Geometric Structures on Manifolds

Sam Ballas

(joint with J. Danciger and G.-S. Lee)

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Outline

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Geometry According to Klein

Erlangen Program

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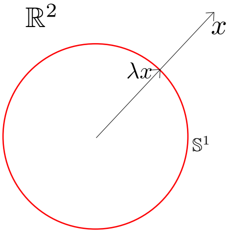
Formally, a *geometry* is a pair (G, X). Typically, $X \subset \mathbb{RP}^n$ and $G \subset \mathrm{PGL}_{n+1}(\mathbb{R})$

The Projective Sphere

Let

- $\mathbb{S}^n := (\mathbb{R}^{n+1} \backslash \{0\})/(x \sim \lambda x)$, $\lambda > 0$ and
- $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) := \{A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid \det(A) = \pm 1\}$



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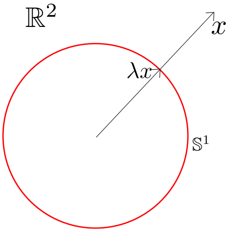

$(\mathrm{SL}^{\pm}_{n+1}(\mathbb{R}), \mathbb{S}^n)$ is convenient because

- $\mathbb{S}^n$ is simply connected and orientable
- No need to work with equivalence classes in $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$
- $(\mathrm{SL}^{\pm}_{n+1}(\mathbb{R}), \mathbb{S}^n)$ is a double cover of $(\mathrm{PGL}_{n+1}(\mathbb{R}), \mathbb{RP}^n)$

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    - Hyperbolic geometry
    - More exotic geometries
2. Provides natural hierarchy for geometries
    - (X', G') is a *subgeometry* of (X, G) if $X' \subset X$ and $G' \subset G$
    - e.g. Euclidean geometry is a subgeometry of affine geometry

Examples

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- Comes with a Riemannian metric coming from the Euclidean inner product on $\mathbb{R}^{n+1}$
- Geometry is *homogeneous* i.e. $O(n+1)$ acts transitively on $\mathbb{S}^n$.

Examples

Affine/Euclidean geometry

- Every hyperplane H in $\mathbb{R}^{n+1}$ gives rise to a decomposition of $\mathbb{S}^n = \mathbb{R}^n_+ \sqcup \mathbb{S}^{n-1} \sqcup \mathbb{R}^n_-$ into affine parts and an ideal part.

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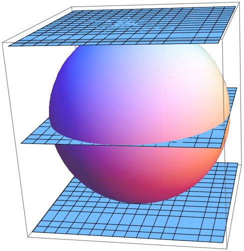

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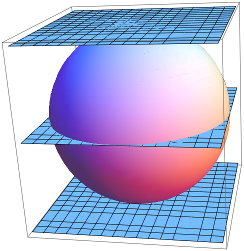

- A component of $\mathbb{S}^n \backslash \overline{H}$ is called an *affine patch*.

Examples

Affine/Euclidean geometry

$\mathbb{R}^n \cong \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$ (affine patch).

- Affine geometry

$$\mathrm{Aff}(\mathbb{R}^n) \cong \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in \mathrm{GL}_n(\mathbb{R}), b \in \mathbb{R}^n \right\}$$

  - Well defined notion of lines, parallelism, and convexity.

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$$\mathrm{Isom}(\mathbb{R}^{\mathrm{n}}) \cong \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}(\mathbb{R}^n) \mid A \in \mathrm{O}(n) \right\}$$

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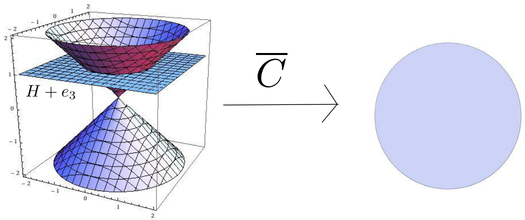
These geometries are also homogeneous.

