A GEOMETRIC DESCRIPTION OF THE $\mathrm{PSL}_4(\mathbb{R})$-HITCHIN COMPONENT

SAMUEL A. BALLAS

Abstract. These notes form a rough outline of the correspondence between the $\mathrm{PSL}_4(\mathbb{R})$-Hitchin component and convex foliated projective structures from [33, 3].

1. Introduction

Let Σ be an orientable hyperbolic surface. The *Teichmüller space* of Σ, denoted $\mathcal{T}^2(\Sigma)$, consists of conjugacy classes of discrete and faithful representations from $\Gamma := \pi_1(\Sigma)$ into $\mathrm{PSL}_2(\mathbb{R})$. This set is well known to have the topology of a ball of dimension $3\,|\chi(\Sigma)|$. There is a well known unique irreducible representation, ρ_n, from $\mathrm{PSL}_2(\mathbb{R})$ into $\mathrm{PSL_n}(\mathbb{R})$ coming from the natural action of $\mathrm{PSL}_2(\mathbb{R})$ on the symmetric product, $\mathrm{Sym}^{n-1}(\mathbb{R}^2) \cong \mathbb{R}^n$. In his article [66, 6], Hitchin showed that the the component of the space of conjugacy classes, $\mathfrak{X}(\Gamma, \mathrm{PSL_n}(\mathbb{R}))$, containing the image of $\mathcal{T}^2(\Sigma)$ under ρ_n is also a ball of dimension $(n^2-1)\,|\chi(\Sigma)|$. This component is typically referred to as the *Hitchin component* and we will denote it as $\mathcal{T}^n(\Sigma)$.

For small values of n these Hitchin component can be thought of as moduli spaces of geometric structures on manifolds. For $n = 2$, the Hitchin component is the same as Teichmüller space which allows us to identify $\mathcal{T}^2(\Sigma)$ with the space of marked hyperbolic structures on Σ. More specifically, let $[\rho] \in \mathcal{T}^2(\Sigma)$. Since $\mathrm{PSL}_2(\mathbb{R})$ can be identified with the orientation preserving isometry group of $\mathbb{H}^2$ we can identify the quotient $\mathbb{H}^2/\rho(\Gamma) \cong \Sigma$, thus giving a marked hyperbolic structure on Σ. Conversely, the holonomy representation of any hyperbolic structure is a discrete and faithful representation from Γ into $\mathrm{PSL}_2(\mathbb{R})$, and equivalent marked hyperbolic structures have conjugate holonomy.

When $n = 3$, work of Goldman [22, 2] and Goldman-Choi [11, 1] show that the space $\mathcal{T}^3(\Sigma)$ can be identified with the moduli space of marked convex projective structures on Σ. Roughly speaking, a convex projective structure on Σ is a realization of Σ as $\Omega/\rho(\Gamma)$, where Ω is a convex set that sits inside of an affine patch (see Exercise 1) of $\mathbb{RP}^2$ and $\rho : \Gamma \to \mathrm{PSL}_3(\mathbb{R})$ is a discrete and faithful representation. For example, when ρ factors through $\mathrm{PSL}_2(\mathbb{R})$ the set Ω can be taken to be the unit disk in $\mathbb{R}^2 \subset \mathbb{RP}^2$ (See Exercise 2). While it is easy to associate a conjugacy class of discrete faithful representations with a marked projective structure, it is much more difficult to start with a representation and find an appropriate convex projective structure.

Exercise 1. *Let $\mathbb{RP}^n$ be the quotient of $\mathbb{R}^{n+1}\backslash\{0\}$ by the action of $\mathbb{R}^\times$ by scaling and let H be a hyperplane $\mathbb{RP}^n$. $\mathbb{RP}^n\backslash H$ is called an* affine patch*. The purpose of this exercise is to justify this terminology.*

(1) *Show that an affine patch can be identified with the affine space $\mathbb{R}^n$.*

(2) *Show that the subgroup* $\mathrm{PGL_{n+1}}(\mathbb{R})$ *consisting of elements that preserve* H *is equivariantly isomorphic (with respect to the identification from part (1)) to the affine group of matrices of the form*

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix},$$

where $A \in \mathrm{GL}_n(\mathbb{R})$ *and* $b \in \mathbb{R}^n$.

