1. Introduction

Inverses.

Given a function f, we define an inverse function f^{-1} so that

$$f^{-1}(x) = y \Rightarrow f(y) = x.$$

You already know several inverse functions. The inverse of a function "undoes" a given function. For instance, subtraction is the inverse of addition and division of multiplication. More precisely, if

$$f(x) = 5x,$$

i.e. if our function is multiplication by 5, then

$$f^{-1}(x) = x/5,$$

i.e. our inverse is division by 5.

Notice that

$$5x = 35$$

is the same as saying

$$x = 35/5,$$

or

$$f(x) = 35$$

is equivalent to

$$x = f^{-1}(35).$$

Another way to put this is: f^{-1} answers the question

$$5 \times \ what = 35?$$

What if

$$f(x) = x^3?$$

We define the function

$$\sqrt[3]{\ }$$

to be the inverse. How do we determine

$$\sqrt[3]{x}?$$

We ask ourselves what we cube to get x. For example, what is

$$\sqrt[3]{8}?$$

Let us set

$$\sqrt[3]{8} = x.$$

What we then know is that

$$x^3 = 8.$$

There is only one number which gives us 8 when cubing and that is 2.

Similarly, consider

$$g(x) = 2^x.$$

We write the inverse of g as

$$\log_2 .$$

So what is

$$\log_2 8?$$

Again, if

$$\log_2 8 = x,$$

then

$$2^x = 8.$$

We then know x must be 3. Logs and roots are completely analogous to each other.

The next three statements are 3 different ways of saying the exact same thing:

$$2^3 = 8,$$

$$\sqrt[3]{8} = 2,$$

$$\log_2 8 = 3.$$

More generally, the following are equivalent for any positive $x \neq 1$:

$$x^y = z,$$

$$\sqrt[y]{z} = x,$$

$$\log_x z = y.$$

What is

$$\log_4 64?$$

Well, we want x such that

$$4^x = 64.$$

$$x = 3.$$

What is

$$\sqrt[3]{125}?$$

We want x such that

$$x^3 = 125.$$

$$x = 5.$$

(Note I am not doing anything clever to find the x that satisfies these equations, just guessing and checking.)

2. Fake Inverses

Moving on, let
$$f(x) = x^2.$$
What is
$$f^{-1}(4)?$$
Well, in this case the inverse of f is poorly defined. For there are 2 possibilities: $x = 2$ and $x = -2$. Both of these numbers give 4 when squared. The problem with this function as opposed to multiplying or cubing is that it is not one-to-one. Two inputs give the same output. But you know the $\sqrt{\ }$ function and you often think of it as being the opposite of squaring.
$$\sqrt{\ }$$
is a fake inverse. What I mean is that
$$\sqrt{x} = y$$
does imply
$$x = y^2;$$
however
$$x = y^2$$
does not mean
$$\sqrt{x} = y.$$
As
$$4 = (-2)^2,$$
but
$$\sqrt{4} \neq -2.$$
How we define a fake inverse of a function is by looking at our choice and choosing one.

For
$$f(x) = x^2,$$
there are (usually) two possibilities. We choose the positive (nonnegative) one.

So we define
$$\sqrt{x}$$
to be the nonnegative number y so that
$$y^2 = x.$$
This is especially relevant as we are about to look at trigonometric functions. And of course if I asked you what x makes
$$\sin(x) = 0,$$
you would respond: "Well, it could be that
$$x = 0$$
or
$$x = \pi$$
or
$$x = 2\pi$$

or

$$x = 3\pi$$

or...", and you would go on like that for a while.

3. Trig inverses

So define

$$\sin^{-1}(x)$$

to be y so that

$$\sin(y) = x$$

and

$$-\pi/2 \leq y \leq \pi/2.$$

So what is

$$\sin^{-1}(0).$$

Well, of all the choices we named, only 0 is in the appropriate range. So

$$\sin^{-1}(0) = 0.$$

How about

$$\sin^{-1}(-1/2)?$$

We recall sin gives the y coordinate on the unit circle. So the question is how far around the circle do we walk until we reach $-1/2$. Walking forward (counterclockwise) we first reach it just before 2π, at $11\pi/6$. But that is out of our desired range. Instead, we go backward (clockwise). So

$$\sin^{-1}(-1/2) = \pi/6$$

We must make similar choices of range for the other functions. For $\tan^{-1}$ we use

$$[-\pi/2, \pi/2]$$

as well. For $\cos^{-1}$ we use

$$[0, \pi].$$

On exams, I will mainly be interested in you being able to plug in 0,1, and -1 to these functions. For $\sin^{-1}$ I am telling you the y-coordinate and asking where on the circle it lies. For $\cos^{-1}$ I am telling you the x-coordinate and telling you where on the circle it lies. And

$$\tan^{-1}$$

I am giving you the ratio of the two, which is harder. But note that

$$\tan(1) = y$$

means

$$\tan(y) = 1$$

which means

$$\frac{\sin(x)}{\cos(x)} = 1,$$

or

$$\sin(x) = \cos(x).$$

Thinking about triangles I realize this happens when

$$x = \pi/4.$$

Now you try.
What is

$$\sin^{-1}(-1), \tan^{-1}(-1), \cos^{-1}(-1)?$$

Think about it.

WORD OF WARNING.

$$\sin^{-1}(x)$$

should NOT be confused with

$$(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc(x).$$

These are very different despite similarities of notation.