extended to the optimal range $s \geq (k-4)/4k$ and $s \geq 2r > 0$.

Our first result gives a significantly simplified proof and slightly stronger version of these results. We shall concentrate in the case of the mKdV equation $k = 2$ in (1.1).

Theorem 1. *Let $u \in C([-T,T] : \dot{H}^{1/4}(\mathbb{R}))$ denote the solution of the IVP (1.1) with $k = 2$ provided by Theorem A. If u_0, $|x|^r u_0 \in L^2(\mathbb{R})$ with $r \in (0, 1/8]$, then*

$$\begin{aligned} &u \in C([-T,T] : H^{1/4}(\mathbb{R}) \cap L^2(|x|^{2r}dx)),\\ &\textit{and}\\ &\||x|^r u\|_{L^5_x L^{10}_T} < \infty. \end{aligned} \tag{1.10}$$

For any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^{1/4}(\mathbb{R}) \cap L^2(|x|^{2r}dx)$ such that the map $\widetilde{u}_0 \to \widetilde{u}(t)$ from V into the class defined by (1.4) and (1.10) with T' instead of T is smooth.

Moreover, if in addition $u_0 \in Z_{s,r'}$ with $s > 1/4$ and $s \geq 2r' > 2r$, then the solution

$$u \in C([-T,T] : Z_{s,r'})$$

with

$$\|J_x^s\, \partial_x u\|_{L^\infty_x L^2_T} + \|J_x^s u\|_{L^5_x L^{10}_T} + \|J_x^{s-1/4}\partial_x u\|_{L^{20}_x L^{5/2}_T} + \||x|^{r'} u\|_{L^5_x L^{10}_T} < \infty.$$

Remarks: (a) We observe that Theorem 1 guarantees that the persistent property in the weighted space $Z_{s,r}$ holds in the same time interval $[-T,T]$ given by Theorem A, where T depends only on $\|D_x^{1/4}u_0\|_2$ (see (2.10)).

(b) In [6] Ginibre and Tsutsumi obtained results concerning the uniqueness and existence (in an appropriate class) of local solutions of the IVP (1.1) with $k = 2$ and data u_0 in the weighted space $L^2((1+|x|)^{1/4}dx)$. Theorem 1 shows that for data $u_0 \in Z_{1/4,1/8}$ the solution provided by Theorem A and that obtained in [6] agree.

(c) Our simplification of the proof of Theorem 1 comes from the use of a new point-wise formula deduced by Fonseca, Linares, and Ponce in [5]. Roughly, this formula extends the operator Γ in (1.6), (1.8) and (1.9) to the case of fractional weights $|x|^r$, $r \in (0,1)$. More precisely, the following result was established in [5] (Lemma 1.2) :

Lemma A. ([5]) *Let $\{U(t) \, : \, t \in \mathbb{R}\}$ be the unitary group of operators defined in* (1.8). *If*

$$u_0 \in Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \qquad s \geq 2r \quad \textit{with} \quad r \in (0,1),$$

then for all $t \in \mathbb{R}$ and for almost every $x \in \mathbb{R}$

$$|x|^r U(t)u_0(x) = U(t)(|x|^r u_0)(x) + U(t)\{\Psi_{t,r}(\widehat{u}_0)(\xi)\}^{\vee}(x), \tag{1.11}$$

with

$$\|\Psi_{t,r}(\widehat{u}_0)\|_2 \leq c(1+|t|)(\|u_0\|_2 + \|D^{2r}u_0\|_2). \tag{1.12}$$

The proof of Lemma A given in [5] is a consequence of a characterization of the Sobolev space

$$L^{\alpha,p}(\mathbb{R}^n) = (1-\Delta)^{-\alpha/2}L^p(\mathbb{R}^n), \quad \alpha \in (0,2), \;\; p \in (1,\infty),$$

due to E. M. Stein, see [19].

(d) As in [17] the result in Theorem 1 extends to the local solutions of the IVP (1.1) with $k \geq 4$ the optimal range of the parameters s, r accordingly to remark (a) after Theorem C, i.e. $s \geq 2r > 0$ with $s \geq (k-4)/2k$. This will be clear from our proof of Theorem 1 given below, so we omit the details. For the cases $k = 1$ and $k = 3$ a weaker version of these results was proven in [18]. The main difference between the cases $k = 2, 4, 5, ...$ and $k = 1, 3$ is that for the later the "optimal" well-posedness results are based on the spaces $X_{s,b}$ which makes fractional weights difficult to handle.