Exercise 2. *Let* $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$ *be the standard bilinear form of signature* $(2,1)$ *on* $\mathbb{R}^3$. *Let* $C = \{x \in \mathbb{R}^3 | \langle x, x\rangle < 0\}$ *be the cone of vectors with negative self pairing. The image,* $P(C)$*, of* C *in* $\mathbb{RP}^2$ *serves as a model for* $\mathbb{H}^2$.

(1) *Let* H *be the image of the* x_1x_2-*plane in* $\mathbb{RP}^2$. *Show that in the affine patch defined by* H *that* $P(C)$ *can be identified with the unit disk. This model is know as the* Klein Model
(2) *Let* K *be the image of the plane* $x_3 - x_2 = 0$ *in* $\mathbb{RP}^2$. *Show that in the affine patch defined by that* $P(C)$ *can be identified with the set* $v > u^2$. *(Hint, use coordinates* $u = x_1$, $v = x_3 + x_2$, $w = x_3 - x_2$ *and use inhomogeneous coordinates* $w = 1$*). This model is known as the paraboloid model.*

Our goal in this lecture to explain a correspondence between the space $\mathcal{T}^4(\Sigma)$ and certain types of projective structures on the unit tangent bundle, $S\Sigma$, of Σ. This result is originally due to Guichard and Wienhard [33, 3]. The rough idea is that there is an $\mathbb{R}$ action on $S\Sigma$ given by the geodesic flow. This flow gives rise to a pair of foliations, $\mathcal{F}$ and $\mathcal{G}$, which are referred to as the *stable foliation* and *geodesic foliation*, respectively. We will show that representations in $\mathcal{T}^4(\Sigma)$ correspond to projective structures in which these foliations can be realized in a geometrically meaningful way inside of $\mathbb{RP}^3$. When $[\rho] \in \mathcal{T}^4(\Sigma)$ factors through $\mathrm{PSL}_2(\mathbb{R})$ then we can regard these projective structures as projective realizations of the familiar $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ structures on these manifolds.

2. $S\Sigma$ and the Geodesic Flow

In this section we will discuss important properties of $M := S\Sigma$ and the action of the geodesic flow. Let $\overline{\Gamma} = \pi_1(M)$, then $\overline{\Gamma}$ is a central (non-split) extension of Γ that fits into the following short exact sequence.

$$0 \to \mathbb{Z} \to \overline{\Gamma} \to \Gamma \to 1. \tag{2.1}$$

The manifold M has an important regular cover, $\overline{M}$, corresponding from the $\mathbb{Z}$ subgroup in (2.1). Furthermore, the foliations $\mathcal{F}$ and $\mathcal{G}$ lift to foliations $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ of $\overline{M}$. The foliations $\mathcal{F}$ and $\mathcal{G}$ can also be lifted to the universal cover, $\tilde{M}$. These lifts will be denoted $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$, respectively. We now give two descriptions of $\overline{M}$.

Let $[\rho] \in \mathcal{T}^2(\Sigma)$ be a (conjugacy class of) representation. Choose and identification of $\tilde{\Sigma}$ with $\mathbb{H}^2$ that is equivariant with respect to ρ, such an identification is called a *uniformization*. The uniformization allows us to equivariantly identify $\overline{M}$ with the unit tangent bundle $S\mathbb{H}^2$. With this in mind we can think of a leaf of $\overline{\mathcal{G}}$ as an oriented geodesic in $\mathbb{H}^2$ and a leaf of $\overline{\mathcal{F}}$ as the union of all oriented geodesics with a common positive endpoint on $\partial\mathbb{H}^2$. Since

