DIHEDRAL LONG ROOT A-PACKETS OF p-ADIC G_2 VIA THETA CORRESPONDENCE

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ABSTRACT. We construct local Arthur packets associated with a dihedral long root A-parameter of a split reductive group of type G_2 over a nonarchimedian local field of characteristic zero. The construction relies on an exceptional correspondence for the pair $(\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}, G_2)$.

CONTENTS

1. Introduction

Let G be a connected reductive linear algebraic group over a number field F. In [Art89, Art90], J. Arthur has given a conjectural description of the constituents of the square integrable automorphic representations $\mathcal{A}_2(G)$ of G. The conjecture predicts that there is a decomposition

$$\mathcal{A}_2(G) = \bigoplus_\psi \mathcal{A}_{2,\psi}$$

where each $\mathcal{A}_{2,\psi}$ is (to a first approximation) a near equivalence class of representations and the sum runs over equivalence classes of discrete A-parameters ψ. A-parameters are admissible maps

$$\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G,$$

where L_F denotes the conjectural Langlands group of F and ${}^L G = \hat{G}(\mathbb{C}) \rtimes W_F$ denotes the L-group of G. Here $\hat{G}(\mathbb{C})$ is the complex dual group of G and W_F is the Weil group of F. We say that ψ is discrete if the component group S_ψ, defined as the centralizer of the image of ψ in $\hat{G}(\mathbb{C})$ modulo the center of $\hat{G}(\mathbb{C})$, is finite. Fix such an A-parameter ψ. Arthur's conjecture gives a more precise description of the constituents of $\mathcal{A}_{2,\psi}$. It first describes the local components of the representations appearing in $\mathcal{A}_{2,\psi}$, via the so-called *local A-packets*, and then determines which combinations of such local representations appear globally, using the *global A-packets* and the *multiplicity formula*.

Let v be a place of F, let F_v be the completion of F at v and denote by L_{F_v} the group $W_{F_v} \times \mathrm{SU}_2(\mathbb{C})$, where W_{F_v} is the Weil group of F_v. We can pre-compose ψ with a fixed embedding $L_{F_v} \hookrightarrow L_F$ to obtain the local A-parameter $\psi_v : L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$. Define the local component group of ψ_v as

$$S_{\psi_v} = \pi_0 \left(Z_{\hat{G}(\mathbb{C})}(\mathrm{Im}(\psi_v)) Z(\hat{G}(\mathbb{C})) / Z(\hat{G}(\mathbb{C})) \right),$$

where $Z_{\hat{G}(\mathbb{C})}(\mathrm{Im}(\psi_v))$ denotes the centralizer in $\hat{G}(\mathbb{C}) \subset {}^L G$ of the image of ψ_v, $Z(\hat{G}(\mathbb{C}))$ denotes the center of $\hat{G}(\mathbb{C})$ and π_0 denotes the group of connected components. Arthur predicted that to each irreducible representation η_v of S_{ψ_v}, we can attach a unitarizable finite length (possibly reducible, possibly zero) representation π_{η_v} of $G(F_v)$. The collection

$$A_{\psi_v} = \{ \pi_{\eta_v} \mid \eta_v \in \mathrm{Irr}(S_{\psi_v}) \}$$

is the local A-packet associated to ψ_v. There are several requirements on the representations in A_{ψ_v}. One of them is that if we let 1_v denote the trivial representation of S_{ψ_v}, then for all but finitely many v, π_{1_v} is the unramified representation with Satake parameter

$$s_{\psi_v} = \psi_v \left(\Phi_v \times \begin{pmatrix} q_v^{-1/2} & 0 \\ 0 & q_v^{1/2} \end{pmatrix} \right).$$

Here Φ_v denotes a geometric Frobenius element at v and q_v is the size of the residue field at v. Given the local A-packets, we define the global A-packet associated to ψ as

$$A_\psi = \left\{ \pi = \otimes_v' \pi_{\eta_v} \mid \pi_{\eta_v} \in A_{\psi_v} \text{ for all } v \text{ and } \eta_v = 1_v \text{ for all but finitely many } v \right\}.$$

Note that this is a set of nearly equivalent representations of $G(\mathbb{A}_F)$, which is indexed by irreducible representations of $S_{\psi,\mathbb{A}_F} = \prod_v S_{\psi_v}$. For a given $\eta = \otimes_v \eta_v$, where $\eta_v \in \mathrm{Irr}(S_{\psi_v})$ and $\eta_v = 1_v$ for all but finitely many v, set $\pi_\eta = \otimes'_v \pi_{\eta_v}$. Arthur constructed a quadratic character ϵ_ψ of S_ψ and used it to determine the multiplicity of each representation $\pi_\eta \in A_\psi$ appearing in $\mathcal{A}_{2,\psi}$. This yields the multiplicity formula

$$\mathcal{A}_{2,\psi} = \bigoplus_\eta m_\eta \pi_\eta, \quad \text{where} \quad m_\eta = \frac{1}{\# S_\psi} \left(\sum_{s \in S_\psi} \epsilon_\psi(s) \eta(s) \right).$$

It is worth mentioning that the conjecture also predicts when all the representations in $\mathcal{A}_{2,\psi}$ are tempered, which should occur when the A-parameter ψ is tempered (if ψ restricted to $\mathrm{SL}_2(\mathbb{C})$ is trivial). On the other hand, nontempered A-parameters lead to local A-packets that can contain both tempered and nontempered representations. An important feature about nontempered A-parameters is that they usually factor through subgroups $^L H \subset {}^L G$. It is then expected that we can construct the representations in the (local and global) A-packets associated to ψ from representations of H.

From now on suppose that G is a split exceptional group of type G_2 over F. Note that $^L G$ can be replaced by $G(\mathbb{C})$ and that there are 4 different conjugacy classes of morphisms $\mathrm{SL}_2(\mathbb{C}) \to G(\mathbb{C})$ corresponding to the 4 nontrivial unipotent conjugacy classes in $G(\mathbb{C})$. They give rise to 4 families of nontempered A-parameters ψ for G_2: if $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ corresponds to the regular orbit, if $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ corresponds to the subregular orbit, if $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ gives the short root SL_2 in G_2 and if $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ gives the long root SL_2 in G_2. For the first three mentioned families of nontempered A-parameters, Arthur's conjecture has been verified; see [GGJ02] and [GG06]. This work is part of a larger project initiated at the 2022 Arizona Winter School which aims to verify Arthur's conjecture for the so-called *dihedral long root A-parameters*, a type of nontempered A-parameters for G_2 which belong to the fourth mentioned family. In particular, in this paper we construct the local nonarchimedean A-packets associated to dihedral long root A-parameters. We proceed to briefly introduce dihedral long root A-parameters and then we summarize how the corresponding local A-packets are constructed. For that we use a theta lift from $\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ to G_2 arising from the exceptional theta correspondence for $(\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2$ studied in [GS04], [GS23] and especially in [BS].

Let K be a quadratic extension of F and c be the nontrivial element in the Galois group $\mathrm{Gal}(K/F)$. Denote by χ a character of $\mathbb{A}_K^\times / K^\times$ which is conjugate symplectic, *i.e.*, $\chi_{|\mathbb{A}_F^\times} = \omega_{K/F}$, where $\omega_{K/F}$ is the quadratic character attached to the extension K/F. Moreover, suppose that $\chi^c \neq \chi$. Note that we can regard χ as a character of W_K. Let $\tau = \otimes'_v \tau_v$ be the representation of $\mathrm{GL}_2(\mathbb{A}_F)$ obtained from χ by automorphic induction, namely τ is the automorphic representation with L-parameter

$$\rho_\tau = \mathrm{Ind}_{W_K}^{W_F} \chi.$$

Note that the central character of τ is $\omega_{K/F} \cdot \chi_{|\mathbb{A}_F^\times} = 1$. Therefore, we regard ρ_τ as a representation of PGL_2. Denote by $\mathrm{SL}_{2,l}$ (resp. $\mathrm{SL}_{2,s}$) the long (resp. short) root SL_2 inside G_2. We define the long root A-parameter of G associated with τ as

$$\psi_{\tau,l} : L_F \times \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow W_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\rho_\tau \times \mathrm{id}} \mathrm{SL}_{2,s}(\mathbb{C}) \times_{\mu_2} \mathrm{SL}_{2,l}(\mathbb{C}) \subset G(\mathbb{C}).$$

There is a subgroup $\mathrm{SL}_{3,l}(\mathbb{C}) \subset G(\mathbb{C})$ corresponding to the root system of type A_2 formed by the six long roots of G. Its normalizer inside $G(\mathbb{C})$ is isomorphic to $\mathrm{SL}_{3,l}(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$. This is at the same time isomorphic to the L-group of PU_3, the projective unitary group in three variables associated to the extension K/F.

$$N_{G(\mathbb{C})}(\mathrm{SL}_{3,l}(\mathbb{C})) \simeq \mathrm{SL}_{3,l}(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} =^L \mathrm{PU}_3.$$

In Section 4 we verify that $\psi_{\tau,l}$ can be conjugated to take values in that normalizer. We can therefore define the restriction

$$\psi_\chi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to^L \mathrm{PU}_3.$$

As discussed above, the fact that the A-parameter $\psi_{\tau,l}$ factors through an A-parameter of PU_3 suggests that we can obtain the local A-packets for $\psi_{\tau,l}$ from the local A-packets for ψ_χ. We explain this phenomenon when v is a nonarchimedean place of F which is nonsplit in K, as this is the most interesting case.

Fix a place of K above v, denote by K_v the completion of K at that place and by χ_v the corresponding local component of χ. The local component group of $\psi_{\chi,v}$ has two elements. Therefore, the local A-packet of PU_3 associated to ψ_{χ_v} has the form

$$A_{\psi_{\chi,v}} = \left\{ \sigma_v^+, \sigma_v^- \right\},$$

where σ_v^+ (resp. σ_v^-) corresponds to the trivial (resp. nontrivial) representation of the local component group. These representations can be obtained as theta lifts using the classical local theta correspondence between the unitary groups U_1 and U_3, as we explain in Section 4. In particular, σ_v^+ is nontempered and σ_v^- is supercuspidal.

On the other hand, the local component group of $\psi_{\tau,l,v}$ has either two elements if $\chi_v^2 \neq 1$ or one element otherwise. Therefore, we expect that the local A-packet associated to $\psi_{\tau,l}$ has the form

$$A_{\psi_{\tau,l,v}} = \begin{cases} \{\pi_v^+, \pi_v^-\} \text{ if } \chi_v^2 \neq 1, \\ \{\pi_v^+\} \text{ if } \chi_v^2 = 1. \end{cases}$$

Here, π_v^+ corresponds to the trivial representation of $S_{\psi_{\tau,l,v}}$ and, when the local component group has two elements, π_v^- is the representation corresponding to the nontrivial representation of $S_{\psi_{\tau,l,v}}$. Denote by Θ_{PU_3} the exceptional big theta lift from PU_3 to G considered in [BS] and by θ_{PU_3} the corresponding small theta lift. Let Q_1 be the non-Heisenberg parabolic subgroup of $G(F_v)$, which has a Levi subgroup isomorphic to $\mathrm{GL}_2(F_v)$, and denote by $i_{Q_1}^G$ the normalized parabolic induction from Q_1 to $G(F_v)$. The main result of this paper is the following.

Theorem 1.1. *Let v be a non-archimedean place of F which is nonsplit in K. Let $A_{\psi_{\chi,v}} = \{\sigma_v^+, \sigma_v^-\}$ be the local A-packet associated to the A-parameter ψ_χ of PU_3, where σ_v^+ is nontempered and σ_v^- is supercuspidal. Define $\pi_v^\pm = \theta_{\mathrm{PU}_3}(\sigma^\pm)$. Then:*

(1) *The representation π_v^+ is the unique nonzero irreducible quotient of $i_{Q_1}^G(|\det|_{F_v}^{1/2} \tau_v)$, where $|\ |_{F_v}$ denotes the normalized absolute value on F_v.*

(2) *If $\chi_v^2 \neq 1$, the representation π_v^- is nonzero, irreducible and tempered.*

(3) *If $\chi_v^2 = 1$, the representation π_v^- is zero.*

Note that as a consequence of the main theorem, we obtain a natural construction of the elements of the local A-packet $A_{\psi_{\tau,l,v}}$ as the nonzero lifts of the elements of $A_{\psi_\chi,v}$ via θ_{PU_3}. Using a similar strategy, we construct the local A-packet $A_{\psi_{\tau,l,v}}$ when v is a non-archimedean place of F that is split in K (see Theorem 4.12).

The first point of the main theorem is a consequence of the work of [BS] on lifts of nontempered representations. To prove the second point we compute Fourier–Jacobi periods of the minimal representation used to define θ_{PU_3} to obtain the following non-vanishing criterion: if the contragradient of a local representation of PU_3, when restricted to a suitable two-variable unitary subgroup, has a quotient with trivial central character, then this representation has nonzero theta lift to G. We then verify that π_v^- satisfies this condition if $\chi_v^2 \neq 1$ using a see-saw argument. In fact, this same reasoning shows that π_v^+ is nonzero in all cases, giving an alternative proof of the nonvanishing part in the first point of the theorem. The key ingredient to prove the third point of the theorem is that the twisted coinvariant spaces for π_v^- corresponding to generic characters of the unipotent of the Heisenberg group of $G(F_v)$ vanish. This is proven using that for every such character we have:

- The explicit description of π_v^+ allows to verify that the twisted coinvariant space for π_v^+ is 1-dimensional.
- The twisted coinvariant space for $\pi_v^+ \oplus \pi_v^-$ can be related to a sum of toric periods for $\sigma_v^+ \oplus \sigma_v^-$. Since the representations $\sigma_v^\pm$ are theta lifts of characters in U_1, the non-vanishing of these periods can be expressed in terms of local epsilon factors, as it is done in [BFG$^+$]. Moreover, in [loc. cit.], it is proven that exactly one of these toric periods contributes to the sum with precisely dimension 1.

We expect that a similar construction should yield the local A-packets in the archimedean case, but the theta correspondence in the archimedean case has not been fully analysed yet. We refer the reader to [BHLHS], where a definition of the local A-packets in this setting, which can be related to theta lifts in some cases, is proposed.

The p-adic construction presented in this article is used in [loc. cit.] to construct global dihedral long root A-packets and, under the hypothesis that $L(\chi, 1/2) \neq 0$ and certain conditions regarding the archimedean construction, verify the multiplicity formula.

We conclude the introduction with notes about the organisation of this paper. In Section 2, we establish notation for unitary groups and recall essential results about the classical local theta correspondence between U_1 and U_3. In Section 3, we introduce the exceptional group of type G_2, the local exceptional theta correspondence between this group and $\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$, and review results about this correspondence that we will need later. In Section 4, we describe the construction of the local A-packets associated to $\psi_{\tau,l}$ for all finite primes. In Section 5, we prove nonvanishing of the representation π_v^- in the case $\chi_v^2 \neq 1$. Finally, in Section 6, we prove the vanishing of π_v^- in the case $\chi_v^2 = 1$.

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2. Theta correspondence for unitary groups

In this section, we work out the local theta correspondence for the dual pair (U_1, U_3). We start by recalling some general structure theory for unitary groups in Section 2.1, followed by a description of the theta correspondence for general unitary groups in Section 2.2. We then apply this theory to the dual pair (U_1, U_3) in Section 2.3 for nonsplit places, and (GL_1, GL_3) in Section 2.4 for split places.

2.1. **Definitions.** Let F be a nonarchimedean local field of characteristic zero and let K be a quadratic field extension with $\mathrm{Gal}(K/F) = \langle c \rangle$. Let $\omega_{K/F}$ be the nontrivial quadratic character of $F^\times / N_{K/F}(K^\times)$. A finite-dimensional Hermitian (resp. skew-Hermitian) space over K is a finite dimensional vector space over K equipped with a nondegenerate sesquilinear form $\langle\,,\,\rangle$ satisfying $\langle v, w \rangle^c = \langle w, v \rangle$ (resp. $\langle v, w \rangle^c = -\langle w, v \rangle$). We adopt the convention that sesquilinear forms are linear on the first variable and conjugate-linear on the second variable.

For each positive integer n, there are two isomorphism classes of Hermitian spaces of dimension n over K. Given a Hermitian space of dimension n over K, its isomorphism class is determined by an invariant known as the discriminant, which we now define.

Let V be a Hermitian space of dimension n over K. Let $\{v_1, v_2, \ldots, v_n\}$ be a K-basis of V and let $\Phi = (\langle v_i, v_j \rangle)$ be the matrix of inner products of the basis elements. Then, the discriminant of V, which we denote by $\mathrm{disc}(V)$, is defined by

$$\mathrm{disc}(V) = (-1)^{n(n-1)/2} \det(\Phi) \in F^\times / N_{K/F}(K^\times).$$

The sign character that classifies V is given by

$$\epsilon(V) = \omega_{K/F}(\mathrm{disc}(V)) \in \{\pm 1\}. \tag{1}$$

If W is a skew-Hermitian space of dimension m, we can define the discriminant using the same procedure, but now we get

$$\mathrm{disc}(W) \in \delta^m F^\times / N_{K/F}(K^\times),$$

where δ denotes any trace-zero element in $K^\times$. We can again attach a sign to W by setting

$$\epsilon(W) = \omega_{K/F}(\delta^{-m} \mathrm{disc}(W)) \in \{\pm 1\}, \tag{2}$$

but note that, if m is odd, this definition depends on our choice of δ.

For an n-dimensional Hermitian or skew-Hermitian space V, we denote by $\underline{U}(V)$ the corresponding unitary group. This is an algebraic group over F. Choose a K-basis of V and let Φ be the matrix of the (skew-)Hermitian form on V with respect to this basis. Then, for any

F-algebra R, the R-points of $\underline{U}(V)$ can be described as

$$\underline{U}(V)(R) = \{g \in \mathrm{GL}_n(K \otimes_F R) : g\Phi g^\dagger = \Phi\}.$$

We are primarily interested in the group of F-points of $\underline{U}(V)$, which we denote by $U(V)$.

Observe that a Hermitian form becomes skew-Hermitian after multiplication by a trace-zero element in $K^\times$ and vice versa, without changing the associated unitary groups. Therefore, from now on we focus on Hermitian spaces.

Let V be an n-dimensional Hermitian space with Hermitian form $\langle\,,\,\rangle$. Let $a \in F^\times$ and let V^a denote the Hermitian space with the same underlying space V equipped with the Hermitian form $a\langle\,,\,\rangle$. From the above description of the associated unitary groups, it is clear that $\underline{U}(V^a) = \underline{U}(V)$.