As a consequence of Theorem B and our proof of Theorem 1 we obtain the following global version of Theorem 1:

Theorem 2. *Let $s \geq 1/4$ and $T^* > 0$. If $u_0 \in Z_{s,r}$ with $s \geq 2r > 0$, then the solution u of the IVP* (1.1) *with $k = 2$ provided by Theorem 1 extends to the time interval $[-T^*, T^*]$ with*

$$u \in C([-T^*, T^*] \, : \, Z_{s,r}).$$

2. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1:

We shall restrict our attention to the most interesting case $s = 1/4$ and $r = 1/8$, i.e. $u_0 \in Z_{1/4,1/8}$.

We begin with a brief review of the argument used in the proof of Theorem A in [11]. The details of this proof will be used latter to complete the proof of Theorem 1.

First, let us assume that

$$u_0 \in \dot{H}^{1/4}(\mathbb{R}).$$

For $w : \mathbb{R} \times [-T, T] \to \mathbb{R}$ with T to be fixed below, define

$$\begin{aligned} \mu_1^T(w) = &\|D_x^{1/4}w\|_{L^\infty_T L^2_x} + \|\partial_x w\|_{L^{20}_x L^{5/2}_T} + \|D_x^{1/4}w\|_{L^5_x L^{10}_T} \\ &+ \|D_x^{1/4}\, \partial_x w\|_{L^\infty_x L^2_T} + \|w\|_{L^4_x L^\infty_T}. \end{aligned} \tag{2.1}$$

Denote by $\Phi(v)=\Phi_{u_0}(v)$ the solution of the linear inhomogeneous IVP

$$\partial_t u+\partial_x^3 u+v^2\partial_x v=0 \qquad u(x,0)=u_0(x). \tag{2.2}$$

The idea is to apply the contraction principle to the integral equation version of the IVP (2.2), i.e.

$$u(t)=\Phi(v(t))=U(t)u_0-\int_0^t U(t-t')(v^2\,\partial_x v)(t')dt'. \tag{2.3}$$

From the linear estimates concerning the group $\{U(t)\,:\,t\in\mathbb{R}\}$ established in [11] one has that

$$\mu_1^T(U(t)u_0)\le c_0\|D_x^{1/4}u_0\|_2, \qquad \forall\, T>0. \tag{2.4}$$

Here and below c_0 will denote a universal constant whose value may change (increase) from line to line. Hence,

$$\begin{aligned}&\mu_1^T(\int_0^t U(t-t')v^2\partial_x v(t')dt')\\ &\le c_0\|D_x^{1/4}(v^2\partial_x v)\|_{L^1_T L^2_x}\le c_0 T^{1/2}\|D_x^{1/4}(v^2\partial_x v)\|_{L^2_x L^2_T}.\end{aligned} \tag{2.5}$$

Using the calculus of inequalities in the Appendix in [11] (Theorem A.8) one gets that

$$\begin{aligned}&\|D_x^{1/4}(v^2\,\partial_x v)\|_{L^2_x L^2_T}\\ &\le c_0\|D_x^{1/4}(v^2)\|_{L^{20/9}_x L^{10}_T}\|\partial_x v\|_{L^{20}_x L^{5/2}_T}\;+\;c_0\|v^2\|_{L^2_x L^\infty_T}\|D_x^{1/4}\,\partial_x v\|_{L^\infty_x L^2_T}\\ &\le c_0\|v\|_{L^4_x L^\infty_T}\|D_x^{1/4}v\|_{L^5_x L^{10}_T}\;\|\partial_x v\|_{L^{20}_x L^{5/2}_T}\;+\;c_0\|v\|^2_{L^4_x L^\infty_T}\|D_x^{1/4}\,\partial_x v\|_{L^\infty_x L^2_T}\\ &\le c_0(\mu_1^T(v))^3.\end{aligned} \tag{2.6}$$

Inserting the estimates (2.4), (2.5), and (2.6) in the integral equation (2.3) it follows that

$$\begin{aligned}\mu_1^T(\Phi(v))&\le c_0\|D_x^{1/4}u_0\|_2+c_0\int_0^T\|D_x^{1/4}(v^2\,\partial_x v)\|_2(t)dt\\ &\le c_0\|D_x^{1/4}u_0\|_2+c_0\,T^{1/2}\|D_x^{1/4}(v^2\,\partial_x v)\|_{L^2_x L^2_T}\\ &\le c_0\|D_x^{1/4}u_0\|_2+c_0\,T^{1/2}(\mu_1^T(v))^3.\end{aligned} \tag{2.7}$$