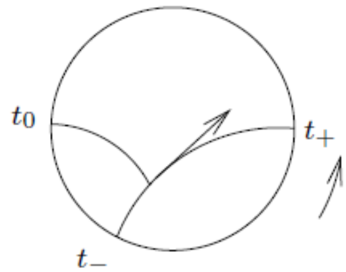

FIGURE 1. The identification between $\overline{M}$ and $\partial\Gamma^{3+}$. Picture from [33, 3]

$\mathrm{PSL}_2(\mathbb{R})$ acts simply transitively on $S\mathbb{H}^2$ we can further identify $\overline{M}$ with $\mathrm{PSL}_2(\mathbb{R})$. Under this identification we see that $M = \rho(\Gamma)\backslash\mathrm{PSL}_2(\mathbb{R})$. Additionally, if we let

$$A = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}, \text{ and } P = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \right\},$$

be *Cartan* and *parabolic* subgroups of $\mathrm{PSL}_2(\mathbb{R})$, then the leaves of the geodesic and stable foliations can be identified with right orbits of A and P, respectively.

The uniformization also allows us to identify the boundary of Γ, $\partial\Gamma$, with $\partial\mathbb{H}^2 \cong \mathbb{RP}^1$. Furthermore, this identification gives an action of Γ on $\partial\Gamma$ that is ρ-equivariant. This boundary gives us another way to describe $\overline{M}$, $\overline{\mathcal{F}}$, and $\overline{\mathcal{G}}$. $\partial\Gamma$ inherits an orientation coming from the orientation on $\mathbb{H}^2$ and we can identify $\overline{M}$ with the set $\partial\Gamma^{3+}$ of pairwise distinct, positively oriented triples of points in $\partial\Gamma$. To see this observe that a point of $\overline{M}$ can be thought of as an oriented geodesic, L, in $\mathbb{H}^2$ and a point, x, on L. Such a point can be identified with (t_+, t_0, t_-), where t_+ and t_- are the positive and negative endpoints of L, respectively, and t_0 is the endpoint of the geodesic that intersects L perpendicularly and passes through x and makes the above triple positively oriented (see Figure 1). This identification is also equivariant with respect to the action of Γ and allows us to identify $\overline{\mathcal{F}}$ with $\partial\Gamma$ and $\overline{\mathcal{G}}$ with $\partial\Gamma^{(2)} := \partial\Gamma^2\backslash\Delta$.

3. Convex Foliated Projective Structures

In this section we describe the geometric structures whose moduli space $\mathcal{T}^4(\Sigma)$ describes. We begin with a description of projective structures. Roughly speaking, a projective structure on a manifold is a way to locally identify the manifold with $\mathbb{RP}^n$ in such a way that the transition functions are locally elements of $\mathrm{PGL}_{n+1}(\mathbb{R})$. As such projective structures can be described in terms of atlases of charts. However, we will take a more global (but equivalent) point of view in our definition. Let N be an $n-$manifold. A *projective structure* consists of a pair (dev, hol), where $hol : \pi_1(N) \to \mathrm{PGL}_{n+1}(\mathbb{R})$ is a representation and dev is a $hol-$equivariant local homeomorphism from $\tilde{N}$ to $\mathbb{RP}^n$. Furthermore, we say that (dev_1, hol_1) and (dev_2, hol_2) are *equivalent* projective structures on N if there exists a homeomorphism $h : N \to N$ that is isotopic to the identity and an element $g \in \mathrm{PGL}_{n+1}(\mathbb{R})$ such that

- $dev_1 \circ \tilde{h} = g \circ dev_2$, where $\tilde{h}$ is a lift of h to $\tilde{N}$, and
- $hol_2 = g^{-1} hol_1 g$.

Let $\mathcal{P}(N)$ be the set of equivalence classes of projective structures on N. The above discussion shows that we have a map

$$hol : \mathcal{P}(N) \to \mathfrak{X}(\pi_1(N), \mathrm{PGL_{n+1}}(\mathbb{R})).$$

The map *dev* is called the *developing map* of the structure and the representation *hol* is called the *holonomy* of the structure.