If $n = 2m$ for a positive integer m, we have that $V \simeq V^a$ from the definition of the discriminant, and it can be proved that non-isomorphic Hermitian spaces yield non-isomorphic unitary groups. When $m = 1$ and $\epsilon(V) = 1$, we can choose a basis $\{e_1, e_2\}$ so that $V = Ke_1 \oplus Ke_2$ and

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 0 \text{ and } \langle e_1, e_2 \rangle = 1.$$

We call this 2-dimensional space the hyperbolic plane and denote it by $\mathbb{H}$. More generally, when $\epsilon(V) = 1$, then $V \simeq \mathbb{H}^m$. We say that such a V is split and the corresponding unitary group is quasi-split. In this case, we can choose a basis for V for which the Hermitian form is given by

$$\Phi = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Once we fix such a basis, the subgroup of upper-triangular matrices in $\underline{U}(V)$ defines a Borel subgroup, which we sometimes refer to as the standard Borel subgroup. Let B' denote the F-points of this Borel subgroup and let $T' \subset B'$ denote the F-points of the maximal torus consisting of diagonal matrices. Then T' consists of the elements of the form

$$t(a_1, \ldots, a_m) = \mathrm{diag}\left(a_m, ..., a_1, (a_1^c)^{-1}, ..., (a_m^c)^{-1}\right), \quad \text{with } a_1, \ldots, a_m \in K^\times,$$

so we get an identification $T' \simeq (K^\times)^m$. If $\epsilon(V) = -1$, then V is isomorphic to the orthogonal direct sum of $m - 1$ hyperbolic planes and an anisotropic two-dimensional Hermitian space. In this case, V is nonsplit and the corresponding unitary group is not quasi-split.

If $n = 2m+1$, then for an element $a \in F^\times$ which is not a norm from $K^\times$, the Hermitian spaces V and V^a are not isomorphic. It follows that there is only one isomorphism class of unitary groups in n-variables, which are always quasi-split. We denote any element in this isomorphism class by $\underline{U}_n$, and its F-points by U_n. Then V is isomorphic to the orthogonal direct sum of m hyperbolic planes and a line. We can choose a basis of V for which the corresponding Hermitian

form is given by

$$\Phi = \begin{pmatrix} 0 & 0 & \cdots & 0 & b \\ 0 & 0 & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b & \cdots & 0 & 0 \\ b & 0 & \cdots & 0 & 0 \end{pmatrix},$$

with the class of b in $F^\times / \mathrm{N}_{K/F}(K^\times)$ determined by the discriminant of V. Once we fix such a basis, the subgroup of upper-triangular matrices in $\underline{\mathrm{U}}(V)$ defines a Borel subgroup, which we sometimes refer to as the standard Borel subgroup. Let B' denote the F-points of this Borel subgroup and let $T' \subset B$ denote the F-points of the maximal torus consisting of diagonal matrices. Then T' consists of the elements of the form

$$t(a_0, a_1, \ldots, a_m) = \mathrm{diag}\left(a_m, ..., a_1, a_0, (a_1^c)^{-1}, ..., (a_m^c)^{-1}\right), \quad \text{with } a_0 \in K^1, \, a_1, \ldots, a_m \in K^\times,$$

so we get an identification $T' \simeq (K^\times)^m \times K^1$.

2.2. Theta correspondence for unitary groups.

Let F be a nonarchimedian local field of characteristic zero and let K/F be a quadratic field extension. Let V be a Hermitian space over K of dimension n and let W be a skew-Hermitian space over K of dimension m. We can regard $V \otimes_K W$ as a vector space over F equipped with the symplectic form

$$\frac{1}{2}\mathrm{Tr}_{K/F}(\langle \, , \, \rangle_V \otimes_K \langle \, , \, \rangle_W).$$

Let $\mathrm{Sp}(V \otimes_K W)$ be the symplectic group associated with this symplectic space. Then we have a natural map

$$\iota : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Sp}(V \otimes_K W),$$

and $\mathrm{U}(V)$ and $\mathrm{U}(W)$ form a reductive dual pair inside $\mathrm{Sp}(V \otimes_K W)$. The aim of this section is to describe the theta correspondence for this pair.

Fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. Let $\mathrm{Mp}(V \otimes_K W)$ be the metaplectic group associated with the symplectic space $V \otimes_K W$, which for us will be an S^1-cover of the symplectic group $\mathrm{Sp}(V \otimes_K W)$. Let ω_ψ denote the Weil representation of $\mathrm{Mp}(V \otimes_K W)$ corresponding to the character ψ.

After fixing two characters χ_V, χ_W of $K^\times$ such that

$$\chi_{V|F^\times} = \omega_{K/F}^n \text{ and } \chi_{W|F^\times} = \omega_{K/F}^m,$$

the work of Kudla [Kud94] provides a morphism

$$\tilde{\iota}_{\chi_V, \chi_W, \psi} : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Mp}(V \otimes_K W)$$

lifting the natural map $\iota : \mathrm{U}(V) \times \mathrm{U}(W) \to \mathrm{Sp}(V \otimes_K W)$. Hence, we can consider the representation of $\mathrm{U}(V) \times \mathrm{U}(W)$ obtained as the pullback of ω_ψ by $\tilde{\iota}_{\chi_V, \chi_W, \psi}$. We will denote this representation by $\Omega_{V,W,\chi_V,\chi_W,\psi}$, or simply by Ω. We use this representation to describe the theta correspondence between $\mathrm{U}(W)$ and $\mathrm{U}(V)$.

Definition 2.1. Let π be an irreducible smooth representation of U(W). The maximal π-isotypic quotient of Ω is

$$\Omega / \bigcap_{f \in \mathrm{Hom}_{\mathrm{U}(W)}(\Omega, \pi)} \ker(f).$$

This is a representation of U(V) × U(W) and can be written as $\Theta(\pi) \boxtimes \pi$, where $\Theta(\pi)$ is a smooth representation of U(V). We denote by $\Theta_{V,W,\chi_V,\chi_W,\psi}(\pi)$, or simply by $\Theta(\pi)$, the representation of U(V) obtained from π following this procedure. It is called the *big theta lift* of π.

The following theorem was a conjecture of Howe [How79]. For odd residue characteristic, a proof was given by Waldspurger [Wal90]. The assumption on the residue characteristic was removed by Gan–Takeda [GT16].

Theorem 2.2 (Howe duality theorem). *Let $\pi, \pi' \in \mathrm{Irr}(\mathrm{U}(W))$.*

(1) *If $\Theta(\pi)$ is nonzero, it has a unique irreducible quotient. We denote it by $\theta_{V,W,\chi_V,\chi_W,\psi}(\pi)$, or simply by $\theta(\pi)$. It is called the small theta lift of π.*
(2) *If $\theta(\pi) \simeq \theta(\pi') \neq 0$, then $\pi \simeq \pi'$.*

It is useful to study theta lifts by considering certain families of Hermitian spaces called Witt towers. Recall that $\mathbb{H}$ denotes the hyperbolic plane. We say that two Hermitian spaces V and V' belong to the same Witt tower if there exist integers $k, l \geq 0$ such that

$$V \oplus \mathbb{H}^k \simeq V' \oplus \mathbb{H}^l.$$

Let V_1 and V_1' denote two non-isomorphic 1-dimensional Hermitian spaces and let V_2 denote a 2-dimensional anisotropic Hermitian space. There are two Witt towers of even-dimensional Hermitian spaces

$$\mathcal{W}_0' = \{\mathbb{H}^k \ : \ k \geq 0\}, \quad \mathcal{W}_0 = \{V_2 \oplus \mathbb{H}^k \ : \ k \geq 0\},$$

and two Witt towers of odd-dimensional Hermitian spaces

$$\mathcal{W}_1' = \{V_1' \oplus \mathbb{H}^k \ : \ k \geq 0\}, \quad \mathcal{W}_1 = \{V_1 \oplus \mathbb{H}^k \ : \ k \geq 0\}.$$

We continue to denote by W a fixed m-dimensional skew-Hermitian space. Fix a character χ_W satisfying the condition stated above, i.e., such that $\chi_{W|F^\times} = \omega_{K/F}^m$. Since the parity of the dimension is the same for elements in a fixed Witt tower, we can choose the same splitting character χ_V to define the corresponding theta lifts. Thus, we fix characters χ_{odd} and χ_{even} such that $\chi_{\mathrm{odd}|F^\times} = \omega_{K/F}$ and $\chi_{\mathrm{even}|F^\times} = 1$.

For an irreducible smooth admissible representation π of U(W), we make the following definitions:

$$n_{\mathcal{W}_0'}(\pi) = \min\{\dim V \ : \ V \in \mathcal{W}_0' \text{ and } \Theta_{V,W,\chi_{\mathrm{even}},\chi_W,\psi}(\pi) \neq 0\};$$
$$n_{\mathcal{W}_0}(\pi) = \min\{\dim V \ : \ V \in \mathcal{W}_0 \text{ and } \Theta_{V,W,\chi_{\mathrm{even}},\chi_W,\psi}(\pi) \neq 0\};$$
$$n_{\mathcal{W}_1'}(\pi) = \min\{\dim V \ : \ V \in \mathcal{W}_1' \text{ and } \Theta_{V,W,\chi_{\mathrm{odd}},\chi_W,\psi}(\pi) \neq 0\};$$
$$n_{\mathcal{W}_1}(\pi) = \min\{\dim V \ : \ V \in \mathcal{W}_1 \text{ and } \Theta_{V,W,\chi_{\mathrm{odd}},\chi_W,\psi}(\pi) \neq 0\}.$$

For the definition of $n_{\mathcal{W}_0'}(\pi)$, we consider that, in the case $V = 0$, the theta lift of π is nonzero if and only if π is the 1-dimensional representation defined by the character $\chi_{\mathrm{even}} \circ i \circ \det_W$,

where i denotes the inverse of the isomorphism $K^\times / F^\times \xrightarrow{\simeq} K^1$ defined by $x \mapsto x/x^c$ and $\det_W$ denotes the natural determinant map on $\mathrm{U}(W)$.

The following result is a special case of [SZ15, Thm. 1.10] (see also the discussion preceding the statement of the theorem).

Theorem 2.3. *Let π be an irreducible smooth representation of $\mathrm{U}(W)$. Then:*

$$n_{\mathcal{W}_0'}(\pi) + n_{\mathcal{W}_0}(\pi) = 2m + 2;$$
$$n_{\mathcal{W}_1'}(\pi) + n_{\mathcal{W}_1}(\pi) = 2m + 2.$$

Moreover, for any Witt tower $\mathcal{W}$ and for any $V \in \mathcal{W}$ with $\dim V \geq n_{\mathcal{W}}(\pi)$, the corresponding theta lift $\Theta(\pi)$ is nonzero. The same results hold if we interchange the role of Hermitian and skew-Hermitian spaces.

2.3. **Theta correspondence for $\mathrm{U}_1 \times \mathrm{U}_3$.** We keep the definitions and the notation from the previous subsection. Let γ be a conjugate-symplectic character of $K^\times$, i.e., such that $\gamma_{|F^\times} = \omega_{K/F}$. Then, we can make the following choice of splitting characters:

$$\chi_V = \gamma^n, \ \chi_W = \gamma^m.$$

We denote by $\Omega_{V,W,\gamma,\psi}$ the pullback to $\mathrm{U}(V) \times \mathrm{U}(W)$ of the Weil representation ω_ψ obtained from this choice of splitting characters. For an irreducible smooth representation π of $\mathrm{U}(W)$, we denote by $\Theta_{V,W,\gamma,\psi}(\pi)$ the corresponding theta lift. Due to the choice of splitting characters, the theta correspondence preserves central characters.

Assume now that W has dimension $m = 1$ and V has dimension $n = 3$. Let V_1 denote the 1-dimensional Hermitian space in the Witt tower of V.

Proposition 2.4. *Let μ be a character of $\mathrm{U}(W)$. Then,*

(1) *if $\Theta_{V_1,W,\gamma,\psi}(\mu\gamma_{|K^1}^{-1}) = 0$, then $\Theta_{V,W,\gamma,\psi}(\mu)$ is a nonzero irreducible supercuspidal representation of $\mathrm{U}(V)$;*

(2) *if $\Theta_{V_1,W,\gamma,\psi}(\mu\gamma_{|K^1}^{-1}) \neq 0$, then $\Theta_{V,W,\gamma,\psi}(\mu)$ is a nonzero, irreducible but not supercuspidal representation of $\mathrm{U}(V)$. Moreover, $\Theta_{V,W,\gamma,\psi}(\mu)$ is a quotient of*

$$i_{B'}^{\mathrm{U}(V)} \left(\gamma| \ |_K^{1/2} \otimes \mu\gamma_{|K^1}^{-1} \right).$$

Proof. This follows from [MVW87, Théorème principal, p. 69]. Indeed, any character of $\mathrm{U}(W)$ is a supercuspidal representation. The appearance of the character $\gamma_{|K^1}^{-1}$ in our statements follows from our different choice of lifting characters. With our choices, the first case above follows from statement 1.b in [*loc. cit.*]. In the second case, we have that $\Theta_{V_1,W,\gamma,\psi}(\mu\gamma_{|K^1}^{-1}) = \mu\gamma_{|K^1}^{-1}$. Therefore, in this case, it follows from statement 1.c in [*loc. cit.*] that $r_{U'}(\Theta_{V,W,\gamma,\psi}(\mu)) = \gamma| \ |_K^{1/2} \otimes \mu\gamma_{|K^1}^{-1}$, where U' denotes the unipotent subgroup of B' and $r_{U'}$ the corresponding Jacquet functor. Thus, an application of (standard) Frobenius reciprocity shows that $\Theta_{V,W,\gamma,\psi}(\mu)$ is a subrepresentation of $i_{B'}^{\mathrm{U}(V)} \left(\gamma| \ |_K^{-1/2} \otimes \mu\gamma_{|K^1}^{-1} \right)$, whereas an application of the Bernstein form of Frobenius reciprocity shows that it is a quotient of $i_{B'}^{\mathrm{U}(V)} \left(\gamma| \ |_K^{1/2} \otimes \mu\gamma_{|K^1}^{-1} \right)$, as desired. $\square$

Remark 2.5. Most of the previous theorem can also be deduced from [Gan22, §2.12].

We will also need a criterion to determine whether $\Theta_{V_1,W,\gamma,\psi}(\mu)$ is nonzero. Fix an element $\delta \in K^\times$ of trace equal to zero. We adapt the sign characters as defined in (1) and (2) to our case. In particular, we set

$$\epsilon(W) := \omega_{K/F}(\delta^{-1}\operatorname{disc}(W)) \in \{\pm 1\}.$$

Theorem 2.6. *Let μ be a character of $\mathrm{U}(W)$. The theta lift $\Theta_{V_1,W,\gamma,\psi}(\mu)$ is nonzero if and only if*

$$\epsilon(V_1)\epsilon(W) = \epsilon_K\left(\frac{1}{2}, \gamma\tilde{\mu}^{-1}, \psi(\operatorname{Tr}_{K/F}(-\delta(\cdot)))\right).$$

where $\tilde{\mu} = \mu \circ i$ and i denotes the inverse of the isomorphism $K^\times/F^\times \xrightarrow{\simeq} K^1$ defined by $x \mapsto x/x^c$.

Proof. This is [Rog92, Proposition 3.4]. Indeed, if V_1 is the 1-dimensional Hermitian space with Hermitian form $(x,y) \mapsto 2xy^c$ and W is the 1-dimensional skew-Hermitian space with skew-Hermitian form $(x,y) \mapsto \delta xy^c$ for an element $\delta \in K^\times$ of trace zero, then it follows from [*loc. cit.*] that the theta lift $\theta_{V_1,W,\gamma,\psi}(\mu)$ is nonzero if and only if

$$\gamma(2\delta)^{-1}\mu(-1)\epsilon_K\left(\frac{1}{2}, \gamma\tilde{\mu}^{-1}, \psi \circ \operatorname{Tr}_{K/F}\right) = 1.$$

Now, in this particular case, we have $\epsilon(V_1) = 2$, $\epsilon(W) = 1$ and

$$\epsilon_K\left(\frac{1}{2}, \gamma\tilde{\mu}^{-1}, \psi(\operatorname{Tr}_{K/F}(-\delta(\cdot)))\right) = \gamma\tilde{\mu}^{-1}(-\delta)\epsilon_K\left(\frac{1}{2}, \gamma\tilde{\mu}^{-1}, \psi \circ \operatorname{Tr}_{K/F}\right)$$

$$= \gamma(-\delta)\mu(-1)\epsilon_K\left(\frac{1}{2}, \gamma\tilde{\mu}^{-1}, \psi \circ \operatorname{Tr}_{K/F}\right),$$

so $\theta_{V_1,W,\gamma,\psi}(\mu)$ is nonzero if and only if

$$\epsilon_K\left(\frac{1}{2}, \gamma\tilde{\mu}^{-1}, \psi(\operatorname{Tr}_{K/F}(-\delta(\cdot)))\right) = \gamma(-2\delta^2),$$

and, since $-\delta^2 \in \mathrm{N}_{K/F}(K^\times)$, the right hand side of the last equation becomes $\gamma(2) = \epsilon(V_1)\epsilon(W)$, so we obtain the result in this case. To obtain the result for arbitrary 1-dimensional Hermitian and skew-Hermitian spaces V_1 and W, we just need to observe that scaling the form on one of these spaces by $a \in F^\times$ amounts to replacing the Weil representation $\Omega_{V,W,\gamma,\psi}$ by $\Omega_{V,W,\gamma,\psi_a}$. $\square$

2.4. **Theta correspondence for $\mathrm{GL}_1 \times \mathrm{GL}_3$.** In this subsection, we briefly review the results that we will later need regarding the theta correspondence for the pair $\mathrm{GL}_1 \times \mathrm{GL}_3$. This is the case that arises if, in the previous setting, we replace the quadratic field extension K/F by the étale quadratic F-algebra $F \times F$. This theta correspondence has been described by Mínguez [Mín08].

Let F be a nonarchimedian local field and fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. Let $M_{n,m}(F)$ denote the space of $n \times m$ matrices with coefficients in F and let $S_{n,m}$ denote the space of locally constant compactly supported $\mathbb{C}$-valued functions on $M_{n,m}(F)$. The Weil representation ω_ψ of the metaplectic group $\mathrm{Mp}_{2nm}(F)$ can be realized on the space $S_{n,m}$. The choice of a character γ of $F^\times$ determines a lifting

$$\tilde{\iota}_{\gamma,\psi} : \mathrm{GL}_n(F) \times \mathrm{GL}_m(F) \longrightarrow \mathrm{Mp}_{2nm}(F)$$

of the natural map

$$\iota : \mathrm{GL}_n(F) \times \mathrm{GL}_m(F) \longrightarrow \mathrm{Sp}_{2nm}(F)$$

defined by

$$(\tilde{\iota}_{\gamma,\psi}(g,h)f)(x) = \gamma(\det(g))^m |\det(g)|_F^{m/2} f(g^T xh) \gamma(\det(h))^n |\det(h)|_F^{n/2}$$

for all $g \in \mathrm{GL}_n(F)$, $h \in \mathrm{GL}_m(F)$, $x \in M_{n,m}(F)$ and $f \in S_{n,m}$. Observe that our conventions differ from those adopted in [Mín08].

