Convex Projective Structures on Non-hyperbolic 3-manifolds

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(joint with J. Danciger and G.-S. Lee)

Higher Teichmüller theory and Higgs bundles
Heidelberg
November 3, 2015

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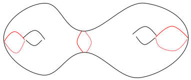

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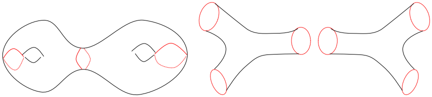

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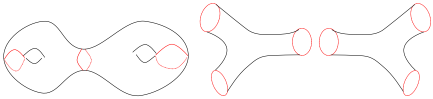

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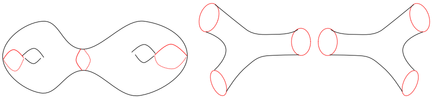

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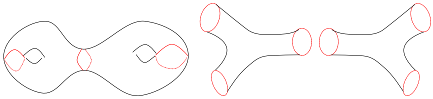

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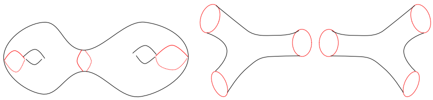

  - Understand hyperbolic structures on pants. (Completely determined by geometry of boundary)
  - Understand how to glue together pants. (Matching problem + twisting)

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- Can put a complete finite volume Thurston geometric structure on each of the pieces.
- If there are multiple geometric pieces, we can't glue them to get a Thurston geometric structure on all of M.
- However, if we allow more general geometric structures then this strategy still works (at least some of the time)

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2. In many cases these pieces behave like pairs of pants (geometry of the pieces are (locally) determined by boundary geometry).
3. Try to glue the pieces together by matching the geometry on the boundary.
4. Analyze the different ways to glue structures with matching boundary geometry.

Projective Space

- $\mathbb{RP}^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.
- Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$ be the obvious projection.
- The automorphism group of $\mathbb{RP}^n$ is $\mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^\times$.

Affine Patches

- Every hyperplane H in $\mathbb{R}^{n+1}$ gives rise to a decomposition of $\mathbb{RP}^n = \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}$ into an affine part and an ideal part.

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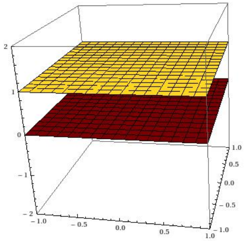

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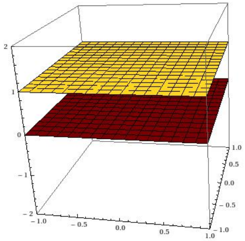
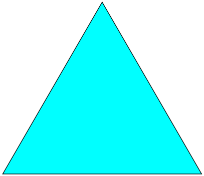


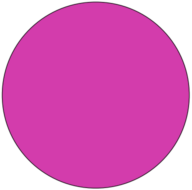
- $\mathbb{RP}^n \backslash P(H)$ is called an *affine patch*.

Convex Projective Domains

- $\Omega \subset \mathbb{RP}^n$ is *properly convex* if it is a bounded convex subset of some affine patch.
- If $\partial\Omega$ contains no non-trivial line segments then Ω is *strictly convex*.



Properly Convex



Strictly Convex

Convex Projecitve Structures

- A *convex projective n-manifold* is a manifold of the form $\Gamma\backslash\Omega$, where $\Omega \subset \mathbb{RP}^n$ is properly convex and $\Gamma \subset \mathrm{PGL}(\Omega)$ is a discrete torsion free subgroup.

