Convex Projective Deformations of the Figure-8 Knot Complement

Sam Ballas
(Joint with Daryl Cooper and Darren Long)

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An Overview

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- Retains many features of hyperbolic geometry.

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- Retains many features of hyperbolic geometry.
- No Mostow rigidity.

Projective Space

- There is a natural action of $\mathbb{R}^\times$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by scaling.
- Let $\mathbb{R}P^n = P(\mathbb{R}^{n+1} \setminus \{0\})$ be the quotient of this action.

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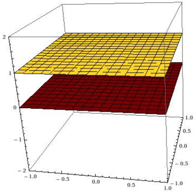
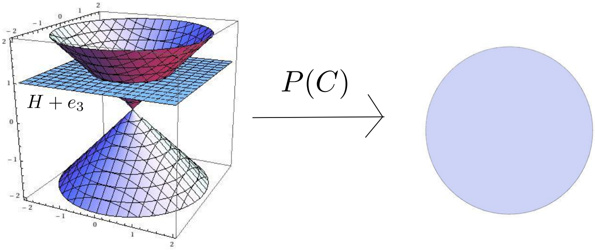
- Let H be a hyperplane in $\mathbb{R}^{n+1}$.
- H gives rise to a splitting of $\mathbb{R}P^n = \mathbb{R}^n \sqcup \mathbb{R}P^{n-1}$ into an affine part and an ideal part (homogeneous coordinates).



The Klein Model

- Let $\langle x, y \rangle = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$ be standard form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$.
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0\}$



Upper Half Space Model

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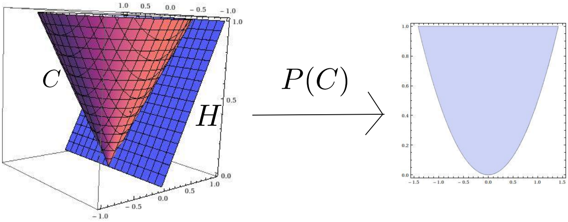

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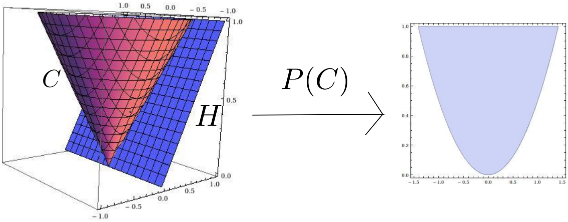
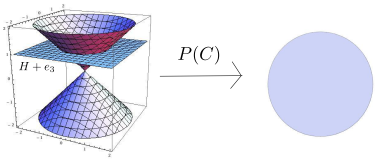


Parabolic translations fixing ∞ will be of the form

$$\begin{pmatrix} 1 & v & \frac{1}{2}|v| \\ 0 & I_{n-1} & v \\ 0 & 0 & 1 \end{pmatrix}$$

Nice Properties of Hyperbolic Space



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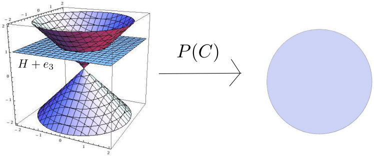

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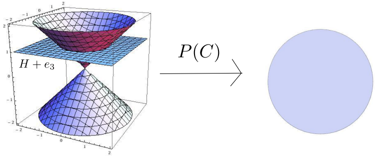

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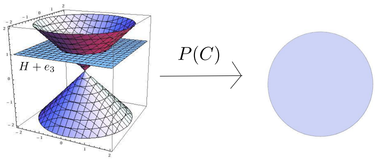

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Convex projective geometry focuses on the geometry of properly (and sometimes strictly) convex domains.

Convex Projective Manifolds

Let M^n be a manifold with $\pi_1(M) = \Gamma$. A *convex projective structure* on M is a pair (Ω, ρ) such that

1. Ω is a properly convex open subset of $\mathbb{R}P^n$.
2. $\rho : \Gamma \to \mathrm{PGL}(\Omega)$ is a discrete and faithful representation.
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- Complete hyperbolic manifolds are examples of strictly convex projective manifolds.