Using the lifting $\tilde{\iota}_{\gamma,\psi}$, we can define the big theta lift $\Theta(\pi)$ of a smooth admissible irreducible representation π of $\mathrm{GL}_m(F)$ and, if it is nonzero, the small theta lift $\theta(\pi)$.

Specialize now to the case $m = 1$ and $n = 3$. The main result in [Mín08] is as follows.

Theorem 2.7. *Let μ be a character of $\mathrm{GL}_1(F)$. Then $\Theta(\mu)$ is nonzero and $\theta(\mu)$ is the Langlands quotient $\pi(\gamma|\ |_F^{1/2}, \mu\gamma^{-2}, \gamma|\ |_F^{-1/2})$ of $i_{B'}^{\mathrm{GL}_3}(\gamma|\ |_F^{1/2} \otimes \mu\gamma^{-2} \otimes \gamma|\ |_F^{-1/2})$, where B' denotes the standard Borel subgroup of GL_3.*

3. THETA CORRESPONDENCE FOR THE PAIR $(\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}, G_2)$

In this section, we collect general results on the theta correspondence for $(\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}, G_2)$. In Section 3.1, we recall the basic facts for the group G_2 and its parabolic subgroups that play a role in this paper. In Section 3.2, we do the same for the group $\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)$. In Section 3.3, we discuss the general theory of theta correspondence for $(\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}, G_2)$. This draws from theta lifting for (PU_3, G_2), which we discuss in Section 3.4 following [BS] closely. Finally, we do the same for the pair (PGL_3, G_2) in Section 3.5, which we use later for the split places.

3.1. **The group G_2.** Let F be a nonarchimedean local field of characteristic zero, with normalized absolute value denoted by $|\ |$. Let G be a split exceptional group of type G_2. Let $B = TU$ be a Borel subgroup of G whose Levi component T is a split maximal torus of G. Denote by

$$\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$$

the corresponding set of positive roots, where α is the short simple root and β is the long simple root. For any root γ we will denote by w_γ the corresponding reflection on the Weyl group of G. We can identify

$$T \simeq F^\times \times F^\times, \ t \mapsto ((2\alpha + \beta)(t), (\alpha + \beta)(t)).$$

Hence, under this identification, if $(t_1, t_2) \in F^\times \times F^\times$, we have $\alpha(t_1, t_2) = t_1 t_2^{-1}$ and $\beta(t_1, t_2) = t_2^2 t_1^{-1}$.

Let Q_1 be the parabolic subgroup corresponding to the root β with Levi decomposition $Q_1 = L_1 U_1$. We have an isomorphism $L_1 \simeq \mathrm{GL}_2$ under the map determined by

$$t \mapsto \mathrm{diag}\left((\alpha + \beta)(t), \alpha(t)\right). \tag{3}$$

The group Q_1 is usually called the three-step parabolic of G_2, as U_1 admits a three-step filtration $U_1 = U_1(1) \supset U_1(2) \supset U_1(3) \supset U_1(4) = 1$ given in [BS, Section 3.1]. Similarly, let Q_2 be the parabolic corresponding to the root α. Consider its Levi decomposition $Q_2 = L_2 U_2$. We have an isomorphism $L_2 \simeq \mathrm{GL}_2$ under the map determined by

$$t \mapsto \mathrm{diag}\left((2\alpha + \beta)(t), (\alpha + \beta)(t)\right). \tag{4}$$

The group Q_2 is usually called Heisenberg parabolic of G_2. For $i \in \{1, 2\}$, We denote by δ_{Q_i} the modular character of Q_i.

3.2. **The group** $\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)$. Let K be a quadratic extension of F, and let U_3 be the unitary group defined in the previous section, which we identify with matrices in $\mathrm{GL}_3(K)$ fixing the Hermitian form determined by

$$\Phi_3 = e = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Let PU_3 be the quotient of U_3 by its center. Equivalently, the group PU_3 is the group of F-points of the group scheme $\underline{\mathrm{U}}_3/\underline{\mathrm{U}}_1$. Denote by B' the standard Borel subgroup of PU_3 consisting of upper triangular matrices with Levi decomposition $B' = T'U'$, where T' is the standard maximal torus of PU_3. We have

$$T' \simeq K^\times$$

via the map

$$\mathrm{diag}(a, b, c) \mapsto \frac{a}{b}.$$

Note that the Galois group $\mathrm{Gal}(K/F)$ acts on PU_3 by acting on its coefficients. We can then consider the semidirect product $\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)$ and the subgroups $B' \rtimes \mathrm{Gal}(K/F)$ and $T' \rtimes \mathrm{Gal}(K/F)$.

3.3. **The dual pair** $(G_2, \mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F))$ **and exceptional correspondence.** We closely follow [BS, §1]. Let $\mathbb{O}$ be an octonion algebra over F and let $J = J_3(K)$ be the set of 3×3 Hermitian matrices with coefficients in K. We can define a structure of Jordan algebra over F on J as follows. Addition on J is given by addition of matrices and regarding multiplication, let e be as above and define $x \circ y = \frac{1}{2}(xe^\# y + ye^\# x)$, where the superindex $\#$ denotes taking the adjoint, so that $xx^\# = \det(x)$. Note that from the expression of e given above, we have $e^\# = e$. The algebra J is equipped with an anti-involution given by the action of the nontrivial element of $\mathrm{Gal}(K/F)$ on the matrix entries. Let

$$G := \mathrm{Aut}(\mathbb{O}),$$

which is a split exceptional group of type G_2. On the other hand, let

$$G' := \mathrm{Aut}(J) \simeq \mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F),$$

where $\mathrm{Gal}(K/F)$ acts on PU_3 by acting on the coefficients. Consider the Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}' = \mathrm{Lie}(G')$. Then,

$$\mathfrak{h} := \mathfrak{g} \oplus \mathfrak{g}' \oplus \mathbb{O}^\circ \oplus J^\circ,$$

has a structure of a simple exceptional Lie algebra over F, where the superscript $\circ$ denotes the trace zero elements. Let $H = \mathrm{Aut}(\mathfrak{h})$, which is disconnected quasi-split of absolute type E_6. By definition of H, we have an inclusion

$$G \times G' \subset H$$

and they form a dual reductive pair. Let Π be the minimal representation of H (see [BS, §1.5] for the definition of minimal representation and [GS05] for its construction). We also use Π to denote the restriction of this representation to $G \times G'$, or the restriction to $G \times \mathrm{PU}_3$. This

representation induces correspondences between certain representations of G' and G, and certain representations of PU_3 and G.

Definition 3.1. Let $\tilde{\sigma}$ be a smooth irreducible representation of G'. Define $\Theta(\tilde{\sigma})$ to be the smooth representation of G such that $\Theta(\tilde{\sigma}) \boxtimes \tilde{\sigma}$ is the maximal $\tilde{\sigma}$-isotypic quotient of Π, viewed as a representation of $G \times G'$. Similarly, let σ be a smooth irreducible representation of PU_3. Define $\Theta_{\mathrm{PU}_3}(\sigma)$ to be the smooth representation of G such that $\Theta_{\mathrm{PU}_3}(\sigma) \boxtimes \sigma$ is the maximal σ-isotypic quotient of Π, viewed as a representation of $G \times \mathrm{PU}_3$.

Remark 3.2. Let σ be a smooth irreducible representation of PU_3. Then, it can be verified that

$$\Theta_{\mathrm{PU}_3}(\sigma) = \Theta\left(\mathrm{Ind}_{\mathrm{PU}_3}^{\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)}(\sigma)\right), \tag{5}$$

where the right-hand side is interpreted as we now explain.

Let σ^c be the representation obtained by twisting σ by the action of the nontrivial element of $\mathrm{Gal}(K/F)$. We then have the following two cases:

(1) Suppose $\sigma \not\simeq \sigma^c$. Then, $\mathrm{Ind}_{\mathrm{PU}_3}^{\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)}(\sigma)$ is irreducible and its restriction to PU_3 is isomorphic to $\sigma \oplus \sigma^c$. In this case, the right-hand side of (5) is the theta lift defined in Definition 3.1, and we have

$$\Theta_{\mathrm{PU}_3}(\sigma) = \Theta_{\mathrm{PU}_3}(\sigma^c) = \Theta\left(\mathrm{Ind}_{\mathrm{PU}_3}^{\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)}(\sigma)\right).$$

(2) Suppose that $\sigma \simeq \sigma^c$. Then, $\mathrm{Ind}_{\mathrm{PU}_3}^{\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)}(\sigma) \simeq \sigma_1 \oplus \sigma_2$ as representations of the group $\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)$, where σ_1, σ_2 are two non-isomorphic irreducible representations of $\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)$ whose restriction to PU_3 is isomorphic to σ. In this case, the right-hand side of (5) is defined as $\Theta(\sigma_1) \oplus \Theta(\sigma_2)$, and we have

$$\Theta_{\mathrm{PU}_3}(\sigma) = \Theta(\sigma_1) \oplus \Theta(\sigma_2).$$

To define a notion of small theta lift of representations of PU_3 and G', we need the following result from [BS].

Theorem 3.3. *Let $\tilde{\sigma}$ be an irreducible representation of G'. Then, $\Theta(\tilde{\sigma})$ is a representation of finite length of G. Similarly, if σ is an irreducible representation of PU_3, we have that $\Theta_{\mathrm{PU}_3}(\sigma)$ is a representation of finite length.*

Proof. The result for $\Theta(\tilde{\sigma})$ is given in Theorem 4.1 (iv) of [BS]. More precisely, the statement for nontempered representations is given in [BS, Proposition 4.3] and it is proven in an analogous way as [BS, Proposition 4.2]. The statement for tempered representations follows from Proposition 4.5, Proposition 4.8 and Proposition 4.15 of [BS]. Finally, the result for $\Theta_{\mathrm{PU}_3}(\sigma)$ can be deduced from there using the relation between theta lifts from G' to G and theta lifts from PU_3 to G described in Remark 3.2. $\square$

Definition 3.4. Let $\tilde{\sigma}$ be a smooth irreducible representation of G'. Define the small theta lift of $\tilde{\sigma}$, denoted by $\theta(\tilde{\sigma})$, to be the maximal semisimple quotient of $\Theta(\tilde{\sigma})$. We similarly define the small theta lift of a smooth irreducible representation σ of PU_3, and denote it by $\theta_{\mathrm{PU}_3}(\sigma)$.

3.4. **Results of Bakić–Savin on the nontempered correspondence.** We now explain a result of [BS] which determines the exceptional theta lift of nontempered representations. It is

described in terms of the following correspondence studied by Roberts in [Rob96]. Let $\tilde{\omega}$ be the representation of $L_1 \times (T' \rtimes \mathrm{Gal}(K/F))$ given (with this same notation) in [BS, Section 3.1]. We recall that $L_1 \simeq \mathrm{GL}_2$ and $T' \simeq K^\times$. The representation $\tilde{\omega}$ defines a big theta lift from $T' \rtimes \mathrm{Gal}(K/F)$ to L_1 given as follows.

Lemma 3.5. *Let $\tilde{\omega}$ be the representation of $L_1 \times (T' \rtimes \mathrm{Gal}(K/F))$ introduced above and consider the lift it induces from $T' \rtimes \mathrm{Gal}(K/F)$ to L_1. Let χ be a character of $K^\times$. Then,*

(1) *If $\chi = \chi^c$, write $\mathrm{Ind}_{K^\times}^{K^\times \rtimes \mathrm{Gal}(K/F)} = \chi_1 \oplus \chi_2$ as $K^\times \rtimes \mathrm{Gal}(K/F)$ representations. Then, exactly one of these two characters has a nonzero big theta lift by the correspondence induced by $\tilde{\omega}$, which is equal to the automorphic induction of χ to $\mathrm{GL}_2(F)$, given by the representation τ introduced in Proposition 6.3.*

(2) *If $\chi \neq \chi^c$, then $\mathrm{Ind}_{K^\times}^{K^\times \rtimes \mathrm{Gal}(K/F)}(\chi)$ is irreducible and its big theta lift by the correspondence induced by $\tilde{\omega}$ is equal to the automorphic induction of χ to $\mathrm{GL}_2(F)$.*

Proof. This result is given in Section 3.4 of [BS]. $\qquad\square$

Let χ be a unitary character of $K^\times$ and $s > 0$. Consider the character $\chi|\mathrm{N}_{K/F}(\ )|^s$ of $K^\times \simeq T'$. We may inflate this character to B' with trivial action on U'. First set

$$I(\chi, s) := i_{B'}^{\mathrm{PU}_3}(\chi|\mathrm{N}_{K/F}(\ )|^s)$$

to denote a principal series representation of PU_3. When $\chi \neq \chi^c$, then $\chi_1 := \mathrm{Ind}_{K^\times}^{K^\times \rtimes \mathrm{Gal}(K/F)} \chi$ is an irreducible representation of $T' \rtimes \mathrm{Gal}(K/F)$. Now $\chi_1|\mathrm{N}_{K/F}(\ )|^s$ is once again inflated to $B' \rtimes \mathrm{Gal}(K/F)$. Set

$$I(\chi_1, s) := i_{B' \rtimes \mathrm{Gal}(K/F)}^{G'} \left(\chi_1|\mathrm{N}_{K/F}(\ )|^s \right). \qquad (6)$$

When $\chi = \chi^c$, then

$$\mathrm{Ind}_{K^\times}^{K^\times \rtimes \mathrm{Gal}(K/F)} \chi = \chi_1 \oplus \chi_2,$$

where we denote by χ_1 the character with a nontrivial theta lift to L_1 via the theta correspondence induced by $\tilde{\omega}$ (see Lemma 3.5). Note that the character χ_2 has then a trivial theta lift via the correspondence induced by $\tilde{\omega}$. We define representations $I(\chi_1, s)$ and $I(\chi_2, s)$ of G' using the same formula as in (6).

Proposition 3.6. *Let χ be a unitary character of $K^\times$ and let $s > 0$ be a positive real number.*

(1) *Let σ_1 be the unique irreducible quotient of $I(\chi_1, s)$ and denote by $|\det|^s \tau$ the representation of L_1 obtained as the big theta lift of the character $\chi_1|\mathrm{N}_{K/F}(\ )|^s$ of $T' \rtimes \mathrm{Gal}(K/F)$ via the correspondence given by $\tilde{\omega}$. Then $\Theta(\sigma_1)$ is a nonzero quotient of $i_{Q_1}^G(|\det|^s \tau)$.*

(2) *If χ is $\mathrm{Gal}(K/F)$-invariant, let σ_2 be the unique irreducible quotient of $I(\chi_2, s)$. Then, $\Theta(\sigma_2) = 0$.*

(3) *Let σ be the unique irreducible quotient of $I(\chi, s)$. Then $\Theta_{\mathrm{PU}_3}(\sigma) = \Theta(\sigma_1)$ is a nonzero quotient of $i_{Q_1}^G(|\det|^s \tau)$.*

Proof. The proof that $\Theta(\sigma_1) \neq 0$ is given in [BS, Proposition 4.2]. The fact that $\Theta(\sigma_1)$ is a quotient of $i_{Q_1}^G(\tau)$ and that $\Theta(\sigma_2) = 0$ is given in [BS, Proposition 4.3]. The third point can be proved from the first two points and Remark 3.2 once we make the following observation. If χ is $\mathrm{Gal}(K/F)$-invariant, then $\mathrm{Ind}_{\mathrm{PU}_3}^{\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)}(\sigma) = \sigma_1 \oplus \sigma_2$. On the other hand, if χ is not

Gal(K/F)-invariant, $\mathrm{Ind}_{\mathrm{PU}_3}^{\mathrm{PU}_3 \rtimes \mathrm{Gal}(K/F)}(\sigma) = \sigma_1$. Alternatively, the third point can be justified in an analogous way as the proof of the first point, but working with the restriction of Π to $G \times \mathrm{PU}_3$ and using the following fact. The big theta lift of $\chi|\mathrm{N}_{K/F}(\ )|^s$ from T' to L_1 induced by the representation $\tilde{\omega}$ is equal to the big theta lift of $\chi_1|\mathrm{N}_{K/F}(\ )|^s$ from $T' \rtimes \mathrm{Gal}(K/F)$ to L_1 induced by the representation $\tilde{\omega}$. $\square$

Remark 3.7. Consider the same notation as in Proposition 3.6. In view of Lemma 3.5, $|\det|^s \tau$ can also be described as the automorphic induction of the character $\chi|\mathrm{N}_{K/F}(\ )|^s$ to $\mathrm{GL}_2(F)$.

3.5. **Theta correspondence for** $\mathrm{PGL}_3 \times G_2$**.** Following [GS04], we state the results we will need regarding the exceptional theta correspondence for $\mathrm{PGL}_3 \times G$. Let H be (the F-points of) a split adjoint linear algebraic group of type E_6 over F. Denote by Π the minimal representation of H as in [GS04, §5]. There is a dual reductive pair $\mathrm{PGL}_3 \times G \hookrightarrow H$ and we can consider the restriction of Π to $\mathrm{PGL}_3 \times G$. Using the representation Π, define the big theta lift $\Theta(\pi)$ of an irreducible admissible representation π of PGL_3 in a similar way as above. The next proposition is a particular case of [GS04, Corollary 9].

Proposition 3.8. *Let* χ_1, χ_2, χ_3 *be characters of* $F^\times$ *satisfying* $\chi_1 \chi_2 \chi_3 = 1$*. For every* i *from* 1 *to* 3*, write* $\chi_i = \mu_i |\ |_F^{s_i}$*, where* μ_i *is unitary and suppose that the characters are ordered so that* $s_1 \geq s_2 \geq s_3$*. Consider the representation of* $L_2 \simeq \mathrm{GL}_2$ *given by* $\tau = \pi(\chi_3^{-1}, \chi_2^{-1})$*. Then,* $\Theta(\pi(\chi_1, \chi_2, \chi_3))$ *is a nonzero quotient of* $i_{Q_2}^G(\tau)$*.*

4. LOCAL DIHEDRAL LONG ROOT A-PACKETS FOR G_2

In this section, we propose a construction of dihedral long root (local) A-packets for the group G_2, following Arthur's conjectures. In Section 4.1, we describe the corresponding A-parameter of G_2 in the global setting. This parameter factors through an A-parameter of $^L\mathrm{PU}_3$. Thus, in Section 4.2, we discuss Arthur's conjectures for $^L\mathrm{PU}_3$ in the local nonarchimedean setting, followed by an explicit construction of the corresponding local A-packets, called Howe–PS packets. We use the results on the theta correspondence for unitary groups outlined in Section 2 for this process. This construction was first given by Gelbart and Rogawski in [GR91] and it gives a unitary version of the original construction of Howe-PS packets, which was done in the orthogonal-symplectic setting. In Section 4.3, we resolve some details regarding the lift of representations from PU_3 to $\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$. Next, in Section 4.4, we describe Arthur's conjectures for the dihedral long root A-parameters of G_2 in the local nonarchimedean setting and propose a construction of the corresponding A-packets. We use the results from Section 3 for this process.