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$$\begin{array}{ccc} \tilde{M} & \xrightarrow[\cong]{\mathrm{Dev}} & \Omega \\ \downarrow & & \downarrow \\ M & \longrightarrow & \Gamma\backslash\Omega \end{array}$$

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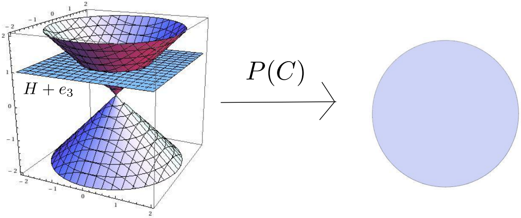
- A marked convex projective structure gives rise to a (conjugacy class of) representation $\rho : \pi_1 M \to \mathrm{PGL}_{n+1}(\mathbb{R})$ called a *holonomy* of the structure and an equivariant diffeomorphism $\mathrm{Dev} : \tilde{M} \to \Omega$ called a *developing map*.

Some Examples

Complete Hyperbolic Manifolds

- Let $\langle x, y\rangle = x_1 y_1 + \ldots x_n y_n - x_{n+1} y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle < 0\}$



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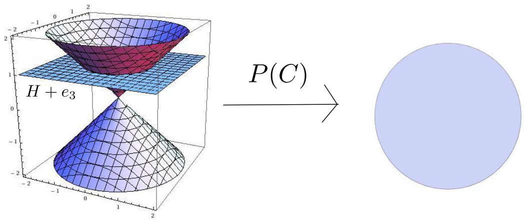
- $P(C) = \mathbb{H}^n$ is the *Klein model* of hyperbolic space.
- $\mathrm{PGL}(\mathbb{H}^n) \cong PO(n, 1) \leq \mathrm{PGL}_{n+1}(\mathbb{R})$



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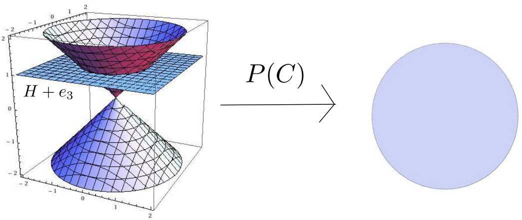
- If Γ is a torsion-free Kleinian group then $\Gamma\backslash\mathbb{H}^n$ is a (strictly) convex projective manifold.



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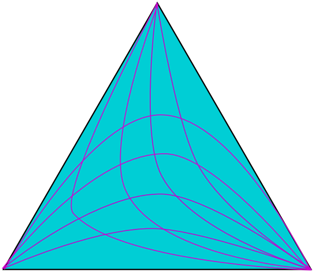
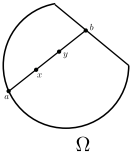
- Let $\Gamma \leq Diag_+ \leq \mathrm{PGL}(\Delta)$ be lattice, then $\Gamma \cong \mathbb{Z}^2$ and $\Gamma \backslash \Delta$ is a torus (a Hex Torus)



Hilbert Metric

Let Ω be a properly convex set and $\mathrm{PGL}(\Omega)$ be the projective automorphisms preserving Ω.



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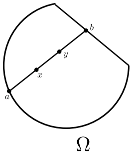

Every properly convex set Ω admits a Hilbert metric given by

$$d_\Omega(x, y) = \log[a : x : y : b] = \log\left(\frac{|x - b|\,|y - a|}{|x - a|\,|y - b|}\right)$$

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Convex projective structures are like Thurston geometric structures, sans homogeneity

Convex Projective Structure in Dimension 3

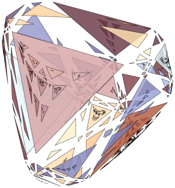
Let $M \cong \Gamma\backslash\Omega$ be a closed indecomposable convex projective 3-manifold.