Some Examples

Hyperbolic geometry

- Let $\langle x, y \rangle = x_1 y_1 + \ldots x_n y_n - x_{n+1} y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C_+ = \{x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0, x_{n+1} > 0\}$



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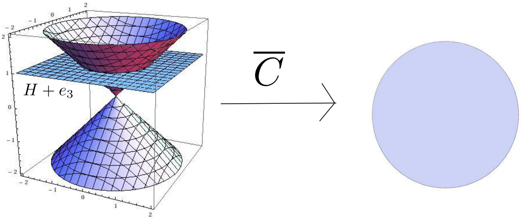
- $\overline{C_+} = \mathbb{H}^n$ is the *Klein model* of hyperbolic space.



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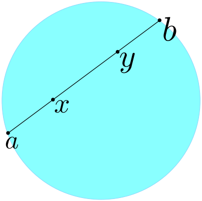
Hyperbolic geometry

The metric on $\mathbb{H}^n$ is given by



$$d_{\mathbb{H}^n}(x, y) = \frac{1}{2}\log([a : x : y : b]) = \frac{1}{2}\log\left(\frac{|b-x|\,|a-y|}{|b-y|\,|a-x|}\right)$$

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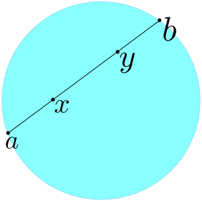

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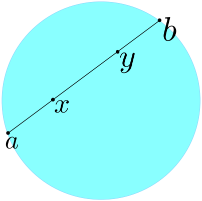

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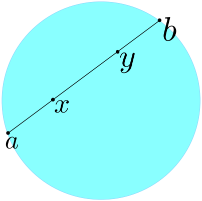

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$\mathrm{Isom}(\mathbb{H}^n) \cong O^+(n, 1) \leqslant \mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ (also homogeneous).

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Properly convex geometry

$\Omega \subset \mathbb{S}^n$ is *properly convex* if $cl(\Omega)$ is a convex subset of an affine patch.

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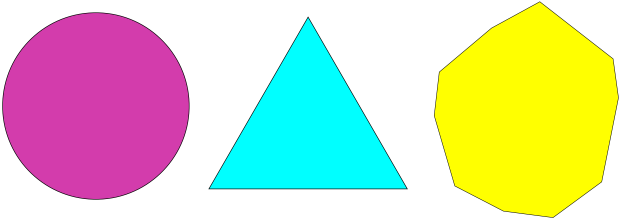

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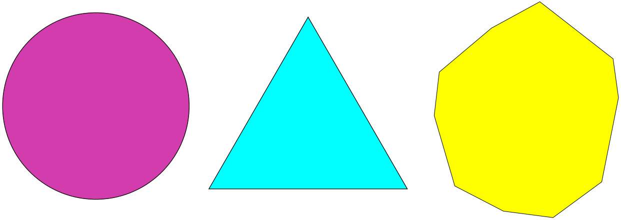

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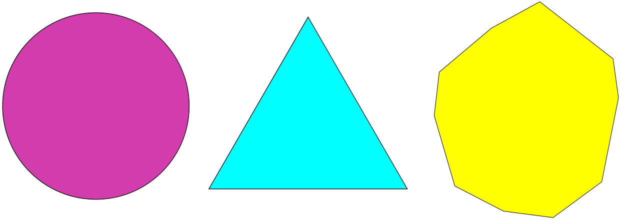

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Often not homogeneous (i.e. $\mathrm{Aut}(\Omega)$ does not act transitively)

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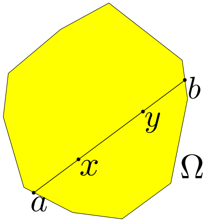
We can define the *Hilbert metric* on Ω by

$$d_\Omega(x, y) = \frac{1}{2}\log([a : x : y : b])$$



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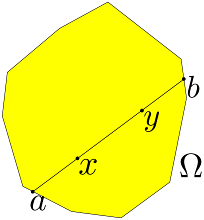