Throughout this section F denotes a number field and L_F denotes the corresponding conjectural Langlands group. We denote by G the F-points of a split group of type G_2 over F.

4.1. **Dihedral long root A-parameters.** Let K/F be a quadratic field extension with Galois group $\mathrm{Gal}(K/F) = \langle c \rangle$. Let χ be a character of $\mathbb{A}_K^\times / K^\times$, which we can also regard as a character of the Weil group W_K. Let τ denote the automorphic representation of GL_2 obtained from χ by automorphic induction, *i.e.*, the automorphic representation with L-parameter

$$\rho_\tau = \mathrm{Ind}_{W_K}^{W_F} \chi.$$

The representation τ is called dihedral with respect to the quadratic field extension K/F. Its central character is given by $\omega_{K/F} \cdot \chi_{|\mathbb{A}_F^\times}$. Assume that the character χ is conjugate-symplectic, i.e.,

$$\chi_{|\mathbb{A}_F^\times} = \omega_{K/F}.$$

Under this assumption, the representation τ has trivial central character and can therefore be regarded as a representation of PGL_2. Alternatively, this assumption implies that $\det(\rho_\tau) = 1$, and therefore ρ_τ defines an L-parameter of PGL_2. We will also assume that $\chi \neq \chi^c$, so that ρ_τ is irreducible and τ is a cuspidal representation.

The root system of G contains pairs of orthogonal roots, always consisting of a short and a long root. A choice of such a pair determines a commuting pair of SL_2 subgroups inside G,

$$(\mathrm{SL}_{2,s} \times \mathrm{SL}_{2,l})/\mu_2 \subset G,$$

where $\mu_2 = \{\pm 1\}$ is embedded diagonally inside $\mathrm{SL}_{2,s} \times \mathrm{SL}_{2,l}$ in the natural way. Here $\mathrm{SL}_{2,s}$ corresponds to the short root and $\mathrm{SL}_{2,l}$ corresponds to the long root. Moreover, the groups $\mathrm{SL}_{2,s}$ and $\mathrm{SL}_{2,l}$ are centralizers of each other inside G. Note that the Langlands dual group of G is $G(\mathbb{C})$ (we abuse notation and denote by $G(\mathbb{C})$ the $\mathbb{C}$-points of our fixed split group of type G_2 over F).

Definition 4.1. The long root A-parameter of G associated with τ is given by

$$\psi_{\tau,l} : L_F \times \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow W_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\rho_\tau \times \mathrm{id}} \mathrm{SL}_{2,s}(\mathbb{C}) \times_{\mu_2} \mathrm{SL}_{2,l}(\mathbb{C}) \subset G(\mathbb{C}).$$

There is a subgroup $\mathrm{SL}_{3,l}(\mathbb{C}) \subset G_2(\mathbb{C})$ corresponding to the root system of type A_2 formed by the six long roots of G. Let $T_s(\mathbb{C})$ denote the diagonal torus of $\mathrm{SL}_{2,s}(\mathbb{C})$. Then $T_s(\mathbb{C}) \times_{\mu_2} \mathrm{SL}_{2,l}(\mathbb{C}) \subset \mathrm{SL}_{3,l}(\mathbb{C})$. The group $\mathrm{SL}_{3,l}(\mathbb{C})$ has index two in its normalizer $N_{G(\mathbb{C})}(\mathrm{SL}_{3,l}(\mathbb{C}))$ inside $G(\mathbb{C})$. Let w denote the standard Weyl element of SL_2. Then $(w, w) \in \mathrm{SL}_{2,s}(\mathbb{C}) \times_{\mu_2} \mathrm{SL}_{2,l}(\mathbb{C})$ belongs to the non-identity component of $N_{G(\mathbb{C})}(\mathrm{SL}_{3,l}(\mathbb{C}))$. Therefore,

$$N_{\mathrm{SL}_{2,s}(\mathbb{C})}(T_s(\mathbb{C})) \times_{\mu_2} \mathrm{SL}_{2,l}(\mathbb{C}) \subset N_{G(\mathbb{C})}(\mathrm{SL}_{3,l}(\mathbb{C})) = \mathrm{SL}_{3,l}(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} \subset G(\mathbb{C}).$$

Note that $^L\mathrm{PU}_3 = \mathrm{SL}_{3,l}(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$. Hence, the parameter $\psi_{\tau,l}$ factors through an A-parameter of PU_3.

Definition 4.2. We denote by ψ_χ the A-parameter of PU_3 given by

$$\psi_\chi : L_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\rho_\tau \times \mathrm{id}} N_{\mathrm{SL}_{2,s}(\mathbb{C})}(T_s(\mathbb{C})) \times_{\mu_2} \mathrm{SL}_{2,l}(\mathbb{C}) \subset \mathrm{SL}_{3,l}(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} =^L \mathrm{PU}_3.$$

Restricted to $L_K \times \mathrm{SL}_2(\mathbb{C})$, the parameter ψ_χ yields a 3-dimensional representation given by

$$\chi^{-2} \oplus \chi \otimes S_2,$$

where χ is regarded as a character of L_K via $L_K \twoheadrightarrow W_K$ and S_2 denotes the standard 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. This information fully determines the equivalence class of ψ_χ given by $\mathrm{SL}_3(\mathbb{C})$-conjugacy.

4.2. **Howe–PS packets for** PU_3. We construct the local Howe–PS packets for PU_3 associated to the A-parameter ψ_χ at nonarchimedean places using the technique of theta lifting. Let v be a nonarchimedean place of F and F_v the completion of F at v. Denote by L_{F_v} the group

$W_{F_v} \times \mathrm{SL}_2(\mathbb{C})$. By fixing an embedding $L_{F_v} \hookrightarrow L_F$, we can precompose the map ψ_χ with the map $L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \hookrightarrow L_F \times \mathrm{SL}_2(\mathbb{C})$ (which is the identity on $\mathrm{SL}_2(\mathbb{C})$) to obtain the local A-parameter

$$\psi_{\chi,v} : L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L \mathrm{PU}_3.$$

Following Arthur's conjectures, the elements of the local A-packet associated with ψ and v are indexed by irreducible representations of the so-called local component group

$$S_{\psi_{\chi,v}} = \pi_0 \left(Z_{\mathrm{SL}_3(\mathbb{C})}(\mathrm{Im}(\psi_{\chi,v}))/Z(\mathrm{SL}_3(\mathbb{C})) \right) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } v \text{ is not split in } K, \\ 1 & \text{if } v \text{ is split in } K, \end{cases}$$

where $Z_{\mathrm{SL}_3(\mathbb{C})}(\mathrm{Im}(\psi_{\chi,v}))$ denotes the centralizer in $\mathrm{SL}_3(\mathbb{C}) \subset {}^L \mathrm{PU}_3$ of the image of $\psi_{\chi,v}$, $Z(\mathrm{SL}_3(\mathbb{C}))$ denotes the center of $\mathrm{SL}_3(\mathbb{C})$, and π_0 denotes the group of connected components. Hence, the local A-packet of PU_3 associated to ψ_χ and v has the form

$$A_{\psi_{\chi,v}} = \begin{cases} \{\sigma_v^+, \sigma_v^-\} & \text{if } v \text{ is not split in } K, \\ \{\sigma_v^+\} & \text{if } v \text{ is split in } K. \end{cases}$$

Here, we denote by σ_v^+ the representation corresponding to the trivial character of $S_{\psi_{\chi,v}} \simeq \mathbb{Z}/2\mathbb{Z}$, and in the case that v does not split in K, we denote by σ_v^- the representation of PU_3 corresponding to the nontrivial character of $S_{\psi_{\chi,v}}$. Moreover, Arthur's conjectures predict that for all but finitely many (nonarchimedean) places, the representation σ_v^+ is the unramified representation whose Satake parameter is

$$s_{\psi_{\chi,v}} = \psi_{\chi,v} \left(\Phi_v \times \begin{pmatrix} q_v^{-1/2} & 0 \\ 0 & q_v^{1/2} \end{pmatrix} \right), \tag{7}$$

where Φ_v denotes a geometric Frobenius at v and q_v denotes the cardinality of the corresponding residue field. Fix a nontrivial additive character $\psi : \mathbb{A}_F/F \to \mathbb{C}^\times$. For each prime v of F, we denote by ψ_v the corresponding (nontrivial) additive character of F_v. To define the representations of PU_3 appearing in $A_{\psi_{\chi,v}}$ for a nonarchimedian prime v of F, we distinguish two cases, according to whether v splits in K or not.

4.2.1. *Nonsplit case.* Suppose that v is not split in K. Denote by K_v the completion of K at the unique place above v. Fix a 3-dimensional Hermitian space V over K_v and identify $\mathrm{U}(V) \simeq \mathrm{U}_3$. Let V_1 denote the 1-dimensional Hermitian space in the Witt tower of V. Fix a trace-zero element $\delta \in K_v^\times$ and recall the definition of sign of a skew-Hermitian space given in Section 2 (which depends on the choice of δ). Finally, let W^+ and W^- be 1-dimensional skew-Hermitian spaces over K_v satisfying

$$\epsilon(V_1)\epsilon(W^+) = \epsilon_K \left(\frac{1}{2}, \chi_v^3, \psi_v \left(\mathrm{Tr}_{K_v/F_v}(-\delta(\cdot)) \right) \right) = -\epsilon(V_1)\epsilon(W^-). \tag{8}$$

Let $\gamma = \chi_v$, which is a conjugate symplectic character. Following Section 2.3, we use this data to define theta correspondences between the unitary groups defined above. In particular, and following the notation of Section 2.3, we can consider the representations $\Omega_{V,W^+,\gamma,\psi_v}$ of $\mathrm{U}(V) \times \mathrm{U}(W^+)$, and $\Omega_{V,W^-,\gamma,\psi_v}$ of $\mathrm{U}(V) \times \mathrm{U}(W^-)$ and the corresponding big theta lifts $\Theta_{V,W^+,\gamma,\psi_v}$, $\Theta_{V,W^-,\gamma,\psi_v}$.

Definition 4.3. We define the following representations of $\mathrm{U}(V) = \mathrm{U}_3$.

(1) Denote by $\mathbb{1}$ the trivial representation of $U(W^+)$. Define $\sigma_v^+ = \Theta_{V,W^+,\gamma,\psi_v}(\mathbb{1})$.

(2) Denote by $\mathbb{1}$ the trivial representation of $U(W^-)$. Define $\sigma_v^- = \Theta_{V,W^-,\gamma,\psi_v}(\mathbb{1})$.

By the choice of our lifting characters, the center of $U(V)$ acts trivially on these representations. Hence, we also view σ_v^+ and σ_v^- as representations of PU_3.

Proposition 4.4. *The representation σ_v^+ is the unique nonzero irreducible quotient of the representation $i_{\bar{B}'}^{PU_3}(\chi_v| \ |_{K_v}^{1/2}) = I(\chi_v, 1/2)$. In particular, if χ_v is unramified, the Satake parameter of σ_v^+ is given by (7).*

Proof. Consider the representation $\Omega_{V_1,W^+,\gamma,\psi_v}$ of $U(V_1) \times U(W^+)$, which induces a theta lift $\Theta_{V_1,W^+,\gamma,\psi_v}$. By (8) and Theorem 2.6, we have that $\Theta_{V_1,W^+,\gamma,\psi}(\gamma_{|K^1}^{-1}) \neq 0$. Therefore, it follows from Proposition 2.4 (2) that, as a representation of U_3, σ_v^+ is an irreducible quotient of

$$i_{B'}^{U(V)}\left(\gamma| \ |_{K_v}^{1/2} \otimes \gamma_{|K_v^1}^{-1}\right).$$

This representation is invariant under the center of $U(V)$, and can therefore be viewed as the representation of PU_3 given by $i_{B'}^{PU_3}(\chi_v| \ |_{K_v}^{1/2})$ (note that we are denoting the Borel of upper-triangular matrices in U_3 and in PU_3 with the same symbol B'). The rest of the assertions follow from there. $\qquad\square$

Proposition 4.5. *The representation σ_v^- is nonzero irreducible and supercuspidal.*

Proof. The proof is similar to the proof of the previous proposition. Consider the representation $\Omega_{V_1,W^-,\gamma,\psi_v}$ of $U(V_1) \times U(W^-)$, which induces a theta lift $\Theta_{V_1,W^-,\gamma,\psi_v}$. In that case, (8) and Theorem 2.6 imply that $\Theta_{V_1,W^-,\gamma,\psi_v}(\gamma_{|K^1}^{-1}) = 0$ and the result follows from Proposition 2.4 (1). $\qquad\square$

4.2.2. *Split case.* Suppose that v is split in K. Choose a place w of K above v, which determines an isomorphism $U_3(F_v) \simeq GL_3(F_v)$. Let $\gamma = \chi_w$. Using this data, we define a theta correspondence for the dual pair $GL_1(F_v) \times GL_3(F_v) \hookrightarrow Sp_6(F_v)$ following Section 2.4. Denote by Θ the theta lift defined by this correspondence.

Definition 4.6. Let $\mathbb{1}$ be the trivial representation of $GL_1(F_v)$. Define the representation $\sigma_v^+ = \theta(\mathbb{1})$. It is a representation of $U_3(F_v) \simeq GL_3(F_v)$ where the center of $U_3(F_v)$ acts trivially, and therefore we view σ_v^+ as a representation of $PU_3(F_v) \simeq PGL_3(F_v)$.

Note that if χ is unramified at v, then it follows from Theorem 2.7 that σ_v^+ is the unramified representation with Satake parameter $s_{\psi_{\chi,v}}$. In particular, this holds for all but finitely many split primes v, as predicted by Arthur's conjecture.

4.3. **From representations of PU_3 to representations of $PU_3 \rtimes \mathbb{Z}/2\mathbb{Z}$.** Consider the same notation as in Section 4.2. Suppose that v is not split in K, let W^+, W^- be the skew-Hermitian spaces introduced in Section 4.2.1 and let $\sigma_v^\pm$ be the representations of PU_3 defined in Definition 4.3. Following Remark 3.2, we are interested in knowing whether the induction of the representations $\sigma_v^\pm$ to the group $G' = PU_3 \rtimes Gal(K_v/F_v)$ is irreducible or decomposes as a direct sum of two non-isomorphic representations. While this question was already studied in Section 3.4 for the case of principal series, the discussion below will be especially relevant to study the irreducibility of the lift of σ_v^- from PU_3 to G via the exceptional theta correspondence.

Recall the following notation introduced in Section 2.1: if W is a skew-Hermitian space equipped with a skew-Hermitian form $\langle\ ,\ \rangle$, we denote W^{-1} the skew Hermitian space with the same underlying space W and skew-Hermitian form given by $-\langle\ ,\ \rangle$. In particular, in this section we will consider the spaces $(W^+)^{-1}$ and $(W^-)^{-1}$.

Proposition 4.7. *Assume that $\chi_v^2 \neq 1$. Then, the representations*

$$\operatorname{Ind}_{\mathrm{PU}_3}^{G'} \sigma_v^+ \quad and \quad \operatorname{Ind}_{\mathrm{PU}_3}^{G'} \sigma_v^-$$

are irreducible.

Proof. Following Remark 3.2, we need to show that, for $? \in \{+, -\}$, we have that $\sigma_v^? \not\simeq (\sigma_v^?)^c$. However, it follows from [HKS96, Lemma 2.1(ii)] that $(\sigma_v^?)^c \simeq (\sigma_v^?)^\vee$, where $(\sigma_v^?)^\vee$ denotes the contragredient of $\sigma_v^?$. Therefore, we need to prove that $\sigma_v^? \not\simeq (\sigma_v^?)^\vee$.

Recall that we take $\gamma = \chi_v$. The representation $\sigma_v^? = \Theta_{V,W^?,\gamma,\psi_v}(\mathbb{1}_{W^?})$ is the isotypic component of the Weil representation

$$\Omega_{V,W^?,\gamma,\psi_v}$$

where the center of $\mathrm{U}(V)$ acts trivially. According to [Gan22, §2.5], the dual of this Weil representation is

$$\overline{\Omega_{V,W^?,\gamma,\psi_v}} \simeq \Omega_{V,W^?,\gamma^{-1},\bar\psi_v} \simeq \Omega_{V,(W^?)^{-1},\gamma^{-1},\psi_v},$$

and thus $(\sigma_v^?)^\vee$ is the isotypic component $\Omega_{V,(W^?)^{-1},\gamma^{-1},\psi_v}$ where the center of $\mathrm{U}(V)$ acts trivially. Equivalently, we have that $(\sigma_v^?)^\vee = \Theta_{V,(W^?)^{-1},\gamma^{-1},\psi_v}(\mathbb{1}_{(W^?)^{-1}})$.

According to [AG17, Theorem 4.3(4)], respectively [AG17, Theorem 4.5(1)], and using the notation in [*loc. cit.*], the L-parameter associated with the representation σ_v^+, respectively σ_v^-, is $\gamma^{-2} \oplus \gamma|\cdot|_{K_v}^{1/2} \oplus \gamma|\cdot|_{K_v}^{-1/2}$, respectively $\gamma^{-2} \oplus \gamma S_2$. On account of the previous paragraph, and using again [*loc. cit.*], for the representations $(\sigma_v^+)^\vee$ and $(\sigma_v^-)^\vee$ the corresponding L-parameters are $\gamma^2 \oplus \gamma^{-1}|\cdot|_{K_v}^{1/2} \oplus \gamma^{-1}|\cdot|_{K_v}^{-1/2}$ and $\gamma^2 \oplus \gamma^{-1}S_2$. Since $\gamma \neq \gamma^{-1}$, we conclude that $\sigma_v^?$ and $(\sigma_v^?)^\vee$ belong to different L-packets and are therefore non-isomorphic. We note that, although the results in [AG17] rely on the local Langlands correspondence, this is known for the group U_3 by the work of Rogawski [Rog90]. $\square$

Proposition 4.8. *Assume that $\chi_v^2 = 1$. Then, the representations*

$$\operatorname{Ind}_{\mathrm{PU}_3}^{G'} \sigma_v^+ \quad and \quad \operatorname{Ind}_{\mathrm{PU}_3}^{G'} \sigma_v^-$$

are not irreducible.

Proof. Following Remark 3.2, we need to show that, for $? \in \{+, -\}$, we have that $\sigma_v^? \simeq (\sigma_v^?)^c$. However, it follows from [HKS96, Lemma 2.1(ii)] that $(\sigma_v^?)^c \simeq (\sigma_v^?)^\vee$, where $(\sigma_v^?)^\vee$ denotes the contragredient of $\sigma_v^?$. Therefore, we need to prove that $\sigma_v^? \simeq (\sigma_v^?)^\vee$.