Theorem (Benoist 2006)

Let M be as above then either

i *M is strictly convex and admits a hyperbolic structure*

ii *M is not strictly convex and contains a finite number of embedded totally geodesic Hex tori. The pieces obtained by cutting along these tori are a JSJ decomposition for M. Furthermore, each piece admits a finite volume hyperbolic structure.*



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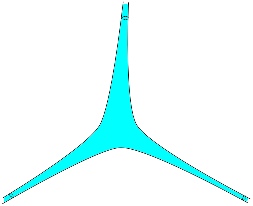

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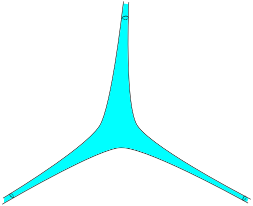

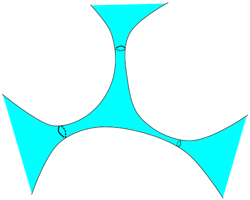

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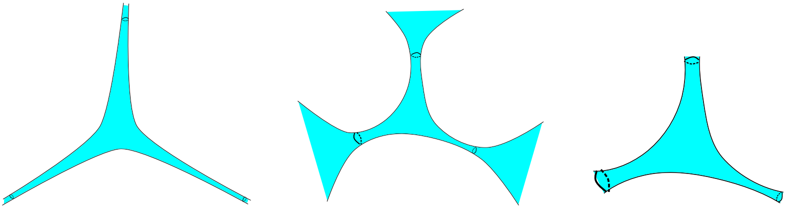
- We can truncate the ends of this infinite volume structure along geodesics to get a structure on a pair of pants $\overline{\mathcal{P}}$.



Deforming Projective Structures

Let N be a finite-volume hyperbolic 3-manifold

- $\mathfrak{B}(N)$ = Space of marked convex projective structures
- $\mathcal{X}(N) = \mathrm{Hom}(\pi_1 N, \mathrm{PGL}_4(\mathbb{R}))/\mathrm{conj}$
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3. When N is non-compact, Hol is a local homeomorphism near $[N_{hyp}]$ onto a subset of $\mathcal{X}(N)$ (Cooper–Long–Tillmann)

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Let N be a finite volume hyperbolic 3-manifold which is infinitesimally rigid rel boundary. Then N admits nearby convex projective structures with totally geodesic boundary.

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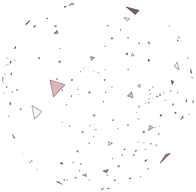

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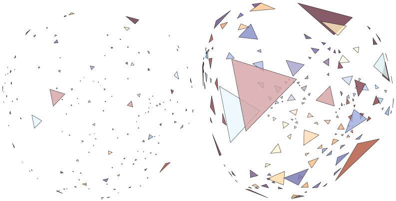

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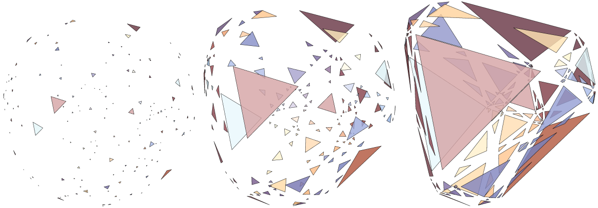

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3. Use a convex hull construction to build a structure with totally geodesic boundary.

More on Step 1

Theorem (folklore,B–D–L)

If N is a 1-cusped finite volume hyperbolic 3-manifold that is infinitesimally rigid rel boundary then $[\rho_{hyp}]$ *is a smooth point of* $\mathcal{X}(N)$*. Furthermore,* $\mathcal{X}(N)$ *is 3-dimensional near* $[\rho_{hyp}]$

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- $\mathcal{X}(\partial N)$ is 6 dimensional (sort of)
- $\mathrm{res} : \mathcal{X}(N) \to \mathcal{X}(\partial N)$ is smooth local embedding near $[\rho_{hyp}]$
- There is a 4-dimensional slice $\mathcal{S} \subset \mathcal{X}(\partial N)$ of generically diagonalizable representations transverse to res at $[\rho_{hyp}]$

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Theorem (folklore,B–D–L)

If N is a 1-cusped finite volume hyperbolic 3-manifold that is infinitesimally rigid rel boundary then $[\rho_{hyp}]$ is a smooth point of $\mathcal{X}(N)$. Furthermore, $\mathcal{X}(N)$ is 3-dimensional near $[\rho_{hyp}]$