A similar argument leads to the estimate

$$\mu_1^T(\Phi(v)-\Phi(\widetilde{v}))\le c_0\,T^{1/2}(\mu_1^T(v)+\mu_1^T(\widetilde{v}))^2\,\mu_1^T(v-\widetilde{v}). \tag{2.8}$$

This basically proves the main part of Theorem A. More precisely, one has that the operator $\Phi=\Phi_{u_0}$ in (2.3) defines a contraction in the set

$$\{v\;:\mathbb{R}\times[-T,T]\to\mathbb{R}\;:\;\mu_1^T(v)\le 2c_0\|D_x^{1/4}u_0\|_2\}, \tag{2.9}$$

with

$$T=\frac{1}{64\,c_0^6\,\|D_x^{1/4}u_0\|_2^4}. \tag{2.10}$$

Hence, the IVP (1.1) with $k=2$ has a unique solution $u\in C([-T,T]:\dot{H}^{1/4}(\mathbb{R}))$ satisfying

$$\mu_1^T(u)\le 2c_0\|D_x^{1/4}u_0\|_2, \tag{2.11}$$

with T as in (2.10).

Now, we assume that

$$u_0 \in H^{1/4}(\mathbb{R}),$$

and define

$$\mu_2^{T_0}(w) = \|w\|_{L^\infty_{T_0} L^2_x} + \|\partial_x w\|_{L^\infty_x L^2_{T_0}} + \mu_1^{T_0}(w),$$

with $\mu_1^{T_0}$ defined in (2.1) and $T_0 > 0$ to be fixed below. By (2.4) one has that

$$\|U(t)u_0\|_{L^\infty_{T_0} L^2_x} + \|\,\partial_x U(t)u_0\|_{L^\infty_x L^2_{T_0}} \le c_0 \|u_0\|_2, \qquad \forall\, T_0 > 0. \tag{2.12}$$

Therefore

$$\begin{aligned}
&\|\int_0^t U(t-t')v^2\partial_x v(t')dt'\,\|_{L^\infty_{T_0} L^2_x} + \|\,\partial_x \int_0^t U(t-t')v^2\partial_x v(t')dt'\,\|_{L^\infty_x L^2_{T_0}} \\
&\le c_0\, \|v^2\partial_x v\|_{L^1_{T_0} L^2_x} \le c_0\, T_0^{1/2}\, \|v^2\partial_x v\|_{L^2_x L^2_{T_0}} \\
&\le c_0\, T_0^{1/2}\, \|v^2\|_{L^2_x L^\infty_{T_0}}\, \|\partial_x v\|_{L^\infty_x L^2_{T_0}} \le c_0\, T_0^{1/2}\, \|v\|^2_{L^4_x L^\infty_{T_0}}\, \|\partial_x v\|_{L^\infty_x L^2_{T_0}} \\
&\le c_0\, T_0^{1/2} (\mu_2^{T_0}(v))^3,
\end{aligned} \tag{2.13}$$

and similarly,

$$\begin{aligned}
&\|\Phi(v) - \Phi(\widetilde{v})\|_{L^\infty_{T_0} L^2_x} + \|\Phi(v) - \Phi(\widetilde{v})\|_{L^\infty_x L^2_{T_0}} \\
&\le c_0\, T_0^{1/2} (\mu_2^{T_0}(v) + \mu_2^{T_0}(\widetilde{v}))^2\, \mu_2^{T_0}(v - \widetilde{v}).
\end{aligned}$$

Hence, collecting the above result one has that the operator $\Phi = \Phi_{u_0}$ defines a contraction in the set

$$\{v\ : \mathbb{R} \times [-T_0, T_0] \to \mathbb{R}\, :\, \mu_2^{T_0}(v) \le 2c_0(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2)\},$$

with

$$T_0 = \frac{1}{64\, c_0^6\, (\|u_0\|_2 + \|D_x^{1/4}u_0\|_2)^4} < T,$$

with T as in (2.10). This proves the existence of a unique solution

$$u \in C([-T_0, T_0]\, :\, H^{1/4}(\mathbb{R})),$$

satisfying

$$\mu_2^{T_0}(u) \le 2c_0(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2),$$

with the map data $\to$ solution smooth.