Recall that we take $\gamma = \chi_v$. The representation $\sigma_v^? = \Theta_{V,W^?,\gamma,\psi_v}(\mathbb{1}_{W^?})$ is the isotypic component of the Weil representation

$$\Omega_{V,W^?,\gamma,\psi_v}$$

where the center of $\mathrm{U}(V)$ acts trivially. According to [Gan22, §2.5], the dual of this Weil representation is

$$\overline{\Omega_{V,W^?,\gamma,\psi_v}} \simeq \Omega_{V,W^?,\gamma^{-1},\bar\psi_v} \simeq \Omega_{V,(W^?)^{-1},\gamma,\psi_v},$$

and thus $(\sigma_v^?)^\vee$ is the isotypic component $\Omega_{V,(W^?)^{-1},\gamma,\psi_v}$ where the center of $U(V)$ acts trivially. Note that we used that $\gamma^{-1} = \gamma$, which holds because $\chi_v^2 = 1$. Also, the condition $\chi_v^2 = 1$ implies that $\chi_v = \chi_v^c$. By Hilbert's Theorem 90, we can write $-1 = a/a^c$ for some $a \in K^\times$. Therefore, we deduce that $\omega_{K/F}(-1) = \chi_v(-1) = \chi_v(a)/\chi_v^c(a) = 1$. In particular, it follows that $-1 \in N_{K/F}(K^\times)$ and hence $(W^?)^{-1} \simeq W^?$, which shows that $\sigma_v^? \simeq (\sigma_v^?)^\vee$. $\qquad\qquad\square$

4.4. Dihedral long root A-packets for G_2.

We discuss local nonarchimedean A-packets associated to the dihedral long root A-parameter introduced in Definition 4.1. As before, we obtain the local A-parameter $\psi_{\tau,l,v}$ at a place v, whose component group $S_{\psi_{\tau,l,v}}$ is given by

$$S_{\psi_{\tau,l,v}} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \rho_{\tau,v} \text{ is irreducible (or, } \tau_v \text{ is a discrete series representation),} \\ 1 & \text{if } \rho_{\tau,v} \text{ is reducible (or, } \tau_v \text{ is not a discrete series representation).} \end{cases}$$

4.4.1. *Nonsplit case.* Let v be a nonarchimedean place of F which does not split in K, and denote by K_v the completion of K at the unique place above v. We continue to use c to denote the generator of $\mathrm{Gal}(K_v/F_v)$. In this case, τ_v is the smooth irreducible representation of PGL_2 with L-parameter $\rho_{\tau,v} = \mathrm{Ind}_{W_{F_v}}^{W_{K_v}} \chi_v$. The representation $\mathrm{Ind}_{W_{F_v}}^{W_{K_v}} \chi_v$ is irreducible if and only if $\chi_v \neq \chi_v^c$. Thus we have the equivalent characterization for a nonsplit v:

$$S_{\psi_{\tau,l,v}} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \chi_v \neq \chi_v^c, \\ 1 & \text{if } \chi_v = \chi_v^c. \end{cases}$$

Arthur's conjectures predict that the local A-packets are as follows:

$$A_{\psi_{\tau,l,v}} = \begin{cases} \{\pi_v^+, \pi_v^-\} & \text{if } \chi_v \neq \chi_v^c, \\ \{\pi_v^+\} & \text{if } \chi_v = \chi_v^c. \end{cases} \tag{9}$$

Here, π_v^+ is indexed by the trivial character of $S_{\psi,l,v}$ and π_v^- by the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$ when it occurs. Arthur's conjectures also predict that, for all but finitely many places, π_v^+ is the unramified principal series representation whose Satake parameter is given by

$$s_{\psi_{\tau,l,v}} = \psi_{\tau,l,v}\left(\Phi_v \times \begin{pmatrix} q_v^{-1/2} & 0 \\ 0 & q_v^{1/2} \end{pmatrix}\right), \tag{10}$$

where, as before, Φ_v denotes a geometric Frobenius at v and q_v denotes the cardinality of the corresponding residue field.

We carry out the construction of the expected local A-packets using the exceptional theta correspondence for the pair $(G_2, \mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z})$ introduced in Section 3.3. Recall that Definition 4.3 gives us the construction of the representations σ_v^+ and σ_v^- of PU_3.

Definition 4.9. With the notation θ_{PU_3} established in Definition 3.4 in mind, we define
  (1) $\pi_v^+ := \theta_{\mathrm{PU}_3}(\sigma_v^+)$,
  (2) $\pi_v^- := \theta_{\mathrm{PU}_3}(\sigma_v^-)$.

Observe that the local A-packet for PU_3 as constructed in Section 4.2.1 always has two elements when v is nonsplit in K, but the local A-packet for G_2 as in (9) is expected to have two elements when $\chi_v \neq \chi_v^c$ and one otherwise. We construct the latter A-packet by lifting the two representations $\{\sigma_v^+, \sigma_v^-\}$. We show in subsequent sections that there is no discrepancy in

the size of the packets; indeed we will see that, while π_v^+ is always nonzero, π_v^- vanishes if and only if $\chi_v = \chi_v^c$. This is summarized in the following theorem.

Theorem 4.10. *The lifting of the local A-packet $\{\sigma_v^+, \sigma_v^-\}$ for a nonsplit place v via the exceptional theta correspondence for the dual pair $(G_2, \mathrm{PU}_3 \rtimes \mathrm{Gal}(K_v/F_v))$ yields the following:*

(1) *π_v^+ is nonzero and it is equal to the unique irreducible quotient of the parabolically induced representation $i_{Q_1}^G(|\det|^{1/2}\tau_v)$. For all but finitely many nonsplit places v, the representation π_v^+ is the unramified principal series with Satake parameter given by* (10).

(2) *If $\chi_v \neq \chi_v^c$, the representation π_v^- is nonzero, irreducible and tempered.*

(3) *If $\chi_v = \chi_v^c$, the representation π_v^- is zero.*

Proof. We prove the first point of the theorem and outline the proofs for the second and third points, which will be completed in the next sections.

(1) According to Proposition 4.4, the representation σ_v^+ is the unique irreducible quotient of $i_{B'}^{\mathrm{PU}_3}(\chi_v| \ |_{K_v}^{1/2}) = I(\chi_v, 1/2)$. Now note that χ_v is unitary (because χ is conjugate symplectic) and that the automorphic induction of $\chi_v| \ |_{K_v}^{1/2}$ is equal to $|\det|^{1/2}\tau_v$. It therefore follows from Proposition 3.6 and Remark 3.7 that $\Theta_{\mathrm{PU}_3}(\sigma_v^+)$ is a nonzero quotient of $i_{Q_1}^G(|\det|^{1/2}\tau_v)$. Since τ_v is a tempered representation, we deduce that $i_{Q_1}^G(|\det|^{1/2}\tau_v)$ has a unique irreducible quotient (see [Mui97, p. 468]), which yields the desired description for $\pi_v^+ = \theta_{\mathrm{PU}_3}(\sigma^+)$.

(2) If $\chi_v \neq \chi_v^c$, it follows from Theorem 5.5 that π_v^- is not zero, and it follows from Proposition 6.1 that it does not have any generic subquotient. Therefore, the fact that π^- is irreducibile and tempered follows from Proposition 4.7, [BS, Remark 4.4] and [BS, Proposition 4.15], as we explain in Theorem 6.2 and in its proof.

(3) If $\chi_v = \chi_v^c$, the vanishing of π_v^- follows from Theorem 6.17.

$\square$

4.4.2. *Split case.* Let v be a nonarchimedean place of F which splits in K and let w and w' be the places of K lying above v. There are natural identifications $F_v \simeq K_w \simeq K_{w'}$. Since χ is conjugate-symplectic, we have that $\chi_{w'} = \chi_w^{-1}$ and χ_w is unitary. Therefore,

$$\rho_{\tau,v} = \chi_w \oplus \chi_{w'} = \chi_w \oplus \chi_w^{-1}$$

and $\tau_v = \pi(\chi_w, \chi_w^{-1})$ is equal to the normalized parabolic induction $i_P^{\mathrm{GL}_2(F_v)}(\chi_w, \chi_w^{-1})$, where here we are using P to denote the Borel of $\mathrm{GL}_2(F_v)$ consisting on upper triangular matrices. Since $\rho_{\tau,v}$ is reducible, it follows that the component group $S_{\psi_{\tau,l,v}}$ is trivial.

The choice of the prime w determines an isomorphism $\mathrm{PU}_3(F_v) \simeq \mathrm{PGL}_3(F_v)$. We construct the predicted singleton A-packet by using theta lifting from $\mathrm{PGL}_3(F_v)$ to $G_2(F_v)$. Recall from Section 4.2.2 that σ_v^+ is the theta lift of the trivial character $\mathbb{1}$ of $F_v^\times$ to $\mathrm{GL}_3(F_v^\times)$ using theta correspondence for the dual pair $(\mathrm{GL}_1, \mathrm{GL}_3)$ and we interpret the result, which has trivial central character, as a representation of PGL_3. More precisely, applying Theorem 2.7, with $\mu = 1$ and $\gamma = \chi_w$,

$$\sigma_v^+ = \pi(\chi_w| \ |_{F_v}^{1/2}, \chi_w^{-2}, \chi_w| \ |_{F_v}^{-1/2}),$$

the Langlands quotient of $i_{B'}^{\mathrm{GL}_3}(\chi_w| \ |_{F_v}^{1/2} \otimes \chi_w^{-2} \otimes \chi_w| \ |_{F_v}^{-1/2})$, where B' is the standard Borel subgroup of GL_3. Let Θ_{PGL_3} stand for big theta lift from $\mathrm{PGL}_3(F_v)$ to $G_2(F_v)$ and let $\tilde{\tau}_v =$

$\pi(\chi_w^{-1}| \ |_{F_v}^{1/2}, \chi_w^2)$, which can be viewed as a representation of Q_2. It follows from Proposition 3.8 that $\Theta_{\mathrm{PGL}_3}(\sigma_v^+)$ is a nonzero quotient of $i_{Q_2}^{G_2}(\tilde{\tau}_v)$, and therefore that it has finite length.

Definition 4.11. Define
$$\pi_v^+ := \theta_{\mathrm{PGL}_3}(\sigma_v^+)$$
to be the maximal semisimple quotient of $\Theta_{\mathrm{PGL}_3}(\sigma_v^+)$.

Theorem 4.12. *The representation π_v^+ is the unique irreducible quotient of $i_{Q_1}^{G_2}(|\det|^{1/2}\tau_v)$. For all but finitely many nonsplit places v, the representation π_v^+ is the unramified principal series with Satake parameter given by* (10).

Proof. Let $\tilde{\tau}_v = \pi(\chi_w^{-1}| \ |_{F_v}^{1/2}, \chi_w^2) = i_P^{\mathrm{GL}_2(F_v)}(\chi_w^{-1}| \ |_{F_v}^{1/2}, \chi_w^2)$ be as above. Then, it follows from Proposition 3.8 that $\Theta_{\mathrm{PGL}_3}(\sigma_v^+)$ is a nonzero quotient of $i_{Q_2}^{G_2}(\tilde{\tau}_v)$. Following a similar procedure as in the proof of Proposition 6.4 below, it can be seen that
$$i_{Q_2}^{G_2}(\tilde{\tau}_v) \simeq i_{Q_1}^{G_2}(|\det|^{1/2}\tau_v).$$
Indeed, this isomorphism follows from (12) and (13) of the proof of Proposition 6.4 after we replace the character μ used there by the character χ_w. Since the representation on the right hand side has a unique irreducible quotient (see [Mui97, p. 468]), the result follows. In fact, similarly as in the case of Remark 6.6 below, it can be proved that $\Theta_{\mathrm{PGL}_3}(\sigma_v^+)$ is equal to the unique irreducible quotient of this representation, given by $i_{Q_2}^{G_2}(\chi_w \circ \det)$. $\qquad\square$

5. Non-vanishing of theta lifts

Let K/F be a quadratic extension of local fields of characteristic zero and consider the notation introduced in Section 3. In this section, we prove the following criterion regarding non-vanishing of theta lifts from PU_3 to G: if the contragredient of a representation τ of PU_3 (viewed as a representation of U_3) has an irreducible quotient with trivial central character when restricted to a suitable two-variable unitary subgroup, then $\Theta_{\mathrm{PU}_3}(\tau) \neq 0$. This criterion is proven by computing Fourier–Jacobi periods: with the notation introduced in Section 3.1, we show that, for any nontrivial character ψ' of $U_1(2)/U_1(3)$,
$$\mathrm{Hom}_{\mathrm{PU}_3}(\Pi_{U_1(2),\psi'}, \tau) \neq 0.$$
The key to prove this result is a description of $\Pi_{U_1(2),\psi'}$ in terms of Weil representations attached to two-variable unitary subgroups and SL_2. In Section 5.1, we provide a suitable description of two-variable unitary subgroups; in Section 5.2, we introduce the symplectic spaces that determine the aforementioned Weil representations, and, in Section 5.3, we prove the criterion.

Fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$ and fix also a conjugate-symplectic character $\chi : K^\times \to \mathbb{C}^\times$. Let σ^+ and σ^- be the representations of PU_3 introduced in Definition 4.3 with respect to these data, and let π^+ and π^- be their corresponding small theta lifts to the group G, as introduced in Definition 4.9. In Section 5.4, we use a see-saw argument to verify that σ^+, and, under the condition that $\chi^2 \neq 1$, also σ^-, satisfy the hypothesis of the non-vanishing criterion. As a consequence, $\pi^+ \neq 0$, and, if $\chi^2 \neq 1$, also $\pi^- \neq 0$, proving the non-vanishing statements in Theorem 4.10.

5.1. **The groups** $U(V_2)$ **and** $U(V_2')$. Let V_2 and V_2' be 2-dimensional Hermitian spaces over K such that V_2 is split and V_2' is nonsplit.

Assume that, with respect to some K-basis, the Hermitian form on V_2 is defined by the Hermitian matrix $\Phi_2 \in H_2(K)$. Then we can describe $U(V_2)$ in the following way:

$$U(V_2) = \{g \in \mathrm{GL}_2(K) : g\Phi_2 g^\dagger = \Phi_2\}.$$

Using this description, the group $U(V_2)$ acts on the 4-dimensional F-space of Hermitian 2×2 matrices $H_2(K)$ by the rule

$$(g, A) \mapsto gAg^\dagger \quad \text{for } g \in U(V_2) \text{ and } A \in H_2(K),$$

and this action preserves the quadratic form defined on $H_2(K)$ by the determinant map. Moreover, it stabilizes the 3-dimensional F-subspace

$$Q = \{A \in H_2(K) : \mathrm{Tr}(A\Phi_2^{-1}) = 0\}.$$

This action defines a homomorphism from $U(V_2)$ to $\mathrm{SO}(Q)$, the F-points of the special orthogonal group associated with the 3-dimensional quadratic F-space Q. The kernel of this homomorphism is the center $Z(U(V_2))$ of $U(V_2)$. This allows us to identify $\mathrm{PU}_2(V_2) = U(V_2)/Z(U(V_2))$ with an index-2 subgroup of $\mathrm{SO}(Q)$, which we will denote by $\mathrm{SO}(Q)^+$. Actually, one can show that $\mathrm{SO}(Q)^+$ consists of the elements in $\mathrm{SO}(Q)$ whose spinor norm belongs to $N_{K/F}(K^\times)/(F^\times)^2 \subset F^\times/(F^\times)^2$.

Similarly, assume that, with respect to some K-basis, the Hermitian form on V_2' is defined by the Hermitian matrix $\Phi_2' \in H_2(K)$. Then we can describe $U(V_2')$ in the following way:

$$U(V_2') = \{g \in \mathrm{GL}_2(K) : g\Phi_2' g^\dagger = \Phi_2'\}.$$

Using this description, the group $U(V_2')$ acts on the 4-dimensional F-space of Hermitian 2×2 matrices $H_2(K)$ by the rule

$$(g, A) \mapsto gAg^\dagger \quad \text{for } g \in U(V_2') \text{ and } A \in H_2(K),$$

and this action preserves the quadratic form defined on $H_2(K)$ by the determinant map. Moreover, it stabilizes the 3-dimensional F-subspace

$$Q' = \{A \in H_2(K) : \mathrm{Tr}(A\Phi_2'^{-1}) = 0\}.$$

This action defines a homomorphism from $U(V_2')$ to $\mathrm{SO}(Q')$, the F-points of the special orthogonal group associated with the 3-dimensional quadratic F-space Q'. The kernel of this homomorphism is the center $Z(U(V_2'))$ of $U(V_2')$. This allows us to identify $U(V_2')/Z(U(V_2'))$ with an index-2 subgroup of $\mathrm{SO}(Q')$, which we will denote by $\mathrm{SO}(Q')^+$. Actually, one can show that $\mathrm{SO}(Q')^+$ consists of the elements in $\mathrm{SO}(Q')$ whose spinor norm belongs to $N_{K/F}(K^\times)/(F^\times)^2 \subset F^\times/(F^\times)^2$.

5.2. **An orbit problem.** We choose the following description of U_3:

$$U_3 = \{g \in \mathrm{GL}_3(K) : g\Phi_3 g^\dagger = \Phi_3\},$$

where

$$\Phi_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and we have $\mathrm{PU}_3 = \mathrm{U}_3/Z(\mathrm{U}_3) = \mathrm{U}_3/\mathrm{U}_1$. Let $J = H_3(K)$, and

$$J_1 = \{g \in J : \ \mathrm{Tr}(g\Phi_3^{-1}) = 1\}.$$

Let Ω_2 be defined as in [BS, §2.1]. Then, as in [$op.\,cit.$, §3.1.1], we have

$$\Omega_2 \cap J_1 = \{g \in J : \ \mathrm{Tr}(g\Phi_3^{-1}) = 1 \ \text{and} \ \mathrm{rank}(g) = 1\}.$$

The group U_3 acts on $\Omega_2 \cap J_1$ by the rule

$$(g, A) \mapsto gAg^\dagger \quad \text{for } g \in \mathrm{U}_3 \text{ and } A \in \Omega_2 \cap J_1,$$

and this descends to an action of PU_3 on $\Omega_2 \cap J_1$. Any matrix $A \in \Omega_2 \cap J_1$ can be written as

$$A = -\lambda u^\dagger u \quad \text{where } u \in M_{1\times 3}(K), \ \lambda \in F^\times \text{ and } -\lambda u\Phi_3 u^\dagger = 1.$$

Observe that, if $\lambda \in \mathrm{N}_{K/F}(K^\times)$, then we can modify u to account for this factor. Fix an element $\lambda_0 \in F^\times$ which is not the norm of an element in $K^\times$. Then, we can decompose the set $\Omega_2 \cap J_1$ into two subsets:

$$(\Omega_2 \cap J_1)_0 = \{-u^\dagger u : \ u \in M_{1\times 3}(K), \ -u\Phi_3 u^\dagger = 1\}$$
$$(\Omega_2 \cap J_1)_1 = \{-\lambda_0 u^\dagger u : \ u \in M_{1\times 3}(K), \ -\lambda_0 u\Phi_3 u^\dagger = 1\}.$$

We claim that these are the orbits for the action of PU_3 on $\Omega_2 \cap J_1$. Indeed, this follows from the observation that, if $-u^\dagger u \in (\Omega_2 \cap J_1)_0$, then the subspace $\langle u \rangle^\perp$ orthogonal to u with respect to Φ_3 is a split 2-dimensional Hermitian K-space, and, if $-\lambda_0 uu^\dagger \in (\Omega_2 \cap J_1)_1$, then $\langle u \rangle^\perp$ is a nonsplit 2-dimensional Hermitian K-space.