As we mentioned before we are interested in projective structures that play well with the foliations coming from the geodesic flow. With this in mind, we say that a projective structure, (dev, hol), on M is *foliated* if the following conditions are satisfied.

- For each leaf $\tilde{g} \in \tilde{\mathcal{G}}$, $dev(\tilde{g})$ is contained in a projective line, and
- For each leaf $\tilde{f} \in \tilde{\mathcal{F}}$, $dev(\tilde{f})$ is contained in a projective plane.

Two foliated projective structures are *equivalent* if they are equivalent as projective structures and the map $h : M \to M$ preserves the foliations $\mathcal{F}$ and $\mathcal{G}$. We denote the set of equivalence classes of foliated projective structures by $\mathcal{P}_f(M)$. We now further refine this notion in order to arrive at the correct geometric structures. Let $C \subset \mathbb{RP}^n$, then C is *convex* if its intersection with every projective line is connected. If C is a convex subset of $\mathbb{RP}^n$ then C is *properly convex* if its closure does not contain a affine line.

Exercise 3. *Show that a subset of $\mathbb{RP}^n$ is properly convex if and only if its closure is contained in an affine patch.*

We can now define the appropriate projective structures. We say that a foliated projective structure on M is *convex* if the image of each leaf of $\tilde{\mathcal{F}}$ under the developing map is a convex set of a projective plane. Additionally, we define a *properly convex* foliated projective structure on M to be a foliated projective structure for which the image of each leaf of $\tilde{\mathcal{F}}$ is mapped to a properly convex subset of a projective plane by the developing map. Let $\mathcal{P}_{pcf}(M)$ subset of $\mathcal{P}_f(M)$ consisting of equivalence classes of properly convex foliated projective structures.

We can now rephrase the correspondence between $\mathcal{T}^4(\Sigma)$ and projective structures in more precise terms. Let $p : \overline{\Gamma} \to \Gamma$ be the projection implicit in (2.1). The map p gives an embedding of $\mathcal{T}^4(\Sigma) \subset \mathfrak{X}(\Gamma, \mathrm{PSL}_4(\mathbb{R})) \subset \mathfrak{X}(\overline{\Gamma}, \mathrm{PSL}_4(\mathbb{R}))$, and the correspondence can be succinctly stated as

Proposition 3.1. *The map hol is a homeomorphism between $\mathcal{P}_{pcf}(M)$ and $\mathcal{T}^4(\Sigma)$.*

Remark 3.2. *Since Γ has trivial center* (2.1) *implies that the center of $\overline{\Gamma}$ is cyclic, and we denote its generator by τ. The above correspondence implies that the holonomy of a properly convex foliated projective structure on M factors through p and thus every such holonomy kills τ.*

4. Examples and Ideas

In this section we will discuss certain examples of properly convex foliated projective structures on M and discuss some of the ideas required to prove Proposition 3.1. Let

$[\rho] \in \mathcal{T}^2(\Sigma)$, then we can define an element of $\mathcal{P}_{pcf}(M)$ as follows. Let $[Q] \in \mathbb{RP}^3$, where $Q = x(x^2+y^2)$ (here we are using the fact that $\mathbb{R}^4 \cong \mathrm{Sym}^3(\mathbb{R}^2)$). Using the fact that $\overline{M} \cong \mathrm{PSL}_2(\mathbb{R})$ we can define a projective structure by letting dev[1] be the map $g \mapsto \rho_4(\rho(g)) \cdot [Q]$, where $g \in \mathrm{PSL}_2(\mathbb{R})$ and letting $hol = \rho_4 \circ \rho$. The vector $Q = (1,0,1,0)$ in the standard basis for $\mathrm{Sym}^3(\mathbb{R}^2)$ and so

$$\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \mapsto \begin{pmatrix} e^{3t/2} & 0 & 0 & 0 \\ 0 & e^{t/2} & 0 & 0 \\ 0 & 0 & e^{-t/2} & 0 \\ 0 & 0 & 0 & e^{-3t/2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{3t/t} \\ 0 \\ e^{-t/2} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \mapsto \begin{pmatrix} a^3 & a^2b & ab^2 & b^3 \\ 0 & a & 2b & 3b^2/a \\ 0 & 0 & 1/a & 3b/a^2 \\ 0 & 0 & 0 & 1/a^3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a^3+ab^2 \\ 2b \\ 1/a \\ 0 \end{pmatrix} = \begin{pmatrix} a^2(a^2+b^2) \\ 2ab \\ 1 \\ 0 \end{pmatrix}.$$