We recall that the L^2-norm of this solution is preserved. Also we observe that formally (2.12)-(2.13) and (2.11) and the integral equation shows that the solution u in the time interval $[-T, T]$ satisfies

$$\begin{aligned}
\|\partial_x u\|_{L^\infty_x L^2_T} &\le c_0\|u\|_2 + c_0\|u^2\partial_x u\|_{L^1_T L^2_x} \\
&\le c_0\|u\|_2 + c_0 T^{1/2}\|u^2\partial_x u\|_{L^2_x L^2_T} \\
&\le c_0\|u\|_2 + c_0 T^{1/2}\|u\|^2_{L^4_x L^\infty_T}\|\partial_x u\|_{L^\infty_x L^2_T} \\
&\le c_0\|u\|_2 + c_0 T^{1/2}(\mu_1^T(u))^2\|\partial_x u\|_{L^\infty_x L^2_T} \\
&\le c_0\|u\|_2 + \frac{1}{2}\|\partial_x u\|_{L^\infty_x L^2_T}.
\end{aligned} \tag{2.14}$$

This gives the *a priori* estimate

$$\|\partial_x u\|_{L^\infty_x L^2_T} \le 2c_0\|u_0\|_2.$$

By uniqueness we have

$$u \in C([-T,T]\,:\,\dot{H}^{1/4}(\mathbb{R})) \cap C([-T_0,T_0]\,:\,H^{1/4}(\mathbb{R})),$$

therefore, using the L^2-conservation law and the *a priori* estimate in (2.14) we can reapply the above argument to extend our solution u to the whole interval $[-T,T]$ with T as in (2.10) to get that

$$u \in C([-T,T]\,:\,H^{1/4}(\mathbb{R})),$$

with

$$\mu_2^T(u) \le 4c_0(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2).$$

Now we turn our attention to the most interesting case in Theorem 1

$$u_0 \in Z_{1/4,1/8} = H^{1/4}(\mathbb{R}) \cap L^2(|x|^{1/4}\,dx),$$

and introduce the notation

$$\mu_3^{\widetilde{T}}(w) = \|\,|x|^{1/8}w(t)\|_{L^\infty_{\widetilde{T}}L^2_x} + \|\,|x|^{1/8}w\|_{L^5_xL^{10}_{\widetilde{T}}},$$

with $\widetilde{T} > 0$ to be fixed below.

From Lemma A (see (1.11)-(1.12)) and the linear estimates in (2.4) it follows that

$$\mu_3^{\widetilde{T}}(U(t)u_0) \le c_0\||x|^{1/8}u_0\|_2 + c_0(1+\widetilde{T})(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2). \tag{2.15}$$

Hence,

$$\begin{aligned}
\mu_3^{\widetilde{T}}(\int_0^t U(t-t')v^2\partial_x v(t')dt') &\le c_0\||x|^{1/8}(v^2\partial_x v)\|_{L^1_{\widetilde{T}}L^2_x}\\
&\quad + c_0(1+\widetilde{T})(\|v^2\partial_x v\|_{L^1_{\widetilde{T}}L^2_x} + \|D_x^{1/4}(v^2\partial_x v)\|_{L^1_{\widetilde{T}}L^2_x})\\
&\le c_0\widetilde{T}^{1/2}\||x|^{1/8}(v^2\partial_x v)\|_{L^2_xL^2_{\widetilde{T}}}\\
&\quad + c_0(1+\widetilde{T})\widetilde{T}^{1/2}(\|v^2\partial_x v\|_{L^2_xL^2_{\widetilde{T}}} + \|D_x^{1/4}(v^2\partial_x v)\|_{L^2_xL^2_{\widetilde{T}}}).
\end{aligned} \tag{2.16}$$

We shall use that

$$\begin{aligned}
&\||x|^{1/8}(v^2\ \partial_x v)\|_{L^2_xL^2_{\widetilde{T}}}\\
&\quad \le c_0\|v\|_{L^4_xL^\infty_{\widetilde{T}}}\||x|^{1/8}v\|_{L^5_xL^{10}_{\widetilde{T}}}\ \|\partial_x v\|_{L^{20}_xL^{5/2}_{\widetilde{T}}}\\
&\quad \le c_0(\mu_1^{\widetilde{T}}(v))^2\,\mu_3^{\widetilde{T}}(v).
\end{aligned} \tag{2.17}$$