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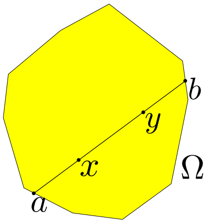

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- $\mathrm{Aut}(\Omega) \subset \mathrm{Isom}(\Omega)$

Geometry on Manifolds

Let

- M be an oriented n-manifold,
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A pair $(f, \Gamma \backslash \Omega)$, where $f : M \to \Gamma \backslash \Omega$ is a diffeomorphism is called a *complete projective structure on M.*
(f is called a *marking*)

Geometry on Manifolds

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\text{Dev}} & \Omega \\ {\scriptstyle \pi_1 M \circlearrowright}\big\downarrow & & \big\downarrow{\scriptstyle \circlearrowleft \Gamma} \\ M & \xrightarrow[\cong]{f} & \Gamma\backslash\Omega \end{array}$$

By lifting f we get a map $\mathrm{Dev} : \widetilde{M} \to \Omega$ called a *developing map*.

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Realize deck transformations geometrically!

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Geometry on Manifolds

Equivalent Structures

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Equivalent structures have conjugate holonomy representations.

$\pi_1 S^2 = 1$, so S^2 is a complete projective (spherical) manifold

Structures on Surfaces

Sphere

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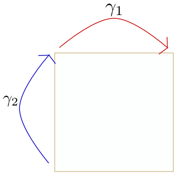
S^2 admits a homogeneous Riemannian metric.

Structures on Surfaces

Torus



Glue the sides by translations γ_1 and γ_2.

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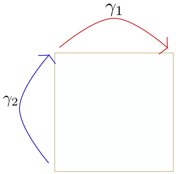

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Let $\Gamma = \langle \gamma_1, \gamma_2 \rangle \subset \text{Isom}(\mathbb{R}^2)$

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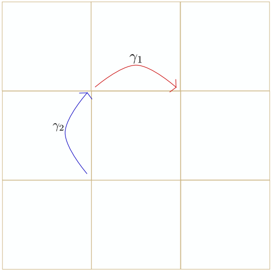

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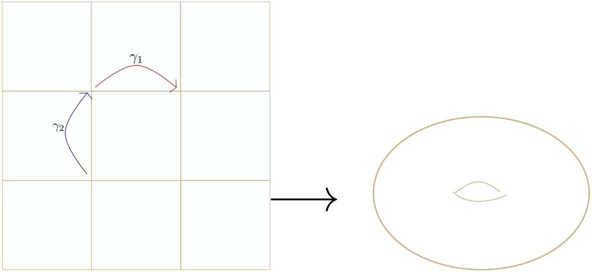

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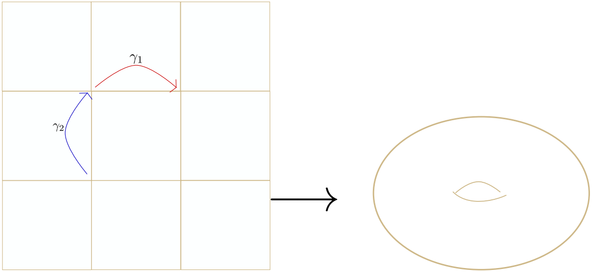

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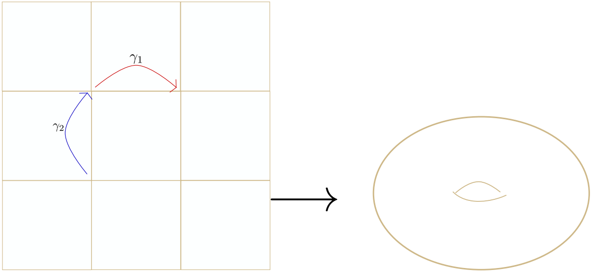

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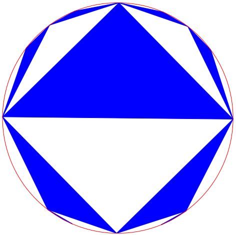
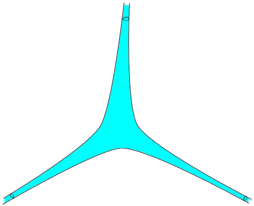
Also homogeneous

