Conditions for the Equivalence of Largeness and Positive vb_1

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Outline

Virtually Haken Conjecture

- Definition

  A compact, orientable, irreducible 3-manifold is *Haken* if it contains an orientable, incompressible, embedded surface. A 3-manifold is *virtually Haken* if it is finitely covered by a Haken manifold.

- Conjecture (Virtually Haken Conjecture)

  Any compact, orientable, hyperbolic 3-manifold is virtually Haken.

Virtual First Betti Number

- Definition
  The *first Betti number* of a manifold M, denoted $b_1(M)$, is the rank of $H_1(M, \mathbb{Q})$, and the *virtual first Betti number* of a manifold M, denoted $vb_1(M)$ is equal to $\max\{b_1(N) \mid N \text{ is a finite cover of } M\}$, and ∞ if no such maximum exists.
- Conjecture (Positive Virtual Betti Number Conjecture)
  For any compact, orientable, hyperbolic 3-manifold, M, $vb_1(M) > 0$, or equivalently $\pi_1(M)$ has infinite abelianization.
- Conjecture (Infinite Virtual Betti Number Conjecture)
  For any compact, orientable hyperbolic 3-manifold, M, $vb_1(M) = \infty$.

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Largeness

- **Definition**
  A group, G, is *large* if a finite index subgroup admits a surjection onto a free non-abelian group. A manifold, M, is large if its fundamental group is large.

- **Conjecture (Largeness Conjecture)**
  The fundamental group of any closed, orientable hyperbolic 3-manifold is large.

Relationships Between the Conjectures

- Largeness

  $\Downarrow$

- Infinite Virtual First Betti Number

  $\Downarrow$

- Positive Virtual First Betti Number

  $\Downarrow$

- Virtually Haken

The Main Theorem

One question that can be asked is when are these conjectures equivalent. The following theorems provides a partial answer.

Theorem (Cooper, Long, Reid 97)

Let M be a compact, orientable, irreducible 3-manifold with non-empty boundary. Then, either M is an I-bundle over a surface with non-negative Euler characteristic or $\pi_1(M)$ is large.

Theorem (Lackenby, Long, Reid 08)

Let O be a 3-orbifold commensurable with a closed, orientable hyperbolic 3-orbifold that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its fundamental group. Suppose that $vb_1 \geq 4$, then $\pi_1(O)$ is large.

Hyperbolic Orbifolds

Definition

A *Kleinian Group* is a discrete subgroup of $PSL_2(\mathbb{C})$

Definition

Given a Kleinian group Γ, we call $O = \mathbb{H}^3/\Gamma$ a *hyperbolic orbifold*

Remark

If Γ contains no torsion this agrees with the standard notion of a hyperbolic 3-manifold

Remark

When we think of O as a topological space we will denote it as $|O|$, and we call this the underlying space.

Orbifold Fundamental Group

If Γ has no torsion, then as a topological space $\Gamma = \pi_1(|O|)$, however this is not the case if Γ contains torsion. So for orbifolds we make the following definition

Definition

If $O = \mathbb{H}^3/\Gamma$ then the *orbifold fundamental group of O* denoted $\pi_1^{orb}(O)$ is equal to Γ.

However, most of the time we just refer to the fundamental group as $\pi_1(O)$

Remark

$\pi_1(|O|) \cong \pi_1^{orb}(O)/ << T >>$, where T is the set of elements that do not act freely on $\mathbb{H}^3$.

Singular Locus

Definition

Given a Kleinian group Γ, the *singular locus of* $O = \mathbb{H}^3/\Gamma$ denoted $sing(O)$ is set of orbits Γx such that x is a fixed point of some $\gamma \in \Gamma$

Definition

Given a hyperbolic orbifold $O = \mathbb{H}^3/\Gamma$ the *order of a point* Γx is the order of the finite group Γ_x.

- In a closed hyperbolic 3-orbifold the singular set is a collection of simple closed curves labelled by integers and trivalent graphs with edges labelled by integers.

Singular Locus cont.

We now focus on decomposing the singular locus of an orbifold

- Definition
  Let $sing^0(O)$ be the components of $sing(O)$ with zero Euler characteristic.
- Definition
  Given a prime p let $sing^0_p(O)$ be the components of $sing(O)$ whose orders are divisible by p, and have zero Euler characteristic.

A Sequence of Subgroups in F_n I

- Let F be a free non-abelian group. Define the following sequence, $L_1 = F$ and $L_{i+1} = [L_i, L_i](L_i)^i$.
- Note that L_{i+1} is characteristic in L_i, and thus normal in F.
- By Schreier index formula $d(L_i) = (d(F) - 1)[F : L_i] + 1$ and $L_i / L_{i+1} = (\mathbb{Z}/i\mathbb{Z})^{d(L_i)}$.