Inserting the estimates (2.15)-(2.17), (2.6), and (2.13) in the integral equation (2.3) it follows that

$$\begin{aligned}
\mu_3^{\widetilde{T}}(\Phi(v)) &\le c_0\||x|^{1/8}u_0\|_2 + c_0(1+\widetilde{T})(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2)\\
&\quad + c_0\int_0^{\widetilde{T}}\||x|^{1/8}(v^2\ \partial_x v)\|_2(t)dt\\
&\quad + c_0\,(1+\widetilde{T})\int_0^{\widetilde{T}}(\|v^2\partial_x v\|_2 + \|D_x^{1/4}(v^2\ \partial_x v)\|_2)dt\\
&\le c_0\||x|^{1/8}u_0\|_2 + c_0(1+\widetilde{T})(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2)\\
&\quad + c_0\,\widetilde{T}^{1/2}(\mu_1^{\widetilde{T}}(v))^2\,\mu_3^{\widetilde{T}}(v) + c_0(1+\widetilde{T})\widetilde{T}^{1/2}(\mu_1^{\widetilde{T}}(v))^3.
\end{aligned} \tag{2.18}$$

At this point, we observe that the estimate (2.18) provides an *a priori* estimate of $\mu_3^T(u)$ with T as in (2.10) and u the solution constructed in (2.1)-(2.11). More precisely, the above argument applied to the solution $u = u(x,t)$ yields the inequalities

$$\begin{aligned}\mu_3^T(u) \le c_0\||x|^{1/8}u_0\|_2 &+ c_0(1+T)(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2)\\ &+ c_0\, T^{1/2}(\mu_1^T(u))^2\, \mu_3^T(u) + c_0(1+T)T^{1/2}(\mu_1^T(u))^3\\ &\le c_0\||x|^{1/8}u_0\|_2 + c_0(1+T)(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2)\\ &+ \frac{1}{2}\,\mu_3^T(u) + \frac{1}{2}\,(1+T)\mu_1^T(u),\end{aligned}$$

by using that

$$c_0 T^{1/2}(\mu_1^T(u))^2 \le 1/2,$$

(see (2.10) and (2.11)). Hence,

$$\begin{aligned}\mu_3^T(u) \le& 2c_0\||x|^{1/8}u_0\|_2\\ &+ 2c_0(1+T)(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2) + (1+T)\mu_1^T(u)\\ \le& 2c_0\||x|^{1/8}u_0\|_2 + 6c_0\left(\|u_0\|_2 + \|D_x^{1/4}u_0\|_2 + \frac{1}{8c_0^3\|D_x^{1/4}u_0\|_2^3}\right)\\ \equiv& M_0^T(u_0).\end{aligned} \tag{2.19}$$

A similar argument to that employed to deduce (2.18) shows that

$$\begin{aligned}\mu_3^{\widetilde{T}}(\Phi(v) - \Phi(\widetilde{v})) \le& c_0\,\widetilde{T}^{1/2}(\mu_1^{\widetilde{T}}(v) + \mu_1^{\widetilde{T}}(\widetilde{v}))^2\,\mu_3^{\widetilde{T}}(v-\widetilde{v})\\ &+ c_0\widetilde{T}^{1/2}\,\mu_1^{\widetilde{T}}(v)\,\mu_3^{\widetilde{T}}(v)\,\mu_1^{\widetilde{T}}(v-\widetilde{v})\\ &+ c_0(1+\widetilde{T})\,\widetilde{T}^{1/2}(\mu_1^{\widetilde{T}}(v) + \mu_1^{\widetilde{T}}(\widetilde{v}))^2\,\mu_1^{\widetilde{T}}(v-\widetilde{v}).\end{aligned}$$

Finally, we define

$$\mu^{\widetilde{T}}(w) = \mu_1^{\widetilde{T}}(w) + \mu_3^{\widetilde{T}}(w),$$

and consider a general data

$$\widetilde{u}_0 \in Z_{1/4,1/8} = H^{1/4}(\mathbb{R}) \cap L^2(|x|^{1/4}dx).$$

Thus, using the notation in (2.2)-(2.3) and collecting the above information we have

