Flexibility and Rigidity of 3-Dimensional Convex Projective Structures

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Thesis Defense

What is Convex Projective Geometry?

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- Retains many features of hyperbolic geometry.
- No Mostow rigidity.

Projective Space

- There is a natural action of $\mathbb{R}^\times$ on $\mathbb{R}^{n+1} \backslash \{0\}$ by scaling.
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- A *Projective Hyperplane* is the projectivization of an $n-$plane in $\mathbb{R}^{n+1}$.
- The automorphism group of $\mathbb{R}P^n$ is $\mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^\times$.

A Splitting of $\mathbb{R}P^n$

- Let H be a hyperplane in $\mathbb{R}^{n+1}$.
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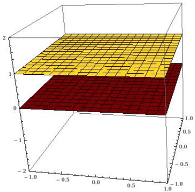

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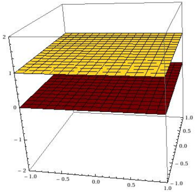
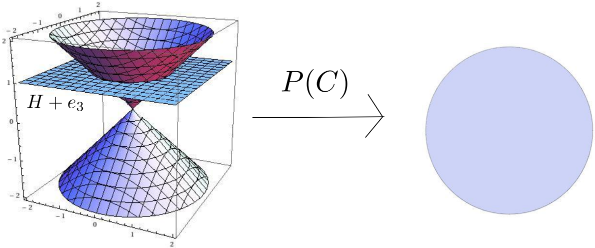


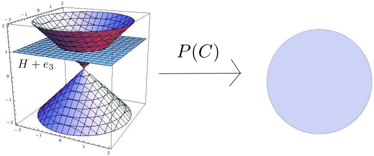
- $\mathbb{R}P^n \backslash P(H)$ is called an *affine patch*.

The Klein Model

- Let $\langle x, y \rangle = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$ be standard form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$.
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0\}$
- $P(C)$ is the Klein model of $\mathbb{H}^n$.
- In the affine patch defined by H it is a disk.



Nice Properties of Hyperbolic Space



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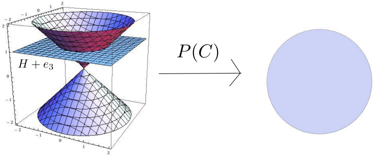

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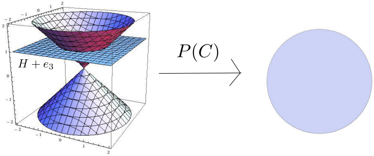

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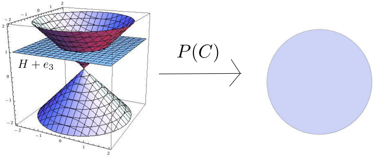

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Convex projective geometry focuses on the geometry of properly (sometimes stictly) convex domains.

Hilbert Metric

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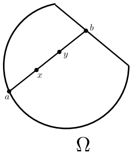

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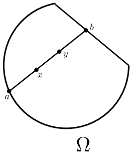

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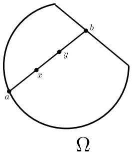

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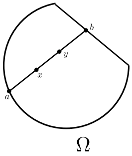

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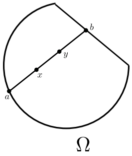

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- Discrete subgroups of $\mathrm{PGL}(\Omega)$ act properly discontinuously on Ω.

Classification of Isometries

If Ω is an open properly convex then $\mathrm{PGL}(\Omega)$ embeds in $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ which allows us to talk about eigenvalues.

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If $\gamma \in \mathrm{PGL}(\Omega)$ then γ is

1. *elliptic* if γ fixes a point in Ω,
2. *parabolic* if γ acts freely on Ω and has all eigenvalues of modulus 1, and
3. *hyperbolic* otherwise

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4. When Ω is strictly convex, parabolic and hyperbolic elements in a common discrete subgroup do not share fixed points.
5. When Ω is strictly convex, a discrete, torsion-free subgroup of elements fixing a geodesic is infinite cyclic.