A Sequence of Subgroups in F_n II

This sequence has the following properties

(i) L_i/L_{i+1} is abelian for each i

(ii) $\lim_{i\to\infty}((\log[L_i : L_{i+1}])/[F : L_i]) = \infty$

(iii) $\limsup_i((d(L_i/L_{i+1}))/[F : L_i]) > 0$.

- If G is large we can pull this sequence back to G and find a sequence $\{G_i\}$ with the same properties.
- It turns out that when G is finitely presented this characterizes large groups.

The Characterization Theorem

Theorem (Lackenby 05)

Let G be a finitely presented group then the following are equivalent

1. *G is large*
2. *there exists a sequence $G_1 \geq G_2 \geq \ldots$ of finite index subgroups of G, each normal in G_1, such that*
   (i) G_i/G_{i+1} *is abelian for every i*
   (ii) $\lim_{i\to\infty}((\log[G_i : G_{i+1}])/[G : G_i]) = \infty$
   (iii) $\limsup_i(d(G_i/G_{i+1})/[G : G_i]) > 0$

The Characterization Theorem

In fact a slightly stronger theorem holds

Theorem (Lackenby 05)

Let G be finitely presented, and suppose that for each natural number i, there is a triple $H_i \geq J_i \geq K_i$ of finite index normal subgroups of G such that

1. H_i/J_i *is abelian for all* i
2. $\lim_{i\to\infty}((\log[H_i : J_i])/[G : H_i]) = \infty$
3. $\limsup_i(d(J_i/K_i)/[G : J_i]) > 0$

Then K_i admits a surjection onto a free non-abelian group for infinitely many i.

The Characterization Theorem

Weaker Version

We will only need the following weaker theorem

Theorem (Lackenby, Long, Reid 08)

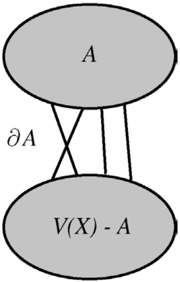
Let G be a finitely presented group, and let $\phi : G \to \mathbb{Z}$ be a surjective homomorphism. Let $G_i = \phi^{-1}(i\mathbb{Z})$, and suppose that for some prime $\{G_i\}$ has linear growth of mod-p homology, then G is large.

Before proceeding we need a few preliminaries.

Graph Boundary

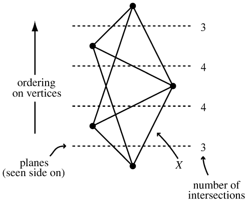
Definition

Given a subset A of vertices of a graph X we define the *boundary of* A, denoted $\partial(A)$, is the set of edges of X that have one vertex in A and the other in A^C.



Width

- **Definition**
  Let X be a finite graph. Given an ordering on $V(X)$, for $1 \leq n \leq |V(X)|$ let D_n be the first n vertices, then the *width of the ordering* is the $\max_n |\partial(D_n)|$. The *width* of the graph X, denoted $w(X)$ is the minimal width over all possible orderings of its vertices.
  - The width of an ordering can be visualized by embedding the graph in $\mathbb{R}^3$ and looking at its intersection with planes.



Schreier Coset Graphs

- **Definition**
  Given a group G with generating set S and a subgroup H of G then the *Schreier coset graph for G/H with respect to S* is the graph, $X(G/H, S)$, with vertex set G/H and edges of the form $\{Hg, Hgs\}$, where $s \in S \cup S^{-1}$.

  Remark
  The width of a Schreier coset graph depends on the choice of generators, however it is still a coarse invariant.

Linear Growth Mod-p Homology

Definition

Given a finitely generated group G we define its Mod-p 1st homology group, denoted $H_1(G, \mathbb{F}_p)$ to be $G/[G,G]G^p$. Given an orbifold, O, $H_1(O, \mathbb{F}_p) = H_1(\pi_1(O), \mathbb{F}_p)$.

Definition

For a finitely generated group G define $d_p(G)$ to be the rank of $H_1(G, \mathbb{F}_p)$.

Definition

Given a sequence $\{G_i\}$ of finite index subgroups of G we say that $\{G_i\}$ has linear growth of Mod-p homology if $\inf_i d_p(G_i)/[G : G_i] > 0$.

The Characterization Theorem

Weaker Version

A reminder of the theorem.

Theorem (Lackenby, Long, Reid 08)

Let G be a finitely presented group, and let $\phi : G \to \mathbb{Z}$ be a surjective homomorphism. Let $G_i = \phi^{-1}(i\mathbb{Z})$, and suppose that for some prime $\{G_i\}$ has linear growth of mod-p homology, then G is large.