Pair of pants

(Poincaré, Fricke–Klein, early 1900’s)





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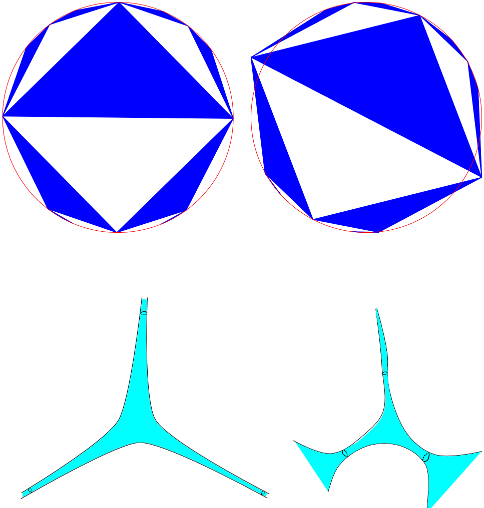

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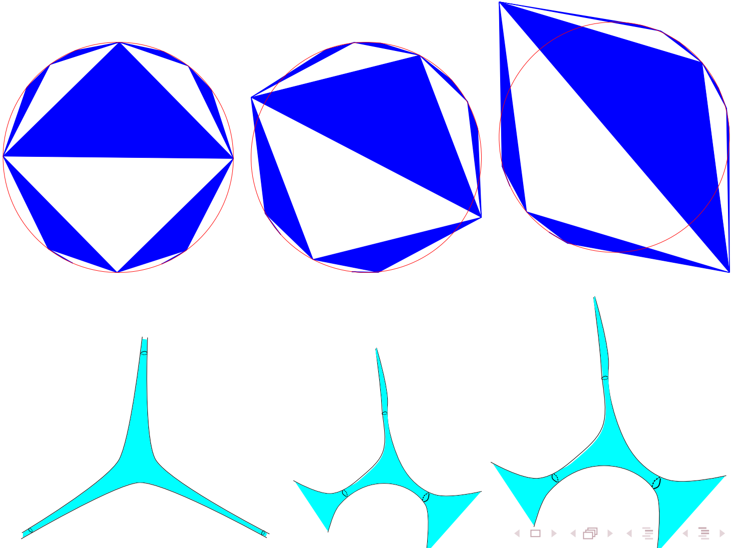
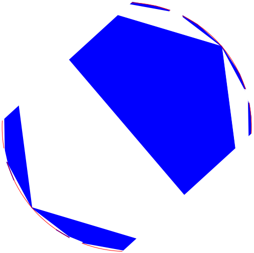
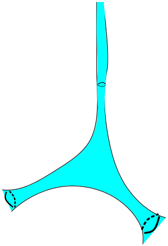


Structures on Surfaces

Other surfaces





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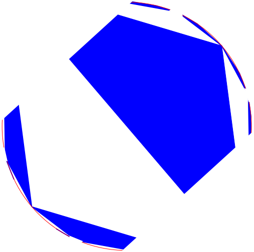
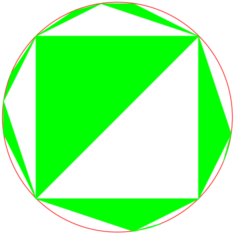
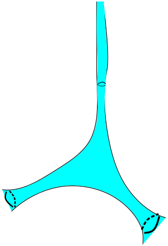
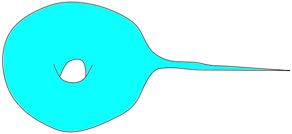


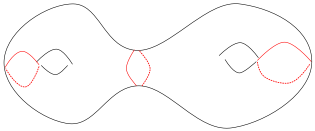






Other surfaces

Every surface of negative Euler characteristic can be decomposed into pairs of pants.