Projective Equivalence

Suppose that $M^n \cong \Omega_i/\rho_i(\Gamma)$ for $i = 1, 2$, then (Ω_1, ρ_1) and (Ω_2, ρ_2) are *projectively equivalent* if there exists $h \in \mathrm{PGL}_{n+1}(\mathbb{R})$ such that $h(\Omega_1) = \Omega_2$ and for each $\gamma \in \pi_1(M)$

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- If (Ω_1, ρ_1) and (Ω_2, ρ_2) are projectively equivalent then $\rho_2(\Gamma) = h\rho_1(\Gamma)h^{-1}$
- Projective equivalence classes of M are in bijective correspondence with $\rho : \Gamma \to \mathrm{PGL}_{n+1}(\mathbb{R})$ that are faithful, discrete, and preserve a properly convex set.

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   - Bending (Johnson-Millson)
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2. How do we know if they exist in general?

The Closed Case

Theorem 1 (Koszul)

Let M be a closed 3-manifold and ρ_0 be the holonomy of a properly convex structure on M. If ρ_t is sufficiently close to ρ_0 in $\mathrm{Hom}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ then ρ_t is the holonomy of a convex projective structure on M

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- Small deformations of holonomy correspond to small deformations of the convex projective structure
- Space of convex projective structures is open inside of $\mathrm{Hom}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$.

Non-Compact Case

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- There are representations near ρ_0 that are not discrete and non-faithful (Dehn surgery space).
- We need to control the behavior near the boundary of M in order to get an analogue of Theorem 1.

Non-Compact Case

Let M be an orientable, non-compact, finite volume hyperbolic 3-manifold, then $M = M_K \sqcup_i C_i$, where M_K is compact and $C_i \cong T^2 \times [1, \infty)$.

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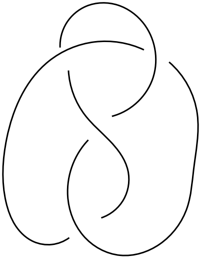
Theorem 2 (Cooper-Long)

Let M be as above and ρ_0 the holonomy of the complete hyperbolic structure on M. Let $\rho_t \in \mathrm{Hom}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ such that

1. *ρ_t is sufficiently close to ρ_0 in $\mathrm{Hom}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$*
2. *For each cusp C, the restriction of ρ_t to $\pi_1(C)$ is the holonomy of a properly convex structure on C that is sufficiently close to the hyperbolic structure on C coming from ρ_0.*

Then ρ_t is the holonomy of a properly convex structure on M.

Figure-8 Deformations



Let M be the figure-8 knot complement and $\Gamma = \pi_1(M)$. Then $\Gamma = \langle \alpha, \beta | \alpha\omega = \omega\beta \rangle$, where α and β are meridians and $\omega = \beta^{-1}\alpha\beta\alpha^{-1}$.

Figure-8 Deformations

Theorem 3 (B)

There is a family ρ_t of nonconjugate representations of Γ into $\mathrm{PGL}_4(\mathbb{R})$.

$$\rho_t(\alpha) \mapsto \begin{pmatrix} 1 & 0 & 1 & t-1 \\ 0 & 1 & 1 & t \\ 0 & 0 & 1 & t+\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_t(\beta) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2+\frac{1}{t} & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

The complete hyperbolic representation occurs at $t = \frac{1}{2}$.

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Let $\pi_1(\partial M) = \langle \mu, \lambda \rangle$. For $t \neq \frac{1}{2}$, after conjugation

$$\rho_t(\mu) = \begin{pmatrix} 1 & 0 & b(t) & \frac{1}{2}b(t)^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_t(\lambda) = \begin{pmatrix} 1 & 0 & 0 & -a(t) \\ 0 & e^{a(t)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $a(t) \to 0$ and $b(t) \to 0$ as $t \to \frac{1}{2}$.