Under the coordinate change $v = a^2(a^2+b^2)$, $u = 2ab$, we see that the above image can be identified with $v > u^2/4$, and is thus properly convex (See Exercise 3). In fact, this set can be identified with $\mathbb{H}^2$ (see Exercise 2). Thus we see that this projective structure is properly convex foliated.

Despite knowing that these structures are properly convex foliated, we do not have a very good idea of what they look like globally. In order to get a more global picture we will try to understand the developing map in terms of the description of $\overline{M}$ as $\partial\Gamma^{3+}$. Let V be a vector space and let $Flag(V)$ denote the flag variety of V. If we think of $\mathbb{R}^2$ as $\mathrm{Sym}^1(\mathbb{R}^2)$, then the *Veronese embedding* gives the following equivariant curve $\xi : \partial\Gamma \cong \mathbb{RP}^1 \to Flag(\mathbb{R}^4)$. Given by $\xi = (\xi^1, \xi^2, \xi^3)$, where

- $\xi^1([S])$ is the line of polynomials divisible by S^3,
- $\xi^2([S])$ is the plane of polynomials divisible by S^2, and
- $\xi^3([S])$ is the hyperplane of polynomials divisible by S.

If we let Ω_ξ be the set of polynomials in $\mathbb{RP}^3$ with a single real root (i.e. they factor over $\mathbb{R}$ into a linear and a quadratic term), then Ω_ξ is the image of dev.

Exercise 4. *Prove that Ω_ξ is the image of dev, namely that Ω_ξ is the* $\mathrm{PSL}_2(\mathbb{R})$ *orbit of* $[Q]$ *under the action $g \cdot [R] = \rho_4(\rho(g)) \cdot [R]$.*

The map ξ allows us to define a family, ξ^1_t of equivariant maps from $\partial\Gamma \to \mathbb{RP}^3$, given by

$$\xi^1_t(t') = \begin{cases} \xi^3(t) \cap \xi^2(t') & \text{if } t \neq t' \\ \xi^1(t) & \text{if } t = t' \end{cases}$$

For each t, the image of ξ^1_t in $\xi^3(t)$ bounds the copy of $\mathbb{H}^2_t$ given by $dev(t)$ (here we are thinking of t as a leaf of $\overline{\mathcal{F}}$). The geodesic leaf $g = (t_+, t_-)$ is taken to the intersection of $\mathbb{H}^2_t$ and the projective line, $\overline{\xi^1(t_+)\xi^1_{t_+}(t_-)}$, connecting $\xi^1(t_+)$ and $\xi^1_{t_+}(t_-)$. The tangent lines to $\mathbb{H}^2_t$ at $\xi^1(t_+)$ and $\xi^1_{t_+}(t_-)$ are $\xi^2(t_+)$ and $\xi^3(t_+) \cap \xi^3(t_-)$, respectively. Furthermore, these

[1]Technically, dev is a lift of this map to $\tilde{M}$.