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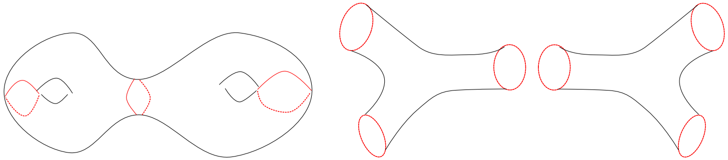

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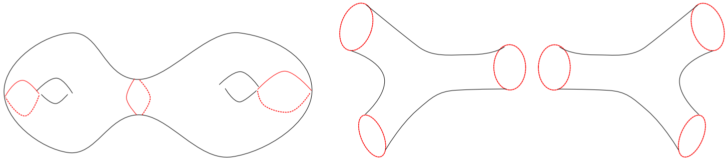

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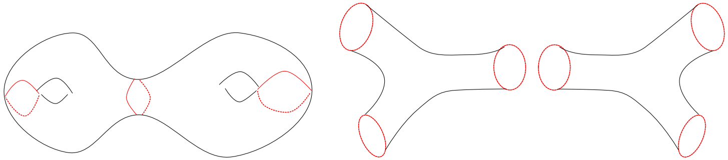

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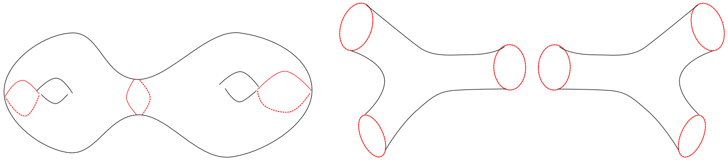

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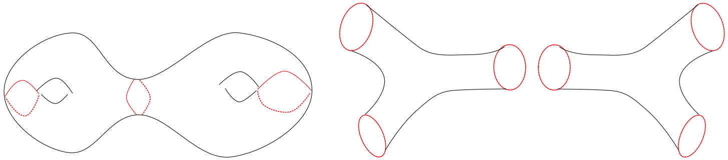

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A surface of genus $g \geqslant 2$ admits $\mathbb{R}^{6g-6}$ hyperbolic structures

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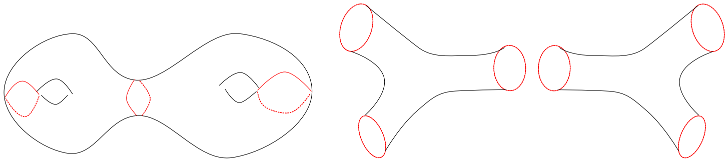

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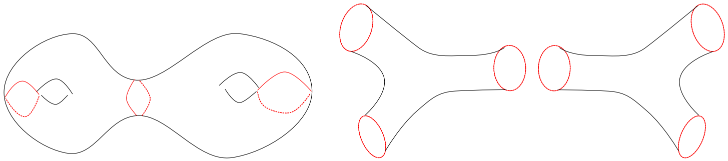

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Can also use this deform/glue strategy to construct structures in dimension 3.

3-manifolds

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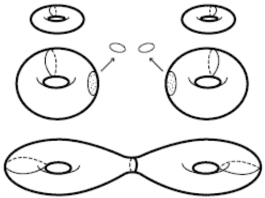

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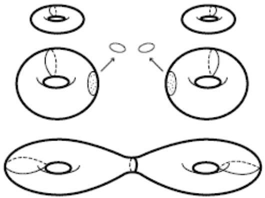

We can find a complete projective structure on $\mathbb{RP}^3$, but we can't extend the structure over the gluing 2 sphere

A 3-manifold M is *prime* if $M \cong M_1 \# M_2$ implies that $M_1 \cong S^3$ or $M_2 \cong S^3$.

3-manifolds

Prime Decomposition

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Does every closed *prime* 3-manifold admit a homogenous complete projective structure?