Proof of Characterization Theorem I

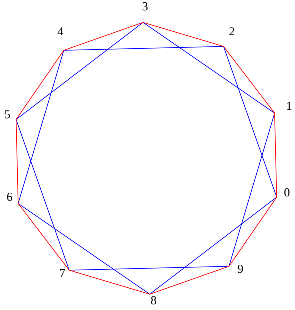
The Weak Version

- The proof of the full characterization theorem requires a few technical lemmas in order to show that $w(X(G/J_i))/[G : J_i] \to 0$.
- Let $H_i = G$, $J_i = \phi^{-1}(i\mathbb{Z})$, where ϕ is the surjective homomorphism to $\mathbb{Z}$, $K_i = [J_i, J_i]J_i^p$, where p is some prime.
- Surjectivity of ϕ gives (1) and (2), and linear growth of Mod-p homology gives (3).

Proof of Characterization Theorem II

The Weak Version

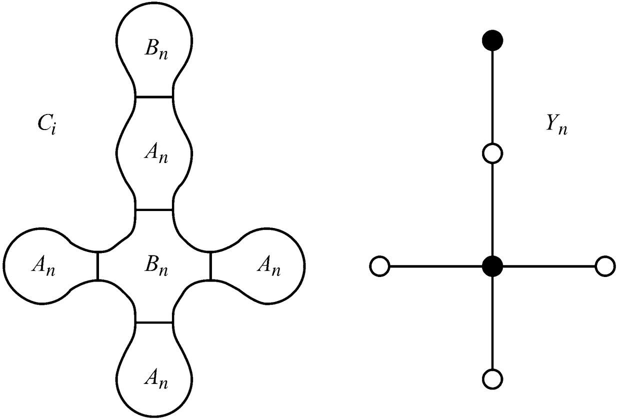
- Since $G/J_i \cong \mathbb{Z}/i\mathbb{Z}$ we get a natural ordering of the vertices of $X(G/J_i)$.
- The mapping to $\mathbb{Z}$ gives a maximum length of generators, and thus a uniform upper bound on $|\partial(D_n)|$.
- Since $[G : J_i] = i$, we see that $w(X(G/J_i))/[G : J_i] \to 0$.



Proof of Characterization Theorem

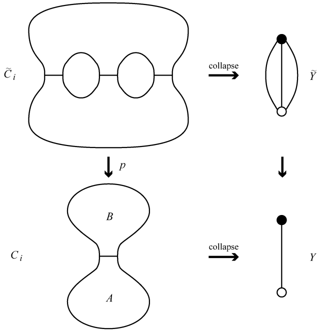
- For sufficiently large i let C_i be a 2-complex with $\pi_1(C_i) = J_i$ and $X(G/J_i, S)$ its 1-skeleton.
- A minimal width ordering on the vertices of $X(G/J_i, S)$ can be extended linearly to all of C_i and then perturbed to an appropriate Morse function, f, on the interior of C_i
- For every $1 \leq n \leq |V(X(G/J_i, S))|$ we can decompose C_i into $A_n = f^{-1}(-\infty, n + 1/2]$ and $B_n = f^{-1}[n + 1/2, \infty)$
- For an appropriate n the fundamental groups of the components of A_n and B_n will have a sufficient number of generators in J_i/K_i

The Decomposition



Proof of Characterization Theorem

- We use this decomposition to collapse C_i to a graph, Y
- Pull this decomposition of C_i back to the covering $\tilde{C}_i$ corresponding to K_i and collapse $\tilde{C}_i$ to a similar graph.
- Since $w(X(G/J_i))/[G : J_i] \to 0$ and $\limsup_i d(J_i/K_i)/[G : J_i] > 0$ there will be vertices with at least 3 edges emanating from them in $\tilde{Y}$.



The Main Theorem

Theorem (Lackenby, Long, Reid 08)

Let O be a 3-orbifold commensurable with a closed, orientable hyperbolic 3-orbifold that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *in its fundamental group. Suppose that* $vb_1 \geq 4$*, then* $\pi_1(O)$ *is large.*

Two Lemmas I

The next result is the main reason why orbifolds with non-empty singular locus are so useful to us.

Lemma

Let O be a compact orbifold, and let p be a prime, then $d_p(O) \geq b_1(sing_p(O))$.

Proof.

- Let M' is the manifold obtained by removing a regular neighborhood of $sing_p(O)$.
- $\pi_1(|O|) \cong \pi_1^{orb}(O)/ << T >>$ and so $d_p(O) = d_p(M')$.
- So by Poincaré duality we have that $d_p(M') \geq \frac{1}{2} d_p(\partial M') \geq b_1(sing_p(O))$.