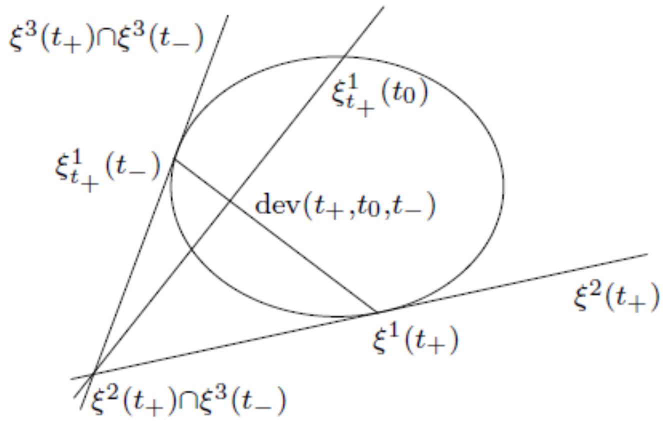

FIGURE 2. The image of the developing map. Picture from [33, 3].

lines intersect in the point $\xi^3(t_-) \cap \xi^2(t_+) = \xi^1_{t_-}(t_+)$. Finally, given t_0 different from t_+ and t_- there is a unique point of intersection between $\overline{\xi^1(t_+)\xi^1_{t_+}(t_-)}$ and $\overline{\xi^1_{t_-}(t_+)\xi^1_{t_+}(t_0)}$. Thus we see that in terms of $\partial\Gamma^{3+}$ that *dev* is defined by

$$(t_+, t_0, t_-) \mapsto \overline{\xi^1(t_+)\xi^1_{t_+}(t_-)} \cap \overline{\xi^1_{t_-}(t_+)\xi^1_{t_+}(t_0)}. \tag{4.1}$$

The discussion of the previous paragraph is illustrated in Figure 2

5. Convex Representations

The proof of Proposition 3.1 relies on the fact that representations in $\mathcal{T}^4(\Sigma)$ can be characterized by a certain convexity property which we now discuss. Work of Labourie and Guichard [77, 755, 5] has shown that $\rho \in \mathcal{T}^4(\Sigma)$ if and only if ρ is a convex representation. A representation is *convex* if we can find a ρ-equivariant curve $\xi^1 : \partial\Gamma \to \mathbb{RP}^3$ such that for $t_1, \dots t_4$ that are pairwise distinct,

$$\xi^1(t_1) \oplus \ldots \oplus \xi^1(t_4) = \mathbb{R}^4.$$

A simple exercise shows that a convex curve in $\mathbb{RP}^2$ bounds a properly convex set. Additionally, a curve $\xi = (\xi^1, \dots, \xi^{n-1}) : \mathbb{RP}^1 \to Flag(\mathbb{R}^n)$ is called *Frenet* (this is sometimes known as hyperconvex) if

(1) For every $(n_1, \dots, n_k)$ such that $\sum_{i=1}^k n_i = n$ and every $t_1, \dots, t_k \in \mathbb{RP}^1$ of pairwise distinct elements the following sum is direct

$$\sum_{i=1}^{k} \xi^{n_i}(x_i) = \mathbb{R}^n.$$

(2) For every $(m_1, \ldots, m_k)$ with $\sum_{i=1}^k m_i = m \leq n$ and every $x \in \mathbb{RP}^1$

$$\lim_{(x_i)\to x} \sum_{i=1}^{k} \xi^{m_i}(x_i) = \xi^m(x),$$

where the limit is taken over $k-$tuples of pairwise distinct points.

It is easy to see that if ξ is Frenet then ξ^1 is a convex curve and that ξ^1 determines ξ via the limit condition. By work of Labourie [77, 7] has shown that If a ρ is convex then it is possible to find a unique ρ-equivariant Frenet curve. Given a non-degenerate bilinear form on $\mathbb{R}^n$ and a curve $f = (f^1, \ldots, f^{n-1}) : \mathbb{RP}^1 \to Flag(\mathbb{R}^n)$, it is possible to define a dual curve $f^\perp = (f^{n-1,\perp}, \ldots, f^{1,\perp}) : \mathbb{RP}^1 \to Flag(\mathbb{R}^{n*})$. Work of Guichard [44, 4] shows that a curve ξ is Frenet if and only if $\xi^\perp$ is Frenet. This duality will be crucial in the proof of Proposition 3.1.