3-manifolds

Geometrization

There are eight 3-dimensional *Thurston* geometries: $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Sol, and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

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$\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have isometries that cannot be realized projectively.
(Flipping the $\mathbb{R}$ factor)

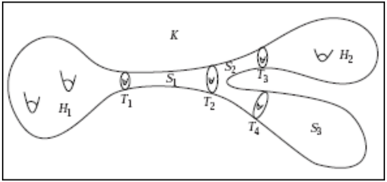
3-manifolds

Geometrization

(Jaco–Shalen ’79, Johannson ’79) Let M be a closed prime 3-manifold. There is a (unique up to isotopy) collection $\mathcal{T}$ of tori such that

$$M \backslash \mathcal{T} = \bigsqcup_i M_i \quad \text{(JSJ decomposition)}$$

each M_i has “nice” topology.



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A generic JSJ piece is hyperbolic.

Virtually, the pieces have homogeneous complete projective structures.

Problem: The homogeneous structures cannot be glued together.
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Lots of symmetries tend to lead to rigid geometry!

Solution: Use less homogeneous, but more flexible geometric structures (properly convex structures).

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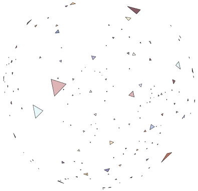
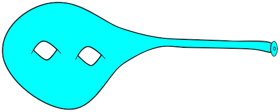
We can find “cusp opening” deformations

3-manifolds

Deforming Convex Projective Structures





3-manifolds

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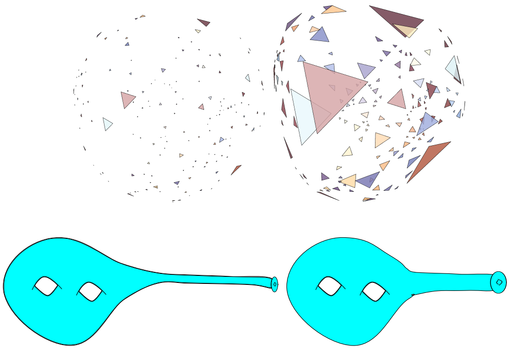

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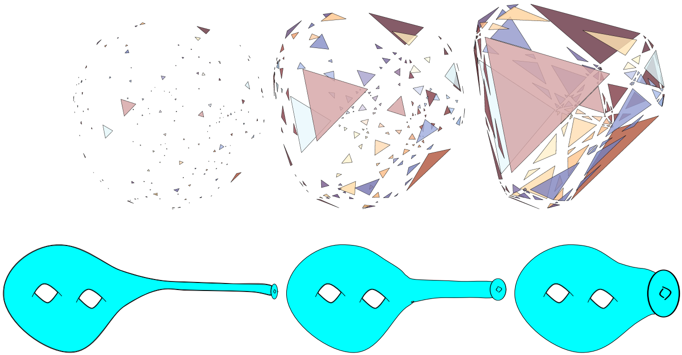

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Analogous to the deformations constructed on pairs of pants we constructed.

The deformations cannot be hyperbolic structures (Mostow rigidity)

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$$H^1(M, \mathfrak{sl}_4) \to H^1(\partial M, \mathfrak{sl}_4)$$

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  (M behaves like a pair of pants)
- Linear condition, so easy to verify
- Common amongst known examples
  (numerically, satisfied by $\sim$ 90% of cusped census manifolds, B–D–L
  as well as some infinite families, Heusener–Porti,'11)

3-manifolds

Gluing

- Let $M_1 \cong \Gamma_1 \backslash \Omega_1$ and $M_2 \cong \Gamma_2 \backslash \Omega_2$ be a properly convex 3-manifolds with *principal* totally geodesic torus boundary components, ∂_1 and ∂_2

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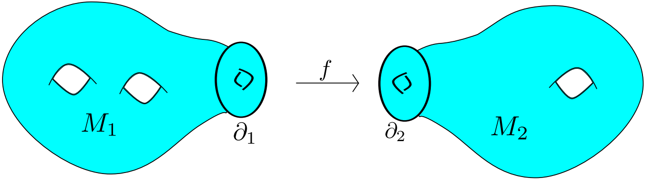