For the orbit $(\Omega_2 \cap J_1)_0$, let $u = (0, 1, 0)$ and choose the element

$$f_0 = -u^\dagger u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Its stabilizer in PU_3 is the subgroup

$$\left\{ \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix} \in \mathrm{PU}_3 \right\} \simeq \left\{ g \in \mathrm{GL}_2(K) : \ g \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} g^\dagger = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\} \simeq \mathrm{U}(V_2).$$

For the last identification, let V_2 be the 2-dimensional K-subspace spanned by $u_1 = (1, 0, 0)$ and $u_3 = (0, 0, 1)$, equipped with the Hermitian form given in this basis by

$$\Phi_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

For the orbit $(\Omega_2 \cap J_1)_1$, let $u' = (1, 0, 1/(2\lambda_0))$ and choose the element

$$f_1 = -\lambda_0 (u')^\dagger u' = \begin{pmatrix} -\lambda_0 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & -1/(4\lambda_0) \end{pmatrix}.$$

Let $u_1' = u'$, $u_2' = (0, 1, 0)$ and $u_3' = (1, 0, -1/(2\lambda_0))$. Then, with respect to this basis,

$$f_1 = \begin{pmatrix} -\lambda_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi_3 = \begin{pmatrix} -1/\lambda_0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/\lambda_0 \end{pmatrix},$$

and therefore the stabilizer of f_1 in PU_3 is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{PU}_3 \right\} \simeq \left\{ g \in \mathrm{GL}_2(K) : g \begin{pmatrix} -1 & 0 \\ 0 & 1/\lambda_0 \end{pmatrix} g^\dagger = \begin{pmatrix} -1 & 0 \\ 0 & 1/\lambda_0 \end{pmatrix} \right\} \simeq \mathrm{U}(V_2').$$

For the last identification, let V_2' be the 2-dimensional K-subspace spanned by u_2' and u_3', equipped with the Hermitian form given in this basis by

$$\Phi_2' = \begin{pmatrix} -1 & 0 \\ 0 & 1/\lambda_0 \end{pmatrix}.$$

As in [BS, §3.1.1], to the element f_0 we attach the subspace

$$\Delta_0^\perp = \left\{ \begin{pmatrix} a & 0 & y \\ 0 & 0 & 0 \\ \bar{y} & 0 & c \end{pmatrix} \in J \right\} \simeq H_2(K).$$

This subspace is invariant by the action of the stabilizer of f_0 in PU_3, and, under its identification with $\mathrm{U}(V_2)$, the action of the stabilizer of f_0 on $\Delta_0^\perp$ agrees with the action described in the previous subsection. It preserves the determinant form on $H_2(K)$ and the decomposition $\Delta_0^\perp \simeq H_2(K) = Q \oplus Z$, with Q defined as above and Z denoting the 1-dimensional subspace of $H_2(K)$ generated by Φ_2. The group $\mathrm{U}(V_2)$ acts trivially on Z, whereas, as described above, its action on Q identifies $\mathrm{PU}(V_2) = \mathrm{U}(V_2)/Z(\mathrm{U}(V_2))$ with an index-2 subgroup of $\mathrm{SO}(Q)$, which we denote by $\mathrm{SO}(Q)^+$.

Similarly, to the element f_1 we attach the subspace $\Delta_1^\perp \subseteq J$, which, with respect to the basis u_1', u_2', u_3' defined above, is given by

$$\Delta_1^\perp = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & y \\ 0 & \bar{y} & c \end{pmatrix} \in J \right\} \simeq H_2(K).$$

This subspace is invariant by the action of the stabilizer of f_1 in PU_3, and, under its identification with $\mathrm{U}(V_2')$, the action of the stabilizer of f_1 on $\Delta_1^\perp$ agrees with the action described in the previous subsection. It preserves the determinant form on $H_2(K)$ and the decomposition $\Delta_1^\perp \simeq H_2(K) = Q' \oplus Z'$, where Q' is defined as above and Z' denotes the 1-dimensional subspace of $H_2(K)$ generated by Φ_2'. The group $\mathrm{U}(V_2')$ acts trivially on Z', whereas, as described above, its action on Q' identifies $\mathrm{PU}(V_2') = \mathrm{U}(V_2')/Z(\mathrm{U}(V_2'))$ with an index-2 subgroup of $\mathrm{SO}(Q')$, which we denote by $\mathrm{SO}(Q')^+$.

5.3. **Non-vanishing criterion for theta lifts.** Let Γ denote a 2-dimensional symplectic space. Then $\Gamma \otimes_F \Delta_0^\perp$ and $\Gamma \otimes_F \Delta_1^\perp$ are 8-dimensional symplectic spaces. Let ω_0 and ω_1 be the Weil representations associated with $\Gamma \otimes_F \Delta_0^\perp$ and $\Gamma \otimes_F \Delta_1^\perp$, respectively, for a fixed nontrivial character $\psi' : F \to \mathbb{C}^\times$. Let $\omega_{0,Q}$, $\omega_{0,Z}$, $\omega_{1,Q'}$ and $\omega_{1,Z'}$ be the Weil representations associated with the symplectic spaces $\Gamma \otimes_F Q$, $\Gamma \otimes_F Z$, $\Gamma \otimes_F Q'$ and $\Gamma \otimes_F Z'$, respectively, for the character ψ'. Then, under the natural homomorphism

$$\mathrm{Mp}(\Gamma \otimes_F Q) \times \mathrm{Mp}(\Gamma \otimes_F Z) \longrightarrow \mathrm{Mp}(\Gamma \otimes_F \Delta_0^\perp),$$

the representation ω_0 pulls back to $\omega_{0,Q} \boxtimes \omega_{0,Z}$, and, under the homomorphism

$$\mathrm{Mp}(\Gamma \otimes_F Q') \times \mathrm{Mp}(\Gamma \otimes_F Z') \longrightarrow \mathrm{Mp}(\Gamma \otimes_F \Delta_1^\perp),$$

the representation ω_1 pulls back to $\omega_{1,Q'} \boxtimes \omega_{1,Z'}$.

Recall that, in Section 3.1, we defined Q_1 as the three-step parabolic of G. It has a Levi decomposition $Q_1 = L_1 U_1$, with Levi subgroup $L_1 \simeq \mathrm{GL}_2$ and unipotent subgroup U_1 equipped with a three-step filtration $U_1 = U_1(1) \supset U_1(2) \supset U_1(3) \supset U_1(4) = 1$ described in [BS, §3.1]. In particular, the quotient $U_1/U_1(3)$ is isomorphic to the Heisenberg group associated with a two-dimensional symplectic space. This yields a natural action of $U_1/U_1(3)$ on $\omega_{0,Z}$ and $\omega_{1,Z'}$ via the Heisenberg representation. From now on, we consider the action of $U_1/U_1(3)$ on $\omega_0 = \omega_{0,Q} \boxtimes \omega_{0,Z}$ and $\omega_1 = \omega_{1,Q'} \boxtimes \omega_{1,Z'}$ given by the trivial representation on the first factors and the Heisenberg representation on the second factors.

There is also a natural action of the metaplectic group Mp_2 on each of the representations $\omega_{0,Q}$, $\omega_{0,Z}$, $\omega_{1,Q'}$ and $\omega_{1,Z'}$. Moreover, the resulting action of Mp_2 on $\omega_0 = \omega_{0,Q} \boxtimes \omega_{0,Z}$ and $\omega_1 = \omega_{1,Q'} \boxtimes \omega_{1,Z'}$ factors through SL_2. From now on, we let $\mathrm{SL}_2 \subset L_1$ act on ω_0 and ω_1 via this action.

Finally, there is a natural action of $\mathrm{SO}(Q)$ (resp. $\mathrm{SO}(Q')$) on $\omega_{0,Q}$ (resp. $\omega_{1,Q'}$), and we let $\mathrm{U}(V_2)$ (resp. $\mathrm{U}(V_2')$) act on $\omega_0 = \omega_{0,Q} \boxtimes \omega_{0,Z}$ (resp. $\omega_1 = \omega_{1,Q'} \boxtimes \omega_{1,Z'}$) via the embedding $\mathrm{U}(V_2)/Z(\mathrm{U}(V_2)) \hookrightarrow \mathrm{SO}(Q)$ (resp. $\mathrm{U}(V_2')/Z(\mathrm{U}(V_2')) \hookrightarrow \mathrm{SO}(Q')$).

It follows from [BS, §3.1.1] that, as a representation of $(\mathrm{SL}_2 \ltimes U_1/U_1(3)) \times \mathrm{PU}_3$,

$$\Pi_{U_1(2),\psi'} = \mathrm{c\text{-}Ind}_{\mathrm{U}(V_2)}^{\mathrm{PU}_3} \omega_0 \oplus \mathrm{c\text{-}Ind}_{\mathrm{U}(V_2')}^{\mathrm{PU}_3} \omega_1.$$

Now we use this description of $\Pi_{U_1(2),\psi'}$ to prove the following result.

Theorem 5.1. *Let τ be a smooth irreducible representation of PU_3 and let $\tau^\vee$ be its contragredient representation. Assume that either of the following two conditions holds:*

- *The restriction of $\tau^\vee$ to $\mathrm{U}(V_2)$ has an irreducible quotient with trivial central character.*
- *The restriction of $\tau^\vee$ to $\mathrm{U}(V_2')$ has an irreducible quotient with trivial central character.*

Then, the theta lift $\Theta_{\mathrm{PU}_3}(\tau)$ is nonzero.

Proof. To prove the non-vanishing of the theta lift, it suffices to prove that the space

$$\mathrm{Hom}_{\mathrm{PU}_3}(\Pi_{U_1(2),\psi'}, \tau) = \mathrm{Hom}_{\mathrm{PU}_3}(\mathrm{c\text{-}Ind}_{\mathrm{U}(V_2)}^{\mathrm{PU}_3} \omega_0, \tau) \oplus \mathrm{Hom}_{\mathrm{PU}_3}(\mathrm{c\text{-}Ind}_{\mathrm{U}(V_2')}^{\mathrm{PU}_3} \omega_0, \tau)$$

is nontrivial.

Assume that the restriction of $\tau^\vee$ to $\mathrm{U}(V_2)$ has an irreducible quotient ρ with trivial central character. Let $\rho^\vee$ be the contragredient of ρ as a smooth representation of $\mathrm{U}(V_2)$. Note that

$\rho^\vee$ can be regarded as a representation of $\mathrm{PU}(V_2) \simeq \mathrm{SO}(Q)^+$. Recall that $\omega_0 = \omega_{0,Q} \boxtimes \omega_{0,Z}$, and $\mathrm{U}(V_2)$ acts on this space via the action of $\mathrm{SO}(Q)^+$ on the factor $\omega_{0,Q}$. Since any irreducible representation of $\mathrm{SO}(Q)$ (and *a fortiori* of $\mathrm{SO}(Q)^+$) appears as a quotient of the Weil representation $\omega_{0,Q}$, so does the representation $\rho^\vee$. Hence, we can regard $\rho^\vee$ as a $\mathrm{U}(V_2)$-quotient of ω_0, and, thus, we can also regard $\text{c-Ind}_{\mathrm{U}(V_2)}^{\mathrm{PU}_3} \rho^\vee$ as a PU_3-quotient of $\text{c-Ind}_{\mathrm{U}(V_2)}^{\mathrm{PU}_3} \omega_0$. Therefore, to conclude the proof in this case we just need to show that τ is a PU_3-quotient of $\text{c-Ind}_{\mathrm{U}(V_2)}^{\mathrm{PU}_3} \rho^\vee$, i.e.

$$\mathrm{Hom}_{\mathrm{PU}_3}(\text{c-Ind}_{\mathrm{U}(V_2)}^{\mathrm{PU}_3} \rho^\vee, \tau) \neq 0.$$

But, dualizing the left-hand side of the previous expression and applying Frobenius reciprocity, it becomes

$$\mathrm{Hom}_{\mathrm{U}(V_2)}(\tau^\vee, \rho),$$

which is nonzero since ρ is a $\mathrm{U}(V_2)$-quotient of $\tau^\vee$.

The case in which the restriction of $\tau^\vee$ to $\mathrm{U}(V_2')$ has an irreducible quotient with trivial central character can be dealt with in a similar way. $\qquad\square$

5.4. **A see-saw argument.** In this subsection, we use a see-saw argument to prove that the contragredient of the representation σ^+ of U_3 defined above, when restricted to a representation of a suitable 2-variable unitary group, has a quotient with trivial central character. We also prove this for the contragredient of σ^- under the assumption that $\chi^2 \neq 1$. Combined with the results in the previous subsection, this will conclude the proof of the non-vanishing of the corresponding theta lifts to the group G.

Recall that χ is a conjugate-symplectic character of $K^\times$. In general, if W is an m-dimensional skew-Hermitian space over K and V is an n-dimensional Hermitian space over K, the choice of characters $\chi_W = \chi^m$ and $\chi_V = \chi^n$ determines a lift $\tilde{\iota}_{\chi,\psi} : \mathrm{U}(V) \times \mathrm{U}(W) \to \mathrm{Mp}(V \otimes_K W)$ of the natural homomorphism $\iota : \mathrm{U}(V) \times \mathrm{U}(W) \to \mathrm{Sp}(V \otimes_K W)$, and we can use this lift to define $\Omega_{V,W} := \tilde{\iota}_{\chi,\psi}^* \omega_\psi$. Throughout this subsection, these are the representations that we use to define theta lifts between different unitary groups.

Let V_3 be a 3-dimensional Hermitian space over K. Assume that its Hermitian form, with respect to some K-basis, is defined by the matrix

$$\Phi_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

As before, let V_2 and V_2' be 2-dimensional Hermitian spaces such that V_2 is split and V_2' is nonsplit. Then, we can write

$$V_3 := V_1 \oplus V_2 \cong V_1' \oplus V_2',$$

where V_1 and V_1' are non-isomorphic 1-dimensional Hermitian spaces over K. Let W and W' be 1-dimensional skew-Hermitian spaces over K such that $\Theta_{V_1,W}(\mathbb{1}_W) \neq 0$ and $\Theta_{V_1',W'}(\mathbb{1}_{W'}) \neq 0$. Observe that, by dichotomy, the spaces W and W' are not isomorphic. More precisely, we have that

$$\epsilon(V_1)\epsilon(W) = \epsilon_K(1/2, \chi, \psi(\mathrm{Tr}_{K/F}(-\delta(\cdot)))) = \epsilon(V_1')\epsilon(W'),$$

where $\delta \in K^\times$ denotes a trace-zero element which we also use to define the signs $\epsilon(W)$ and $\epsilon(W')$ as in (2).

We have that $\Theta_{V_3,W}(\mathbb{1}_W)$ and $\Theta_{V_3,W'}(\mathbb{1}_{W'})$ are both nonzero, because we are in the stable range. Furthermore, it follows from Proposition 2.4 and Theorem 2.6 that

$$\Theta_{V_3,W}(\mathbb{1}_W) \text{ is non-tempered} \iff \epsilon_K(1/2,\chi,\psi(\mathrm{Tr}_{K/F}(-\delta(\cdot))))\epsilon_K(1/2,\chi^3,\psi(\mathrm{Tr}_{K/F}(-\delta(\cdot)))) = 1$$
$$\iff \Theta_{V_3,W'}(\mathbb{1}_{W'}) \text{ is supercuspidal.}$$

In particular, when $\chi^2 = 1$, the statements above always hold.

We define $\sigma = \Theta_{V_3,W}(\mathbb{1}_W)$ and $\sigma' = \Theta_{V_3,W'}(\mathbb{1}_{W'})$. Then, we have that $\{\sigma,\sigma'\} = \{\sigma^+,\sigma^-\}$, and, if $\chi^2 = 1$, then $\sigma = \sigma^+$ and $\sigma' = \sigma^-$.

Proposition 5.2. *With the previous definitions, it always holds that $\Theta_{V_2,W}(\mathbb{1}) \neq 0$, and*

$$\Theta_{V_2',W}(\mathbb{1}_W) \neq 0 \text{ if and only if } \chi^2 \neq 1.$$
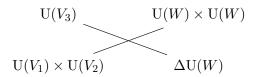
These statements also hold if we replace W by W'.

Proof. Observe that V_2' is the first element in its Witt tower, whereas V_2 is the second element in its Witt tower, the first one being a zero-dimensional Hermitian space. The trivial character on $U(W)$ (or on $U(W')$) is said to lift in dimension zero if and only if $\chi^2 = 1$. Therefore the result follows from the conservation relation in Theorem 2.3. $\square$

We denote by $\sigma^\vee$ and $(\sigma')^\vee$ the contragredient of the smooth PU_3-representations σ and σ', respectively.

Proposition 5.3. *The restriction of the representation $\sigma^\vee$ to $U(V_2)$ has a unique irreducible quotient with trivial central character.*

Proof. Consider the see-saw diagram

$$\begin{array}{ccc} U(V_3) & & U(W) \times U(W) \\ & \times & \\ U(V_1) \times U(V_2) & & \Delta U(W) \end{array}$$

and recall that we choose $\chi_{V_i} = \chi^i$ and $\chi_W = \chi$ as splitting characters. This choice is compatible with respect to the see-saw diagram above. Let $G_1 = U(V_1) \times U(V_2)$ and $H_1 = \Delta U(W)$. Let τ be an irreducible representation of $U(V_2)$ with trivial central character. Then, we have

$$\mathrm{Hom}_{G_1}\left(\sigma, \mathbb{1}_{U(V_1)} \boxtimes \tau\right) \cong \mathrm{Hom}_{G_1 \times H_1}\left(\Omega, \mathbb{1}_{U(V_1)} \boxtimes \tau \boxtimes \mathbb{1}_{U(W)}\right)$$
$$\cong \mathrm{Hom}_{H_1}\left(\mathbb{1}_{U(W)} \otimes \Theta_{V_2,W}(\tau), \mathbb{1}_{U(W)}\right).$$

Since τ has trivial central character, the big theta lift $\Theta_{V_2,W}(\tau)$ is zero unless $\tau = \theta_{V_2,W}(\mathbb{1}_{U(W)})$, which is nonzero because of Proposition 5.2. In this case, $\Theta_{V_2,W}(\tau) = \mathbb{1}_{U(W)}$ and therefore the space above is one-dimensional.