5.1. **convex implies properly convex foliated.** We begin by showing that an element $[\rho] \in \mathcal{T}^4(\Sigma)$ gives rise to a properly convex foliated projective structure on M. By the previous paragraph we see that ρ is a convex representation and thus we can find a ρ-equivariant flag curve ξ. We begin by using this curve to define a family of ρ-equivariant lower dimensional flag curves. Define $\xi_t : \mathbb{RP}^1 \to Flag(\xi^3(t))$ by

$$\xi_t(t') = \begin{cases} (\xi^3(t) \cap \xi^2(t'), \xi^3(t) \cap \xi^3(t')) & \text{if } t \neq t' \\ (\xi^1(t), \xi^2(t)) & \text{if } t = t' \end{cases} \tag{5.1}$$

For each $t \in \mathbb{RP}^1$ the curve ξ_t is also Frenet. This is proven by showing that $\xi_t^\perp$ is Frenet and using the basic fact that if W and V are linear subspace then $(V + W)^\perp = V^\perp \cap W^\perp$. For example, to show that $\xi_t^\perp$ is Frenet we need to show that $(\xi_t^{2,\perp}(t_1) + \xi_t^{1,\perp}(t_2) = \mathbb{R}^3$ for all distinct pairs t_1, t_2. To show this we observe that

$$(\xi_t^{2,\perp}(t_1)+\xi_t^{1,\perp}(t_2))^\perp = \xi_t^2(t_1)\cap\xi_t^1(t_2) = \xi^3(t)\cap\xi^3(t_1)\cap\xi^2(t_2) = (\xi^{3,\perp}(t)+\xi^{3,\perp}(t_1)+\xi^{2,\perp}(t_2))^\perp = \{0\},$$

with the last equality coming from the fact that $\xi^\perp$ is Frenet. We now define a developing map using the formula in (4.1). For each t the image of ξ_t^1 bounds is convex and thus bounds a properly convex subset C_t of $\xi^3(t)$. The Frenet properties of ξ this new developing map has all the same nice properties as the map given to us in the previous example by the Veronese embedding.

5.2. **properly convex foliated implies convex.** The more difficult direction is to show that given a properly convex foliated projective structure on M that the holonomy representation is a convex representation. Details can be found in [33, 3] and we simply outline the key ideas. Suppose that we have such a structure with holonomy ρ. Since we know that the image of a leaf of $\tilde{\mathcal{F}} \cong \tilde{\Gamma}$ under the developing map is contained in a projective plane we get a map[2] $\xi^3 : \partial\tilde{\Gamma} \to \mathbb{RP}^{3*}$taking $t \in \partial\tilde{\Gamma}$ to the projective plane containing $dev(t)$. The first thing that we have to do is to show that the map ξ^3 descends to a map defined on $\partial\Gamma$ (this

[2]Here we are implicitly identifying the space of projective planes in $\mathbb{RP}^3$ with $\mathbb{RP}^{3*}$ using a non-degenerate bilinear form.

part is highly non-trivial and comprises a good chunk of [33, 3]). We must then show that this map is convex. Namely that for $t_1, \ldots, t_4$ pairwise distinct points that

$$\xi^3(t_1) + \ldots + \xi^3(t_4) = \mathbb{R}^{4*},$$

or equivalently that

$$\xi^3(t_1) \cap \ldots \cap \xi^3(t_4) = \emptyset. \tag{5.2}$$

The fact that these planes do not have a common intersection can be viewed geometrically. Fix t_1, then the fact that the intersection from (5.2) is empty is equivalent to the three lines lines $\xi^3(t_1) \cap \xi^3(t_i)$, $2 \leq i \leq 4$, not intersecting. Let C_{t_1} be the properly convex set that is the image of the developing map restricted to plane $\xi^3(t_1)$. Then it can be shown (with a good deal of work) that the domain C_t is *strictly convex* (contains no line segments in its boundary) and that the lines $\xi^3(t_1) \cap \xi^3(t_i)$ for $2 \leq i \leq 4$ are tangent lines to C_{t_1} at distinct points, and thus do not intersect. Try drawing tangents to the domain in Figure 2 to convince yourself that these lines must be disjoint.

6. Projective Duality

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