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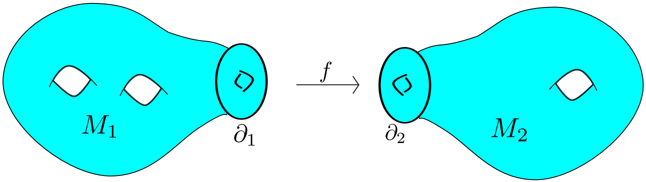

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If there exists $g \in \mathrm{SL}_4^{\pm}(\mathbb{R})$ such that $f_* : \pi_1\partial_1 \to \pi_1\partial_2$ is induced by conjugation by g then there is a properly convex structure on N such that the inclusion $M_i \hookrightarrow N$ is a projective embedding.

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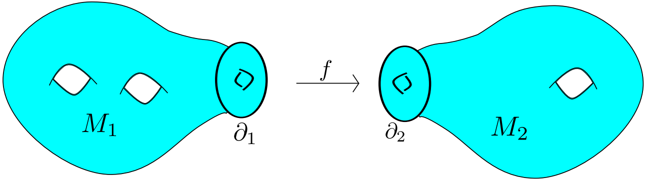

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We can glue M_1 and M_2 if their boundary geometry matches

- If $M_1 = M_2$ and f is the identity then N admits a properly convex structure.

3-manifolds

Gluing Examples

- If $M_1 = M_2$ and f is the identity then N admits a properly convex structure.
- If M_1 and M_2 can be built from regular ideal hyperbolic tetrahedra then they can often be glued.

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We need $f^*(Y_2) \cap Y_1 \neq \emptyset$ to satisfy matching condition.

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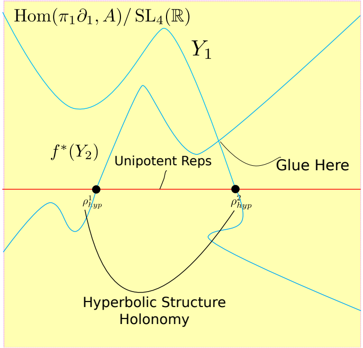

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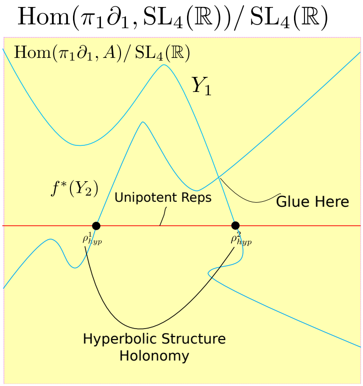

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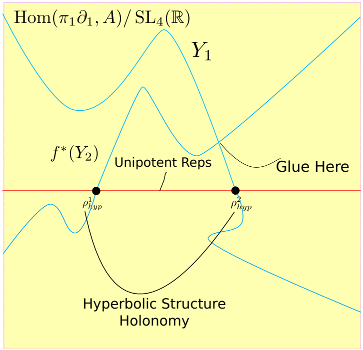

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Blue curves are Lagrangians in a symplectic (yellow) manifold.

Applications

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Is the converse of Benoist’s theorem true?

Linear Representations

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Can help to effectivize various virtual properties of 3-manifold groups

Applications

Thin Groups

A group $\Gamma \subset \mathrm{SL}_4(\mathbb{R})$ is *thin* if it is an infinite index subgroup of a lattice and is Zariski dense.
Such groups have connections to

- Expander families
- Superstrong approximation properties
- Diophantine problems

Theorem 4 (B)

Let M be the complement of the figure-eight knot in S^3. Then there is a 1-parameter family, M_t of finite volume properly convex deformations of the complete hyperbolic structure on M.

Theorem 5 (B–Long)

Let $\rho_t : \pi_1 M \to \mathrm{SL}_4(\mathbb{R})$ be a holonomy of M_t then there are infinitely many specializations of t so that $\rho_t(\pi_1 M)$ contains a thin subgroup.

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Can try to specialize so that the image (virtually) lives in a lattice.

Thank you