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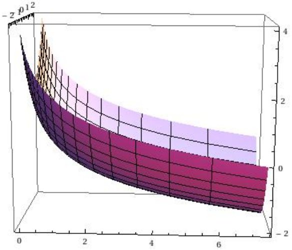
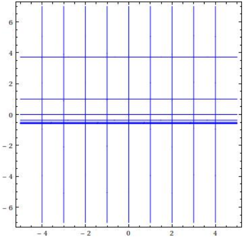
$\langle \rho_t(\mu), \rho_t(\lambda) \rangle$ preserves a properly convex domain



Cross sections are affine tori.



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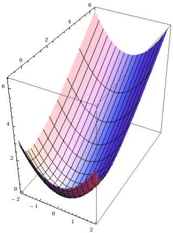
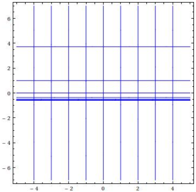
We can further conjugate to prevent this collapse so that

$$t = 1$$





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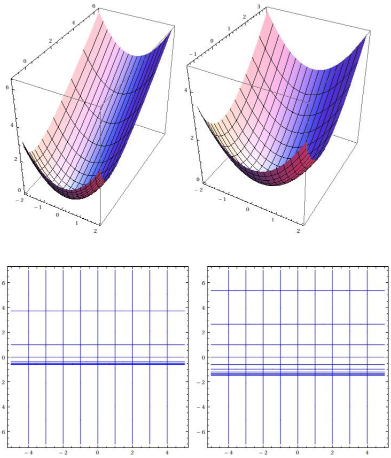

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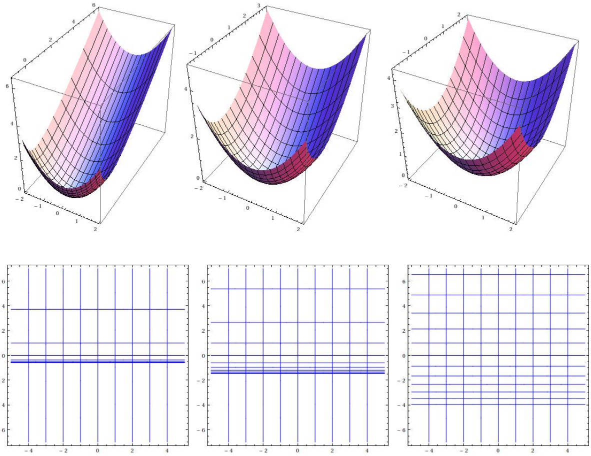

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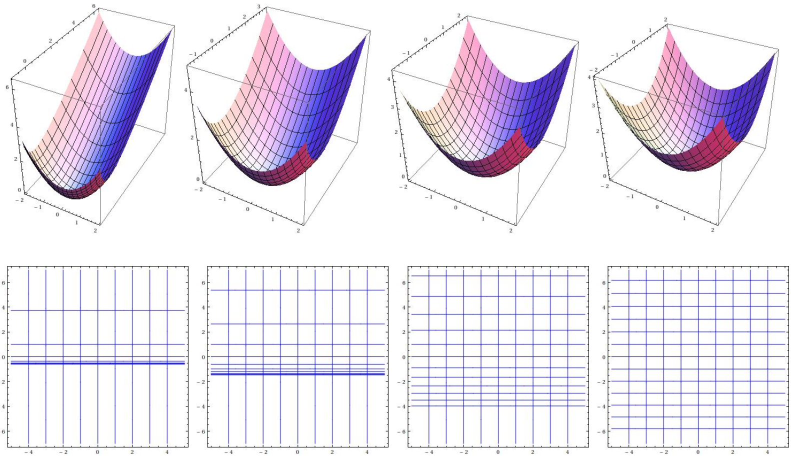

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Theorem 4 (B-Cooper-Long)

The representations ρ_t are holonomies of convex projective structrures on the figure-8 knot complement.