Convex Projective Manifolds

Let M^n be a manifold with $\pi_1(M) = \Gamma$. A *convex projective structure* on M is a pair (Ω, ρ) such that

1. Ω is a properly convex open subset of $\mathbb{R}P^n$.
2. $\rho : \Gamma \to \mathrm{PGL}(\Omega)$ is a discrete and faithful representation.
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- When Ω is an ellipsoid then $\mathrm{PGL}(\Omega) \cong \mathrm{Isom}(\mathbb{H}^n)$ and a complete hyperbolic structure is a strictly convex projective structure.

Projective Equivalence

Suppose that $M^n \cong \Omega_i/\rho_i(\Gamma)$ for $i = 1, 2$, then (Ω_1, ρ_1) and (Ω_2, ρ_2) are *projectively equivalent* if there exists $h \in \mathrm{PGL}_{n+1}(\mathbb{R})$ such that $h(\Omega_1) = \Omega_2$ and for each $\gamma \in \pi_1(M)$

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- Let $\mathfrak{X}(\Gamma, \mathrm{PGL}_{n+1}(\mathbb{R}))$ be the set of conjugacy classes of representations from Γ to $\mathrm{PGL}_{n+1}(\mathbb{R})$. Projective equivalence classes of M are in bijective correspondence with elements of $\mathfrak{X}(\Gamma, \mathrm{PGL}_{n+1}(\mathbb{R}))$ that are faithful, discrete, and preserve a properly convex set.

Mostow Rigidity

Let M^n be a finite volume hyperbolic manifold ($n \geq 3$) and let (Ω_1, ρ_1) and (Ω_2, ρ_2) be two complete hyperbolic structures on M. Mostow rigidity tells us that (Ω_1, ρ_1) and (Ω_2, ρ_2) are projectively equivalent.

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There is a distinguished projective equivalence class of convex projective structures on M consisting of complete hyperbolic structures on M.

Rigidity and Flexibility

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2. How do we know when deformations exist?

A decomposition of M

Let M be an orientable, finite volume, hyperbolic 3-manifold. Then

$$M = M_K \cup (\sqcup_i C_i).$$

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- If ρ_1 is the holonomy of a general convex projective structure on M then $T^2 \times \{x\}$ has the same *affine* structure for each $x \in [1, \infty)$.

Description of the Holonomy

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Lemma 1 (Cooper-Long-Tillman)

Let $\Omega \subset \mathbb{R}P^3$ *be properly convex. If* $\gamma \in \mathrm{PGL}(\Omega)$ *is parabolic then* γ *is conjugate in* $\mathrm{PGL}_4(\mathbb{R})$ *to*

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If $\gamma \in \mathrm{PGL}_4(\mathbb{R})$ is conjugate the above matrix then we say that γ is a *strictly convex parabolic*.

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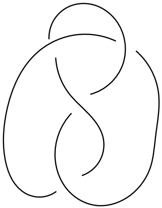
Corollary 3

If ρ is the holonomy of a strictly convex projective structure on M then $[\rho] \in \mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$

Two-Bridge Knots



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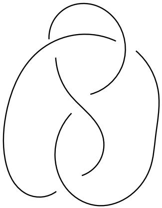

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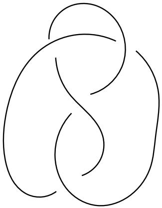

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- α and β can be taken to be meridians
- We want to look for $\rho : \Gamma \to \mathrm{PGL}_4(\mathbb{R})$ where α and β are sent to strictly convex parabolic elements

A Normal Form

By work of Riley it is possible to uniquely conjugate non-commuting parabolic $a, b \in \mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ so that

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix},$$

where z is a non-zero complex number.

A Normal Form

By work of Riley it is possible to uniquely conjugate non-commuting parabolic $a, b \in \mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ so that

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix},$$

where z is a non-zero complex number.
Geometrically, this is done be moving the repsective fixed points of a and b to ∞ and 0

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The minimal polynomial of a strictly convex parabolic is $(x-1)^3$. Therefore, neither A_l and B_u are diagonalizable and so by further conjugating we can assume that

$$A_l = \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix}, \qquad B_u = \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix}$$

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Conjugacies that preserve this form look like

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Each solution to the matrix equation $\rho(\alpha)\rho(\omega) - \rho(\omega)\rho(\beta) = 0$ gives a conjugacy class of representations for the two bridge knot complement.