It follows from the previous discussion that the restriction of σ to $U(V_2)$ has a unique irreducible quotient with trivial central character, which is isomorphic to $\theta_{V_2,W}(\mathbb{1}_{U(W)})$. By [HKS96, Lemma 2.1(ii)], we know that $\sigma^\vee \simeq \sigma^c$, with the notation of Remark 3.2. Therefore, we conclude that the restriction of $\sigma^\vee$ to $U(V_2)$ has a unique irreducible quotient with trivial central character. $\square$

Proposition 5.4. *Assume that $\chi^2 \neq 1$. Then, the restriction of the representation $(\sigma')^\vee$ to $\mathrm{U}(V_2')$ has a unique irreducible quotient with trivial central character.*

Proof. This can be proved by a see-saw argument as in the previous proposition, replacing V_1, V_2, and W, with V_1', V_2', and W', respectively. Note that the hypothesis $\chi^2 \neq 1$ is needed to ensure that the theta lift $\tau = \theta_{V_2', W'}(\mathbb{1}_{\mathrm{U}(W')})$ is nonzero, which follows in this case from Proposition 5.2. When running the previous argument, with $H_1' = \Delta \mathrm{U}(W')$, this yields

$$\mathrm{Hom}_{H_1'} \left(\mathbb{1}_{\mathrm{U}(W')} \otimes \Theta_{V_2', W'}(\tau), \mathbb{1}_{\mathrm{U}(W')} \right) = \mathrm{Hom}_{H_1'} \left(\mathbb{1}_{\mathrm{U}(W')} \otimes \mathbb{1}_{\mathrm{U}(W')}, \mathbb{1}_{\mathrm{U}(W')} \right),$$

which is one-dimensional. $\qquad\square$

Now, an application of Theorem 5.1 yields the following result.

Theorem 5.5. *With the previous definitions, we have the following:*
  (1) *The representation $\Theta_{\mathrm{PU}_3}(\sigma^+)$ of G is nonzero, and thus π^+ is nonzero.*
  (2) *If $\chi^2 \neq 1$, the representation $\Theta_{\mathrm{PU}_3}(\sigma^-)$ of G is nonzero, and thus π^- is nonzero.*

6. Vanishing of theta lifts

Let K/F be a quadratic extension of local fields of characteristic zero. Fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$ and a conjugate-symplectic character $\chi : K^\times \to \mathbb{C}^\times$. Let σ^+ and σ^- be the representations of PU_3 introduced in Definition 4.3 with respect to these data and let π^+ and π^- be their corresponding small theta lifts to the group G, as introduced in Definition 4.9. In this section we will prove some vanishing results for the representation π^-.

In Section 6.1, we prove that π^- does not have any generic subquotient. This result, combined with the non-vanishing of π^- when $\chi^2 \neq 1$ proved in Theorem 5.5, implies that π^- is irreducible if $\chi^2 \neq 1$. This completes the proof of the second part of Theorem 4.10. The rest of the section considers the case when $\chi^2 = 1$ and proves the vanishing of π^- in that situation, completing the proof of the third point of Theorem 4.10. For that, we study the twisted coinvariant spaces of $\pi^\pm$ corresponding to generic characters of the unipotent radical of the Heisenberg parabolic subgroup of G. More precisely, for every such character ψ_E:

  - In Section 6.2, we obtain an explicit description of π^+ which allows us to verify that the twisted coinvariant space for π^+ with respect to ψ_E does not vanish.
  - In Section 6.3, we express the twisted coinvariant space for $\pi^+ \oplus \pi^-$ with respect to ψ_E as a sum of toric periods for $\sigma^+ \oplus \sigma^-$. Since the representations $\sigma^\pm$ are theta lifts of characters in U_1, the non-vanishing of these periods can be expressed in terms of local epsilon factors, as it is done in [BFG+]. Moreover, in [*loc. cit.*], it is proven that exactly one of these toric periods contributes to the sum with precisely dimension 1.

As a consequence, we obtain that the twisted coinvariant spaces for π^- with respect to any generic character of the unipotent radical of the Heisenberg parabolic subgroup of G are zero, from which we deduce that π^- does not have a nontrivial non-generic quotient. Combining this fact with the results in Section 6.1, we conclude the vanishing of π^-.

6.1. **Vanishing of the generic part.** In this subsection, we prove that π^- does not have any generic subquotient. When $\chi^2 \neq 1$, we deduce from this that π^- is irreducible. The proofs rely on results of [BS, §4.3-4.4].

As before, let B' be a Borel subgroup of PU_3 and let U' be its unipotent radical. It has a filtration $0 \subseteq U'(2) \subseteq U'$, where $U'(2) \simeq F$ and $U'/U'(2) \simeq K$. Via the last identification, we can define a character $\psi_{U'}$ of U' corresponding to a nontrivial additive character $\psi \circ \mathrm{Tr}_{K/F}$ of K.

According to [BS, Corollary 4.6], in order to prove the vanishing of the generic part of π^-, it suffices to prove the vanishing of the space of Whittaker periods

$$\mathrm{Hom}_{U'}((\sigma^-)^\vee, \bar{\psi}_{U'}) \subseteq \mathrm{Hom}_{U'(2)}((\sigma^-)^\vee, \mathbb{1}).$$

Note that, as explained in the proofs of Proposition 4.7 and of Proposition 4.8, the representation $(\sigma^-)^\vee$ is the big theta lift of a character of U_1 to a representation of U_3. In particular, we can use the explicit description of the coinvariant spaces of theta lifts in this setting described in [Gan22, Section 2.12, Guided Exercise (ii)] to affirm that

$$(\sigma^-)^\vee_{U'(2)} = (\sigma^-)^\vee_{U'}.$$

Since $(\sigma^-)^\vee$ is supercuspidal, as $(\sigma^-)^\vee \simeq (\sigma^-)^c$ by [HKS96, Lemma 2.1(ii)] and σ^- is super-cuspidal, the right-hand side is equal to zero, implying the vanishing of the space of Whittaker periods.

We write the result as a proposition for reference.

Proposition 6.1. *The representation π^- does not have any generic subquotient.*

From there, we obtain the following result.

Theorem 6.2. *If $\chi^2 \neq 1$, the representation π^- is irreducible and tempered.*

Proof. Note that $\mathrm{Ind}_{\mathrm{PU}_3}^{G'} \sigma^-$ is irreducible (by Proposition 4.7) and tempered (because σ^- is supercuspidal). We also make the following two observations:

- By [BS, Remark 4.4], every irreducible quotient of π^- is tempered.
- By Proposition 6.1, every irreducible quotient of π^- is non-generic.

Now, Theorem 5.5 implies that there exists a nonzero irreducible quotient of π^-. Moreover, [BS, Proposition 4.15] together with these two observations imply that such quotient is unique. Hence, π^- is irreducible and tempered as desired. $\square$

6.2. **Twisted coinvariant spaces for π^+.** We assume from now on that $\chi^2 = 1$. This implies that χ is $\mathrm{Gal}(K/F)$-invariant, hence there exists a character μ of $F^\times$ such that $\chi = \mu \circ \mathrm{N}_{K/F}$. Fix a choice of such character μ. Note that $\mu \neq \mu^{-1}$ as $\mu^2 = \omega_{K/F}$. Denote by P the Borel of $\mathrm{GL}_2(F)$ consisting on upper triangular matrices.

Proposition 6.3. *Consider the same notation as above. We have:*

(1) *The automorphic induction of $\chi|\mathrm{N}_{K/F}(\ )|^{1/2}$ from $K^\times$ to $\mathrm{GL}_2(F)$ is the irreducible representation $|\det|^{1/2}\tau$, where*

$$\tau = i_P^{\mathrm{GL}_2(F)}(\mu, \mu^{-1}).$$

(2) *Regard $|\det|^{1/2}\tau$ as a representation of $L_1 \simeq \mathrm{GL}_2$, i.e.*

$$|\det|^{1/2}\tau = i_{T(U \cap L_1)}^{L_1}(\mu|\ |^{1/2} \otimes \mu^{-1}|\ |^{1/2}). \tag{11}$$

Then, π^+ is the unique irreducible quotient of $i_{Q_1}^G(|\det|^{1/2}\tau)$.

Proof. We start proving the first point. Note that $(\mu \circ \mathrm{N}_{K/F})|\mathrm{N}_{K/F}(\ )|^{1/2}$ is Galois invariant. Thus, the automorphic induction of this character to $\mathrm{GL}_2(F)$ is the irreducible principal series corresponding to the character of the diagonal torus in $\mathrm{GL}_2(F)$ given by $\mu| \ |^{1/2} \otimes \omega_{K/F}\mu| \ |^{1/2}$. Note that we used that, since $\omega_{K/F} \neq | \ |^{\pm 1}$, such principal series is irreducible. Using the relation $\mu^2 = \omega_{K/F}$, we can rewrite the automorphic induction as

$$i_P^{\mathrm{GL}_2(F)}(\mu| \ |^{1/2}, \mu^{-1}| \ |^{1/2}) = |\det|^{1/2} i_P^{\mathrm{GL}_2(F)}(\mu, \mu^{-1}),$$

proving the first point. Since $\pi^+ = \theta_{\mathrm{PU}_3}(\sigma^+)$, and we saw in the proof of Theorem 4.10 that $i_{Q_1}^G(|\det|^{1/2}\tau)$ has a unique irreducible quotient, the second point of the proposition follows from the first one, Proposition 3.6 and Remark 3.7. $\qquad\square$

In view of the previous proposition, the next proposition will provide a description of π^+ in this case.

Proposition 6.4. *The unique irreducible quotient of $i_{Q_1}^G(|\det|^{1/2}\tau)$ is given by $i_{Q_2}^G(\mu \circ \det)$.*

Proof. The key to prove this proposition is to rewrite normalized parabolic inductions for the parabolic Q_1 (resp. Q_2) in terms of the parabolic Q_2 (resp. Q_1) and to use intertwining operators.

Using the isomorphism induced by (3), it can be seen

$$i_{Q_1}^G(|\det|^{1/2}\tau) = i_{Q_1}^G i_{T(U\cap L_1)}^{L_1}(\mu| \ |^{1/2} \otimes \mu^{-1}| \ |^{1/2}) = i_{Q_1}^G i_B^{Q_1}(\mu| \ |^{1/2}(\alpha+\beta) \cdot \mu^{-1}| \ |^{1/2}(\alpha)), \quad (12)$$

where in the last expression we are extending $\mu| \ |^{1/2}(\alpha+\beta) \cdot \mu^{-1}| \ |^{1/2}(\alpha)$ from a character of T to a character of B. Now, using that $i_{Q_1}^G i_B^{Q_1} = i_B^G$, we obtain

$$i_{Q_1}^G(|\det|^{1/2}\tau) = i_B^G\left(\mu| \ |^{1/2}(\alpha+\beta) \cdot \mu^{-1}| \ |^{1/2}(\alpha)\right).$$

This can be rewritten as

$$
\begin{aligned}
i_B^G\left(\mu| \ |^{1/2}(\alpha+\beta) \cdot \mu^{-1}| \ |^{1/2}(\alpha)\right) &= i_B^G\left(\mu^{-1}| \ |^{1/2}(2\alpha+\beta) \cdot \mu^2(\alpha+\beta)\right) \\
&= i_{Q_2}^G i_{T(U\cap L_2)}^{L_2}\left(\mu^{-1}| \ |^{1/2} \otimes \mu^2\right),
\end{aligned}
\quad (13)
$$

where in the last equality we proceeded in a similar way as in (12) and used the isomorphism induced by (4). By Proposition 1.1 (i) of [Mui97], the representation of $L_2 \simeq \mathrm{GL}_2$ given by $i_{T(U\cap L_2)}^{L_2}\left(\mu^{-1}| \ |^{1/2} \otimes \mu^2\right)$ is irreducible and we have an isomorphism

$$i_{T(U\cap L_2)}^{L_2}\left(\mu^{-1}| \ |^{1/2}, \mu^2\right) \simeq i_{T(U\cap L_2)}^{L_2}\left(\mu^2, \mu^{-1}| \ |^{1/2}\right).$$

Hence,

$$
\begin{aligned}
i_{Q_2}^G i_{T(U\cap L_2)}^{L_2}\left(\mu^{-1}| \ |^{1/2} \otimes \mu^2\right) &\simeq i_{Q_2}^G i_{T(U\cap L_2)}^{L_2}\left(\mu^2 \otimes \mu^{-1}| \ |^{1/2}\right) \\
&\simeq i_B^G\left(\mu^2(2\alpha+\beta) \cdot \mu^{-1}| \ |^{1/2}(\alpha+\beta)\right) \simeq i_B^G\left(\mu| \ |^{1/2}(\alpha+\beta) \cdot \mu^2(\alpha)\right),
\end{aligned}
$$

where we used that $2\alpha + \beta = \alpha + (\alpha + \beta)$. Now, we can use Proposition 1.1 (i) of [Mui97] to obtain

$$i_B^G\left(\mu| \ |^{1/2}(\alpha+\beta) \cdot \mu^2(\alpha)\right) \simeq i_{Q_1}^G i_{T(U\cap L_1)}^{L_1}\left(\mu| \ |^{1/2} \otimes \mu^2\right) \simeq i_{Q_1}^G i_{T(U\cap L_1)}^{L_1}\left(\mu^2 \otimes \mu| \ |^{1/2}\right).$$

Proceeding in a similar way as above, we deduce

$$i_{Q_1}^G i_{T(U \cap L_1)}^{L_1} \left(\mu^2 \otimes \mu | \ |^{1/2} \right) \simeq i_B^G \left(\mu^2(\alpha + \beta) \cdot \mu | \ |^{1/2}(\alpha) \right)$$

$$\simeq i_B^G \left(\mu | \ |^{1/2}(2\alpha + \beta) \cdot \mu | \ |^{-1/2}(\alpha + \beta) \right) \simeq i_{Q_2}^G i_{T(U \cap L_2)}^{L_2} \left(\mu | \ |^{1/2} \otimes \mu | \ |^{-1/2} \right).$$

By Proposition 1.1 (ii) of [Mui97], we have that $i_{T(U \cap L_2)}^{L_2} \left(\mu | \ |^{1/2} \otimes \mu | \ |^{-1/2} \right)$ has a unique irreducible quotient given by $\mu \circ \det$. It follows that $i_{Q_1}^G (| \det |^{1/2}\tau) = i_{Q_2}^G i_{T(U \cap L_2)}^{L_2} \left(\mu | \ |^{1/2} \otimes \mu | \ |^{-1/2} \right)$ has a quotient given by $i_{Q_2}^G (\mu \circ \det)$. Finally, note that $\mu^2 = \omega_{K/F} \neq 1$ so, in particular, μ is unitary. Hence, Theorem 3.1 (i) of [Mui97] affirms that $i_{Q_2}^G (\mu \circ \det)$ is irreducible, and we are done. $\qquad \square$

We therefore obtain the following corollary.

Corollary 6.5. *Suppose that $\chi^2 = 1$ and let μ be a character of $F^\times$ such that $\chi = \mu \circ \mathrm{N}_{K/F}$. Then,*

$$\pi^+ = i_{Q_2}^G (\mu \circ \det),$$

where Q_2 denotes the Heisenberg parabolic of G_2.

Proof. This follows from Proposition 6.3 and Proposition 6.4. $\qquad \square$

Remark 6.6. Even though we will not need it explicitly, it can be verified that in fact $\Theta_{\mathrm{PU}_3}(\sigma^+) = i_{Q_2}^G (\mu \circ \det)$.

We use the previous corollary to determine the non-vanishing of the twisted coinvariant spaces of π^+.

Proposition 6.7. *Consider the same notation as above. Let ψ' be a character of U_2. We have*

$$\dim \left(\mathrm{Hom}_{U_2}(\pi^+, \psi') \right) \geq 1.$$

Proof. By Corollary 6.5, it is enough to see that

$$\dim \left(\mathrm{Hom}_{U_2}(i_{Q_2}^G (\mu \circ \det), \psi') \right) \geq 1.$$

Using the definition of normalized induction, we have

$$i_{Q_2}^G (\mu \circ \det) = \left\{ f : G_2 \to \mathbb{C} \mid f(qx) = (\mu \circ \det)(q)\delta_{Q_2}^{1/2}(q)f(x) \text{ for } q \in Q_2, \ x \in G \right\},$$

where we consider locally constant functions. The group G acts on the left of this space as follows: if $g \in G$ and $f \in i_{Q_2}^G (\mu \circ \det)$, then $(g \cdot f)(x) = f(xg)$ for every $x \in G$. Restrict the representation $i_{Q_2}^G (\mu \circ \det)$ to a representation of U_2. Let $C_c^\infty(U_2)$ be the regular representation (consisting on smooth functions with compact support) of U_2. Then, we have an injective map of U_2-representations

$$C_c^\infty(U_2) \hookrightarrow i_{Q_2}^G (\mu \circ \det).$$

We proceed to describe it. The Bruhat decomposition of G_2 with respect to the parabolic subgroup Q_2 is

$$G_2 = Q_2 \cup Q_2 w_\beta Q_2 \cup Q_2 w_{\beta\alpha\beta} Q_2 \cup Q_2 w_{\beta\alpha\beta\alpha\beta} Q_2$$

(see [JR97, Page 289]). Hence, we can find the following U_2-invariant subspace of $i_{Q_2}^G(\mu \circ \det)$

$$\left\{ f : Q_2 w_{\beta\alpha\beta\alpha\beta} U_2 \to \mathbb{C} \mid f(qx) = (\mu \circ \det)(q) \delta_{Q_2}^{1/2}(q) f(x) \text{ for } q \in Q_2, \ x \in Q_2 w_{\beta\alpha\beta\alpha\beta} U_2 \right\}$$
$$\simeq \left\{ f : w_{\beta\alpha\beta\alpha\beta} U_2 \to \mathbb{C} \right\},$$

where we again consider locally constant functions. Note that we used that the big cell in the Bruhat decomposition of G_2 given above is the one corresponding to $w_{\beta\alpha\beta\alpha\beta}$ and therefore we have a natural bijection $Q_2 w_{\beta\alpha\beta\alpha\beta} U_2 \simeq Q_2 \times U_2$. If we further restrict the second space to functions of compact support, we obtain the desired U_2-representation subspace of $i_{Q_2}^G(\mu \circ \det)$ isomorphic to the regular representation $C_c^\infty(U_2)$.

Taking (U_2, ψ')-coinvariants, which preserve injectivity by smoothness, we get

$$C_c^\infty(U_2)_{U_2, \psi'} \hookrightarrow \left(i_{Q_2}^G(\mu \circ \det) \right)_{U_2, \psi'}.$$

Since, $C_c^\infty(U_2)$ is the regular representation, we have that $C_c^\infty(U_2)_{U_2, \psi'}$ is 1-dimensional, and the result follows from there. $\square$

Remark 6.8. Consider the same notation as above and suppose in addition that ψ' is generic. It follows from [JR97, Theorem 3] that $\dim\left(\mathrm{Hom}_{U_2}(i_{Q_2}^G(\mu \circ \det), \psi') \right) \leq 1$. Hence, we have $\dim\left(\mathrm{Hom}_{U_2}(i_{Q_2}^G(\mu \circ \det), \psi') \right) = 1$.