- $\mathcal{X}(\partial N)$ is 6 dimensional (sort of)
- res : $\mathcal{X}(N) \to \mathcal{X}(\partial N)$ is smooth local embedding near $[\rho_{hyp}]$
- There is a 4-dimensional slice $\mathcal{S} \subset \mathcal{X}(\partial N)$ of generically diagonalizable representations transverse to res at $[\rho_{hyp}]$
- We get a curve $[\rho_t]$ in $\mathcal{X}(N)$ diagonalizable over $\mathbb{R}$ on $\pi_1 \partial N$.

The Slice

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Define $\rho_{(t,\theta,a,b)} : \pi_1 \partial N \to A = \exp(\mathfrak{a}) \subset \mathrm{PGL}_4(\mathbb{R})$ by

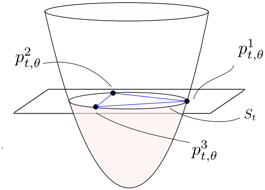
$$\rho_{(t,\theta,a,b)}(\gamma_1) = \exp(x_{t,\theta}), \, \rho_{(t,\theta,a,b)}(\gamma_2) = \exp(a x_{t,\theta} + b y_{t,\theta}).$$

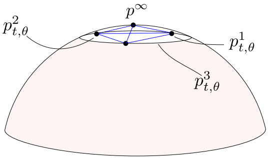
The Slice

Another model for $\mathbb{H}^3$ is

$$\{[x_1 : x_2 : x_3 : 1] \in \mathbb{RP}^3 \mid x_1 > 2(x_2^2 + x_3^2)\}$$

For $t > 0$, let S_t cross-section of $\partial\mathbb{H}^n$ at $x_1 = \frac{1}{4t^2}$.





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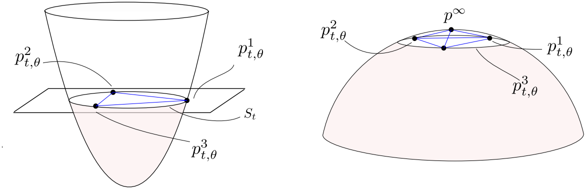

- Using $x_{t,\theta}$ and $y_{t,\theta}$ we construct three complex numbers $\{z^i_{t,\theta}\}_{i=1}^3$ equally spaced on the circle of radius $2t$.

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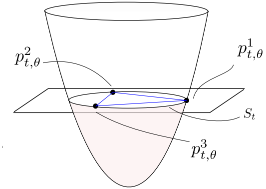

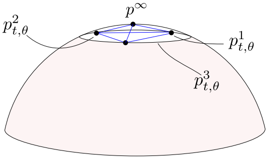

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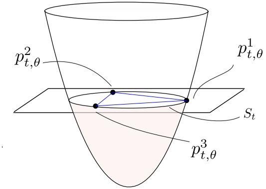

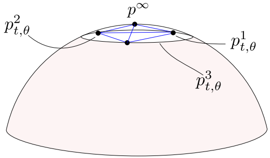

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- Let $C_{t,\theta} \in \mathrm{PGL}_4$ be an element taking the vertices of the standard simplex to $p^1_{t,\theta}, p^2_{t,\theta}, p^3_{t,\theta}$, and p^∞.