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- Locally, the complete hyperbolic structure is the unique strictly convex projective structure on M

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- Is there a general rigidity result for two-bridge knots and links?

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- Small deformations of holonomy correspond to small deformations of the convex projective structure
- To find deformations of convex projective structures we only need to deform the conjugacy class of representations.

Group Cohomology

Let $\rho_t : \Gamma \to \mathrm{PGL}_4(\mathbb{R})$ be a representation, then for $\gamma \in \Gamma$ and $t \in (-\varepsilon, \varepsilon)$ we have

$$\rho_t(\gamma) = (I + z_1(\gamma)t + z_2(\gamma)t^2 + \ldots)\rho_0(\gamma),$$

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- Dimension of $H^1(\Gamma)$ gives an upper bound on the dimension of $\mathfrak{X}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$

Building Representations

Let $\rho_t : \Gamma \to \mathrm{PGL}_4(\mathbb{R})$ be a representation, then for $\gamma \in \Gamma$ and $t \in (-\varepsilon, \varepsilon)$ we have

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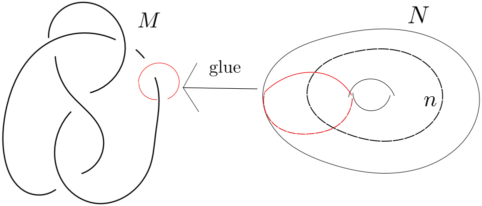
- By a result of Artin, if we can find z_i satisfying the above condition then we can build a convergent family of representations.

Orbifold Surgery



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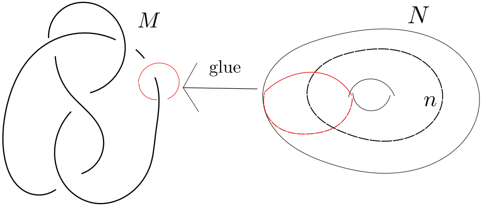

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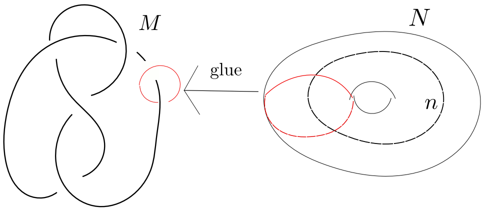

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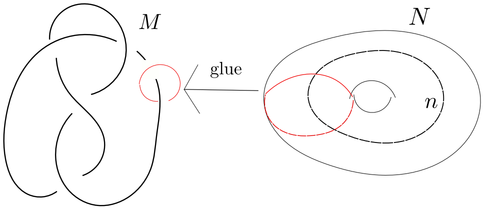

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- ϕ extends to a symmetry $\phi : O_n \to O_n$
- We can use this symmetry to build representations $\rho_t : \Gamma_n \to \mathrm{PGL}_4(\mathbb{R})$

A Flexibility Theorem

Theorem 5 (B)

Let M be the complement of a hyperbolic, amphicheiral knot, and suppose that M is infinitesimally projectively rigid relative to the boundary at the complete hyperbolic structure and the longitude is a rigid slope. Then for sufficiently large n, O_n has a one dimensional space of strictly convex projective deformations near the complete hyperbolic structure.

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By Mayer-Vietoris we have

$$0 \to H^1(O_n) \overset{\iota_1^* \oplus \iota_2^*}{\to} \overset{1}{H^1(M)} \oplus \overset{1}{H^1(N)} \overset{\iota_3^* - \iota_4^*}{\to} \overset{2}{H^1(\partial M)} \cong \overset{1}{E_1} \oplus \overset{1}{E_{-1}} \overset{\Delta^*}{\to} H^2(O_n) \to 0$$

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- Repeat indefinitely to get remaining z_i.

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- There are infinitely many amphicheiral two-bridge knots.