Two Lemmas II

Lemma (Lackenby, Long, Reid 08)

Let O be a compact, orientable 3-orbifold. Suppose that $\pi_1(O)$ admits a surjective homomorphism ϕ onto $\mathbb{Z}$ such that some component of $sing^0_p(O)$ has trivial image, for some prime p, then $\pi_1(O)$ is large.

Proof.

- All torsion dies in $\mathbb{Z}$ so we factor ϕ through $\psi : \pi_1(|O|) \to \mathbb{Z}$.
- Let $|O_i|$ be the covering corresponding to $\psi^{-1}(i\mathbb{Z})$, and let O_i be the corresponding cover of O. Let C be the circle component of $sing^0_p(O)$ with trivial image.
- Every lift of C to the cover $|O_i|$ is a loop, and so $d_p(O_i) \geq \left|sing^0_p(O)\right| \geq [O, O_i]$.

Proof of Main Theorem I

We can now prove the main result

- Let O' be a cover with $b_1(O) \geq 4$, and let O'' be the hyperbolic orbifold containing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, commensurable with O.
- O' and O'' have a common, finite index, hyperbolic cover O''', which in turn has a finite manifold cover M with $b_1 \geq 4$ that regularly covers O''.
- The deck transformations of $M \to O''$ are $G = \pi_1(O'')/\pi_1(M)$, and the quotient of the action of G on M is O''.
- Since $\pi_1(O'')$ contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ some point of its singular locus contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in its local group, and so G contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof of Main Theorem II

- Let h_1 and h_2 generate $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in G, and $h_3 = h_1 h_2$. All three elements are involutions of M, if we let $O_i = M/h_i$ then O_i has non-empty $sing_2^0(O_i)$.
- If $b_1(O_i) \geq 2$ then we can find a homomorphism to $\mathbb{Z}$ from the previous theorem.
- h_i induces an automorphism h_{i_*} on $H_1(M, \mathbb{R})$
- Since h_i is an involution h_{i_*} decomposes $H_1(M, \mathbb{R})$ as a product of eigenspaces.

Proof of Main Theorem III

- $b_1(O_i)$ is the dimension of the 1-eigenspace of h_{i_*}.
- If either $b_1(O_1)$ or $b_1(O_2)$ is at least 2 we are done, otherwise the dimension of the -1-eigenspace of h_{1_*} and h_{2_*} are both at least 3.
- The intersection of these spaces has dimension at least 2, which is contained in the 1-eigenspace of h_{3*}, and thus $b_1(O_3) \geq 2$.

Generalized Triangle Group

I hope to use this theorem to study the following family of groups

$$G_j = \langle a, b \mid a^3, b^3, ((ab)^j (a^{-1} b^{-1}))^2 \rangle$$

These groups contain $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For small values of j many of these groups have been shown to be large by using computer algebra systems to explicitly find finite index subgroups with $b_1 \geq 4$.

Proof of 2 $\Rightarrow$ 1

We need a few lemmas before we proceed

Lemma

Let G group with finite generating set S, and let $H_i \geq J_i$ be f.i. normal subgroups of G. If Σ is the generating set from the Reidermeister-Schreier process then

$$w(X(G/J_i), S) \leq w(X(H_i/J_i), \Sigma) + 2\,|S|\,[G : H_i]$$

Lemma

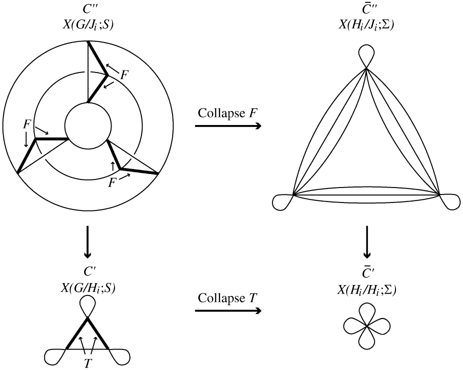
Let A be a finite abelian group with finite generating set Σ, then

$$w(X(A, \Sigma)) \leq \frac{6\,|\Sigma|\,|A|}{\lfloor (|A| - 1)^{1/|\Sigma|} \rfloor}.$$

Proof of 2 ⇒ 1

Proof of First Lemma

- An efficient ordering on $X(H_i/J_i, \Sigma)$ pulls back to an ordering on the components of F.



Proof of 2 $\Rightarrow$ 1

Proof of Second Lemma

- To prove the second lemma we can find an homomorphism from A to S^1 that allows us to efficiently order the vertices of A.
- To do this we find a non-trivial homomorphism where all the generators of A are mapped close to $1 \in S^1$.
- This shows that the images of vertices of ∂D_n under this ordering are close in S^1
- This gives a bound on $|\partial D_n|$ since the images of A are evenly spaced on S^1.