The Slice

Let $\rho'_{(t,\theta,a,b)} = C_{t,\theta}\rho_{(t,\theta,a,b)}C_{t,\theta}^{-1}$

$$\lim_{t\to 0}\rho'_{(t,\theta,a,b)}(\gamma_1) = \begin{pmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$\mathcal{S} = \{[\rho'_{(t,\theta,a,b)}] \mid a, b, \theta \in \mathbb{R}, t \in \mathbb{R}^{\geq 0}\}$

The Slice

Properties

- If $t \neq 0$ then elements of $\mathcal{S}$ are diagonalizable over reals.
- If $z = x + iy$ is the cusp shape of N w.r.t. $\{\gamma_1, \gamma_2\}$ then $res(\rho_{hyp}) = \rho'_{(0,0,x,y)}$.
- $\mathcal{S}$ is transverse to $res(\mathcal{X}(N))$ at $[\rho_{hyp}]$ with 1-dimensional intersection $[\rho_s]$.
- $[\rho_s]$ is diagonalizable over $\mathbb{R}$ for $s \neq 0$.

Gluing Manifolds with Totally Geodesic Boundary

- Let $M_1 \cong \Gamma_1 \backslash \Omega_1$ and $M_2 \cong \Gamma_2 \backslash \Omega_2$ be a properly convex 3-manifolds with principal totally geodesic torus boundary components, ∂_1 and ∂_2

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Theorem (B–D–L)

If there exists $g \in \mathrm{PGL}_4(\mathbb{R})$ such that $f_* : \pi_1 \partial_1 \to \pi_1 \partial_2$ is induced by conjugation by g then there is a properly convex projective structure on M such that the inclusion $M_i \hookrightarrow M$ is a projective embedding.

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Corollary

If N is a 1-cusped hyperbolic 3-manifold that is infinitesimally rigid rel. boundary then 2N admits a properly convex projective structure.

The Matching Problem

Let N_1 and N_2 are infinitesimally rigid rel. boundary hyperbolic 3-manifolds and M be obtained by gluing N_1 and N_2 along their boundaries. Can we find a convex projective structure on M?

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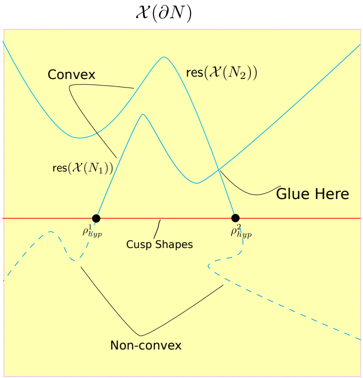

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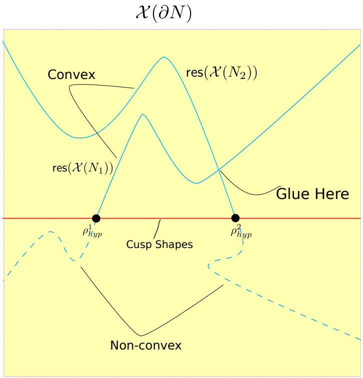

Blue curves $\rightsquigarrow$ Zero locus of A-polynomial

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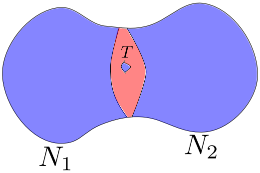

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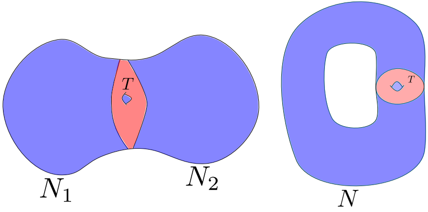

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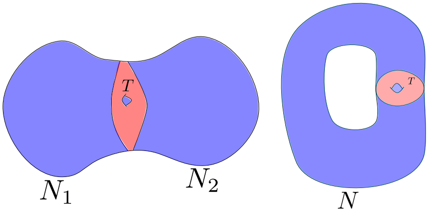

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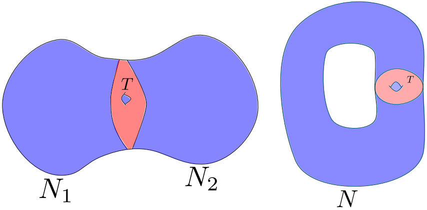

We get “twist coordinates” on $\mathfrak{B}(N)$!

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- What are good "length coordinates"?

Thank you