$$\begin{aligned}\mu^{\widetilde{T}}(\Phi_{\widetilde{u}_0}(v)) \le& c_0\||x|^{1/8}\widetilde{u}_0\|_2 + c_0(1+\widetilde{T})(\|\widetilde{u}_0\|_2 + \|D_x^{1/4}\widetilde{u}_0\|_2)\\ &+ c_0\,\widetilde{T}^{1/2}(1+\widetilde{T})\,(\mu^{\widetilde{T}}(v))^3\\ \le& c_0\||x|^{1/8}\widetilde{u}_0\|_2 + c_0(\|\widetilde{u}_0\|_2 + \|D_x^{1/4}\widetilde{u}_0\|_2)\\ &+ c_0\,\widetilde{T}(\|\widetilde{u}_0\|_2 + \|D_x^{1/4}\widetilde{u}_0\|_2) + c_0\,\widetilde{T}^{1/2}(1+\widetilde{T})\,(\mu^{\widetilde{T}}(v))^3,\end{aligned}$$

and

$$\mu^{\widetilde{T}}(\Phi_{\widetilde{u}_0}(v) - \Phi_{\widetilde{u}_0}(\widetilde{v})) \le c_0\,\widetilde{T}^{1/2}(1+\widetilde{T})(\mu^{\widetilde{T}}(v) + \mu^{\widetilde{T}}(\widetilde{v}))^2\mu^{\widetilde{T}}(v-\widetilde{v}).$$

Defining

$$\delta = \|\widetilde{u}_0\|_2 + \|D_x^{1/4}\widetilde{u}_0\|_2 + \||x|^{1/8}\widetilde{u}_0\|_2,$$

it follows that the operator $\Phi_{\widetilde{u}_0}$ (see (2.3)) defines a contraction in the set

$$\Omega_{\widetilde{T}} = \{v : \mathbb{R}\times[-T,T] \to \mathbb{R} \,:\, \mu^{\widetilde{T}}(v) \le 2c_0\delta\},$$

into itself if

$$c_0\widetilde{T}^{1/2}(1+\widetilde{T})(2c_0\delta)^2 \le 1/10,$$

and

$$c_0\widetilde{T}\delta + c_0\widetilde{T}^{1/2}(1+\widetilde{T})(2c_0\delta)^3 \le c_0\delta.$$

So we need to have

$$c_0\widetilde{T}^{1/2}(1+\widetilde{T})(2c_0\delta)^2 \le 1/10, \quad \text{ and } \quad \widetilde{T} \le 1/5.$$

Hence, it suffices to take

$$\widetilde{T}(\delta) = min\{\,\frac{1}{5}\,;\,\frac{1}{(80)^2\,c_0^6\,\delta^4}\}.$$

This guarantees LWP results in $H^{1/4}(\mathbb{R}) \cap L^2(|x|^{1/8})$ for the IVP (1.1) with $k=2$ in the time interval $[-\widetilde{T},\widetilde{T}]$. We recall that *a priori* we know that in the time interval $[-T,T]$ with T as in (2.10) one has that

$$\sup_{[-T,T]} (\|u(t)\|_2 + \|D_x^{1/4}u(t)\|_2 + \||x|^{1/8}u(t)\|_2) \\ \le \|u_0\|_2 + 2c_0\|D_x^{1/4}u_0\|_2 + M_0^T(u_0),$$

with $M_0^T(u_0)$ defined in (2.19). Thus, taking

$$\delta_0 = \|u_0\|_2 + 2c_0\|D_x^{1/4}u_0\|_2 + M_0^T(u_0),$$

we obtain a uniform estimate for $\widetilde{T}(\delta_0)$ in the whole time interval $[-T,T]$ which allows us to reapply the local existence theorem above $2T/\widetilde{T}(\delta_0)$-times to get the local solution to the whole time interval $[-T,T]$.

Proof of Theorem 2:

We shall consider the most interesting case $s=1/4$, and recall that the L^2-norm of the solution $u(t)$ is preserved.

By Theorem B for any given $T^*>0$ and $u_0 \in H^{1/4}(\mathbb{R})$ one has that the corresponding solution $u=u(x,t)$ of the IVP (1.1) with $k=2$ satisfies

$$u \in C([-T^*,T^*]\,:\,H^{1/4}(\mathbb{R})) \cap ....$$

Let

$$K = \max_{[-T^*,T^*]} \|D_x^{1/4}u(t)\|_2.$$

Following (2.10) we define

$$T' = \frac{1}{64\,c_0^6\,K^4},$$

and split the interval $[-T^*,T^*]$ into $2T^*/T'$ sub-intervals. In each of these sub-intervals we can apply Theorem 1 to get the desired solution to the whole interval $[-T^*,T^*]$.

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