6.3. **Vanishing of the non-generic part.** In this subsection, we show the vanishing of the non-generic part of π^- when $\chi^2 = 1$ by proving that the corresponding generic Fourier coefficients along the unipotent U_2 are all zero.

Recall that $J = J_3(K)$, the set of 3×3 Hermitian matrices defined in Section 3.3. Let $P = MN$ be the Heisenberg maximal parabolic subgroup of H, where H is as defined in Section 3.3, and let $\bar{P} = M\bar{N}$ be the parabolic subgroup opposite to P. Let Z (resp. $\bar{Z}$) be the center of N (resp. $\bar{N}$). Then N/Z and $\bar{N}/\bar{Z}$ are commutative, and we have that $N/Z \cong \mathfrak{n}/\mathfrak{z}$ and $\bar{N}/\bar{Z} \cong \bar{\mathfrak{n}}/\bar{\mathfrak{z}}$, where $\mathfrak{n}, \mathfrak{z}, \bar{\mathfrak{n}}$ and $\bar{\mathfrak{z}}$ denote the Lie algebras of $N, Z, \bar{N}$ and $\bar{Z}$, respectively. According to [GS98, (2.5)], we have the following decompositions:

$$\mathfrak{n}/\mathfrak{z} = F \oplus J \oplus J^* \oplus F^*$$
$$\bar{\mathfrak{n}}/\bar{\mathfrak{z}} = F^* \oplus J^* \oplus J \oplus F.$$

The Killing form on $\mathfrak{h}$ induces a non-degenerate pairing between $N/Z \cong \mathfrak{n}/\mathfrak{z}$ and $\bar{N}/\bar{Z} \cong \bar{\mathfrak{n}}/\bar{\mathfrak{z}}$ which is given by

$$\langle (x, u, u^*, x^*), (y^*, v^*, v, y) \rangle = xy^* + \langle u, v^* \rangle + \langle v, u^* \rangle + yx^*.$$

The following theorem gives a useful description of the minimal representation Π of H.

Theorem 6.9. [MS97, Theorem 7.1] *Let Π be the minimal representation of H. Let Π_Z and Π_N be the maximal Z-invariant and N-invariant quotients of Π. Then Π_Z has a P-equivariant filtration*

$$0 \to C_c^\infty(\bar{\Omega}) \to \Pi_Z \to \Pi_N \to 0,$$

where $\bar{\Omega}$ is the smallest nontrivial M-orbit in $\bar{N}/\bar{Z}$ and $C_c^\infty(\bar{\Omega})$ denotes the space of locally constant, compactly supported functions on $\bar{\Omega}$.

Moreover, the action of P on $C_c^\infty(\bar\Omega)$ is given by

$$\pi(n)f(x) = \psi(\langle n, x\rangle)f(x), \quad n \in N,$$

$$\pi(m)f(x) = |\det(m)|^{1/5}f\left(m^{-1}xm\right), \quad m \in M,$$

where $\det$ denotes the determinant of the representation of M on N/Z.

Now we consider $G_2 \cap P = L_2 U_2 = \mathrm{GL}_2 U_2$, the Heisenberg parabolic subgroup of G_2. Notice that

$$\bar{U}_2/\bar{Z} \cong F^* \oplus Fe^* \oplus Fe \oplus F \subset F^* \oplus J^* \oplus J \oplus F \cong \bar{N}/\bar{Z},$$

where e is an element of J that is invariant under the action of $G' = \mathrm{PU}_3 \rtimes \mu_2$, normalized by $\langle e^*, e\rangle = 3$. An element $(a, b, c, d) \in \bar{U}_2/\bar{Z}$ defines a binary cubic form

$$ax^3 + bx^2y + cxy^2 + dy^3,$$

which defines a cubic separable algebra if and only if the corresponding GL_2-orbit of (a, b, c, d) is generic. Moreover, the restriction of $\langle\,,\,\rangle$ defines a non-degenerate pairing between U_2/Z and $\bar{U}_2/\bar{Z}$. Hence, a point $\bar{n} = (a, b, c, d) \in \bar{U}_2/\bar{Z}$ corresponding to a cubic separable algebra E defines a character $\psi_E(x) \coloneqq \psi(\langle x, \bar{n}\rangle)$ on U_2/Z.

We define the twisted Jacquet module Π_{U_2, ψ_E} to be the maximal quotient of Π on which the action of U_2 is given by the character ψ_E. Theorem 6.9 enables us to describe the PU_3-module Π_{U_2, ψ_E} explicitly as stated in the following proposition, which is essentially a summary of [GS98, Proposition 2.8] and [GS98, Lemma 2.9].

Proposition 6.10. *Let E be a cubic separable F-algebra and let $X_E = \{A \in H_3^e(K) : f_A = f_E\}$, where*

- *$H_3^e(K) \coloneqq \{A \in M_3(K) : A = eA^\dagger e^{-1}\}$;*
- *f_A is the characteristic polynomial of $A \in H_3^e(K)$;*
- *f_E is a fixed monic cubic polynomial such that $E \cong F[x]/(f_E(x))$.*

Then, we have the following isomorphism of PU_3-modules:

$$\Pi_{U_2, \psi_E} \cong C_c^\infty(X_E).$$

Proof. Since ψ_E is not trivial, we have $(\Pi_N)_{U_2, \psi_E} = 0$. Hence, taking (U_2, ψ_E)-coinvariants in the short exact sequence in Theorem 6.9, we obtain that $C_c^\infty(\bar\Omega)_{U_2, \psi_E} \simeq \Pi_{U_2, \psi_E}$.

Now we show that $C_c^\infty(\bar\Omega)_{U_2, \psi_E} \simeq C_c^\infty(X_E)$. For $n \in U_2$ and $f \in C_c^\infty(\bar\Omega)$, we have

$$\pi(n)f(x) = \psi(\langle n, x\rangle)f(x)$$

by Theorem 6.9. It follows from the proof of [GS98, Lemma 2.9] that the natural projection map $C_c^\infty(\bar\Omega) \twoheadrightarrow C_c^\infty(\bar\Omega)_{U_2, \psi_E}$ can be realized as the restriction map $C_c^\infty(\bar\Omega) \twoheadrightarrow C_c^\infty(X_E')$, where

$$X_E' = \{A \in J : f_{Ae^{-1}} = f_E\} \subseteq \bar\Omega.$$

An element $z \in \mathrm{PU}_3$ acts on X_E' by $A \mapsto zAz^\dagger$. The map $A \mapsto Ae^{-1}$ defines a bijection of PU_3-sets between X_E' and X_E, with PU_3 acting on the latter by conjugation. $\square$

In the statement and proof of the following lemma, we regard $H_3^e(K)$, $M_3(K)$ and any cubic separable F-algebra E as Jordan algebras with multiplication defined by

$$x \circ y = \frac{1}{2}(xy + yx).$$

Of course, on E this agrees with the standard multiplication.

Lemma 6.11. *Let E be a cubic separable F-algebra. The following PU_3-sets are in natural bijection:*

(1) *the set X_E defined above, with PU_3 acting by conjugation;*
(2) *the set of embeddings $i : E \hookrightarrow H_3^e(K)$ of Jordan algebras over F, with PU_3 acting by conjugation;*
(3) *the set of embeddings $i : E \otimes_F K \hookrightarrow M_3(K)$ of K-algebras which are compatible with the anti-involution defined by the nontrivial element $c \in \mathrm{Gal}(K/F)$ on each side, given on $E \otimes_F K$ by the natural Galois action and on $M_3(K)$ by $A \mapsto eA^\dagger e^{-1}$, with PU_3 acting by conjugation.*

Proof. An embedding $i : E \hookrightarrow H_3^e(K)$ of Jordan algebras over F is determined by the image of the element $x \in E \cong F[x]/(f_E(x))$. Since i is an embedding, the characteristic polynomial of $i(x)$ must be equal to f_E. This gives a natural bijection between (1) and (2).

Observe that there is an isomorphism of K-algebras

$$M_3(K) \otimes_F K \cong M_3(K) \times M_3(K)$$

given on pure tensors by $A \otimes a \mapsto (aA, a\bar{A})$. Under this isomorphism, the action of the nontrivial element $c \in \mathrm{Gal}(K/F)$, which acts on $M_3(K) \otimes_F K$ via its natural action on the right factor K, is given on $M_3(K) \times M_3(K)$ by $(A, B) \mapsto (\bar{B}, \bar{A})$. Moreover, the anti-involution on $M_3(K) \otimes_F K$ defined by $A \otimes a \mapsto eA^\dagger e^{-1} \otimes a$ corresponds to the anti-involution on $M_3(K) \times M_3(K)$ defined by $(A, B) \mapsto (eB^T e^{-1}, eA^T e^{-1})$. Therefore, the subspace $H_3^e(K) \otimes_F K$ maps to

$$\{(A, eA^T e^{-1}) : A \in M_3(K)\} \subseteq M_3(K) \times M_3(K),$$

and, by projection onto the first factor, we obtain an isomorphism $H_3^e(K) \otimes_F K \cong M_3(K)$ of Jordan algebras over K. Moreover, under this isomorphism, the nontrivial element $c \in \mathrm{Gal}(K/F)$ acts on $M_3(K)$ by $A \mapsto eA^\dagger e^{-1}$, so that $M_3(K)^{\mathrm{Gal}(K/F)}$ is the set of matrices $H_3^e(K) \subseteq H_3^e(K) \otimes_F K \cong M_3(K)$. It follows from these remarks that there is a natural bijection between (2) and (3). $\square$

We now prove that the sets appearing in the previous lemma are always non-empty.

Lemma 6.12. *There exists an embedding $i : E \otimes_F K \hookrightarrow M_3(K)$ of K-algebras compatible with the anti-involution defined by the nontrivial element $c \in \mathrm{Gal}(K/F)$ on each side, given on $E \otimes_F K$ by the natural Galois action and on $M_3(K)$ by $A \mapsto eA^\dagger e^{-1}$.*

Proof. For any $\lambda \in F^\times$, we define the following Hermitian pairing on the K-vector space $E \otimes_F K$. For $x \otimes a, y \otimes b \in E \otimes_F K$, we define

$$\langle x \otimes a, y \otimes b \rangle_\lambda = \lambda ab^c \, \mathrm{Tr}(xy),$$

where $\mathrm{Tr} : E \to F$ denotes the trace form, and we extend the pairing additively. Choosing a suitable λ, the Hermitian space $(E \otimes_F K, \langle\,,\,\rangle_\lambda)$ is isomorphic to the Hermitian space $(V = K^3, \Phi_3 = e)$. This provides an embedding

$$E \otimes_F K \hookrightarrow \mathrm{End}_K(E \otimes_F K) \simeq \mathrm{End}_K(V) = M_3(K).$$

The actions of $c \in \mathrm{Gal}(K/F)$ on $E \otimes_F K$ and on $M_3(K)$ defined in the statement amount to taking the adjoint with respect to the corresponding Hermitian pairing. Therefore, the previous embedding is compatible with the action of c.

$\square$

Given a cubic separable F-algebra E, let $\underline{T}_{E,K}$ denote the algebraic group over F such that, for an F-algebra R,

$$\underline{T}_{E,K}(R) = \{x \in E \otimes_F R \otimes_F K : \ \mathrm{N}_{E \otimes_F R \otimes_F K / E \otimes_F R}(x) = 1\},$$

and let $T_{E,K}$ denote the corresponding group of F-points. An embedding $i' : E \otimes_F K \hookrightarrow M_3(K)$ compatible with the anti-involution defined by the nontrivial element $c \in \mathrm{Gal}(K/F)$ on each side, as described in the previous lemma, defines an embedding of algebraic groups $i : \underline{T}_{E,K} \hookrightarrow \underline{U}_3$. For an F-algebra R, the corresponding map on R-points is the map

$$(E \otimes_F K \otimes_F R)^1 \hookrightarrow \underline{U}_3(R)$$

obtained by restricting the map induced by i' to norm-1 elements. Let $i'_0 : E \otimes_F K \hookrightarrow M_3(K)$ be an embedding compatible with the action of $c \in \mathrm{Gal}(K/F)$ and let i_0 be the corresponding embedding of algebraic groups $\underline{T}_{E,K} \hookrightarrow \underline{U}_3$. Let $\tilde{X}_E$ denote the set of embeddings of algebraic groups $\underline{T}_{E,K} \hookrightarrow \underline{U}_3$ which are conjugate to i_0 by an element of $\mathrm{U}_3(\bar{F})$. The group PU_3 acts on $\tilde{X}_E$ by conjugation.

Proposition 6.13. *Let E be a cubic separable F-algebra. There is a bijection of PU_3-sets between X_E and $\tilde{X}_E$.*

Proof. By Lemma 6.11, we just need to show that there is a bijection of PU_3-sets between $\tilde{X}_E$ and the set of embeddings of K-algebras $i' : E \otimes_F K \hookrightarrow M_3(K)$ which are compatible with the anti-involution defined by the nontrivial element $c \in \mathrm{Gal}(K/F)$ on each side. Such a map can be extended to an embedding of $\bar{F}$-algebras

$$E \otimes_F K \otimes_F \bar{F} \hookrightarrow M_3(K \otimes_F \bar{F})$$

compatible with the action of $c \in \mathrm{Gal}(K/F)$. Giving such a map is equivalent to giving an embedding of $\bar{F}$-algebras

$$\bar{F} \times \bar{F} \times \bar{F} \hookrightarrow M_3(\bar{F}).$$

Therefore, it becomes clear that, after choosing an embedding $K \hookrightarrow \bar{F}$, any two embeddings $i'_1, i'_2 : E \otimes_F K \hookrightarrow M_3(K)$ compatible with the action of $c \in \mathrm{Gal}(K/F)$ are conjugate by an element of $\mathrm{U}_3(\bar{F}) \simeq \mathrm{GL}_3(\bar{F})$. In particular, any embedding $i' : E \otimes_F K \hookrightarrow M_3(K)$ compatible with the action of $c \in \mathrm{Gal}(K/F)$ is conjugate to i'_0 by an element of $\mathrm{U}_3(\bar{F})$. Moreover, if $a \in \mathrm{U}_3(\bar{F})$, then $a \cdot i'_0 \cdot a^{-1}$ defines an embedding $E \otimes_F K \hookrightarrow M_3(K)$ compatible with the action of $c \in \mathrm{Gal}(K/F)$ if and only if the cocycle $\tau \mapsto a^{-1} \cdot \tau(a)$ defines a cohomology class in

$$\ker\left(H^1(F, \underline{T}_{E,K}) \longrightarrow H^1(F, \underline{U}_3)\right).$$

This is the same condition that $a \in \mathrm{U}_3(\bar{F})$ must satisfy in order that $a \cdot i_0 \cdot a^{-1}$ defines an embedding of algebraic groups $\underline{T}_{E,K} \hookrightarrow \underline{\mathrm{U}}_3$. Furthermore, $a \in \mathrm{U}_3(\bar{F})$ fixes i_0' if and only if $a \in i_0(\underline{T}_{E,K}(\bar{F}))$ and thus if and only if it fixes i_0. Hence, it follows that the map defined above sending an embedding $i' : E \otimes_F K \hookrightarrow M_3(K)$ compatible with the action of $c \in \mathrm{Gal}(K/F)$ to an embedding of algebraic groups $i : \underline{T}_{E,K} \hookrightarrow \underline{\mathrm{U}}_3$ provides the desired bijection. $\qquad\square$

Let $\tilde{\mathcal{T}}_{E,K}$ denote set of PU_3-orbits of $\tilde{X}_E$.

Corollary 6.14. *Let E be a cubic separable F-algebra. Then,*

$$\Pi_{U_2,\psi_E} \cong \bigoplus_{i \in \tilde{\mathcal{T}}_{E,K}} \text{c-Ind}_{i(T_{E,K})}^{\mathrm{PU}_3} \mathbb{1}.$$

Proof. It follows from Proposition 6.10 that

$$\Pi_{U_2,\psi_E} \cong C_c^\infty(X_E).$$

Proposition 6.13 establishes a bijection of PU_3-sets between X_E and $\tilde{X}_E$. The stabilizer of an embedding of algebraic groups $i : \underline{T}_{E,K} \hookrightarrow \underline{\mathrm{U}}_3$ for the conjugation action of PU_3 is $i(T_{E,K})$. Therefore, decomposing X_E into PU_3-orbits, we obtain

$$C_c^\infty(X_E) \cong \bigoplus_{i \in \tilde{\mathcal{T}}_{E,K}} \text{c-Ind}_{i(T_{E,K})}^{\mathrm{PU}_3} \mathbb{1}.$$

$\qquad\square$

Corollary 6.15. *Let E be a cubic separable F-algebra. Then,*

$$\dim \mathrm{Hom}_{U_2}(\Theta_{\mathrm{PU}_3}(\sigma^+), \psi_E) + \dim \mathrm{Hom}_{U_2}(\Theta_{\mathrm{PU}_3}(\sigma^-), \psi_E) = 1.$$

Proof. It follows from the previous corollary and an application of Frobenius reciprocity for compact induction that the left-hand side is equal to

$$\sum_{i \in \tilde{\mathcal{T}}_{E,K}} \left(\dim \mathrm{Hom}_{i(T_{E,K})}((\sigma^+)^\vee, \mathbb{1}) + \dim \mathrm{Hom}_{i(T_{E,K})}((\sigma^-)^\vee, \mathbb{1}) \right).$$

Since both σ^+ and σ^- are unitary representations, this sum can be rewritten as

$$\sum_{i \in \tilde{\mathcal{T}}_{E,K}} \left(\dim \mathrm{Hom}_{i(T_{E,K})}(\sigma^+, \mathbb{1}) + \dim \mathrm{Hom}_{i(T_{E,K})}(\sigma^-, \mathbb{1}) \right),$$

which is equal to 1 by [BFG$^+$, Corollary 4.2]. $\qquad\square$

Proposition 6.16. *If $\chi^2 = 1$, the representation π^- does not have a nonzero non-generic subquotient.*

Proof. Combining Proposition 6.7 and the previous corollary, it follows that, for all character ψ_E of U_2 corresponding to a cubic separable F-algebra E as above,

$$\mathrm{Hom}_{U_2}(\pi^-, \psi_E) = 0.$$

Therefore, it follows from [BS, Lemma 4.13] that any non-generic subquotient of π^- is finite-dimensional. Since π^- is tempered by [BS, Remark 4.4], it cannot have any finite-dimensional subquotient, which concludes the proof. $\qquad\square$

Combining Proposition 6.1 and Proposition 6.16, we finally obtain the following.

Theorem 6.17. *If $\chi^2 = 1$, the representation π^- is zero.*

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