# A (Hopefully Gentle) Introduction to Arithmetic Kleinian Groups

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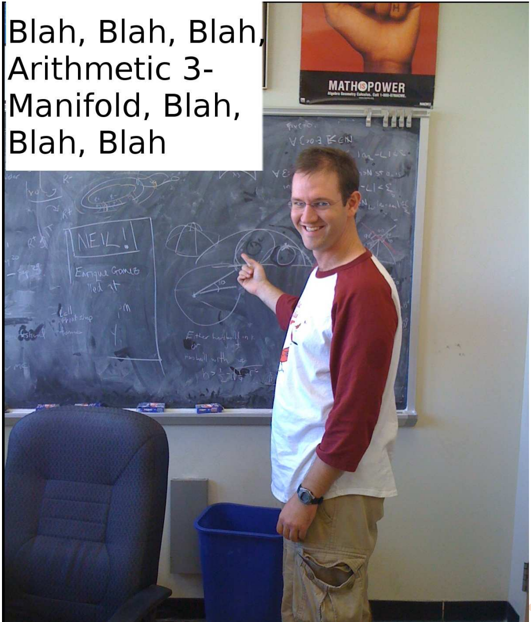
# Dramatic Reenactment



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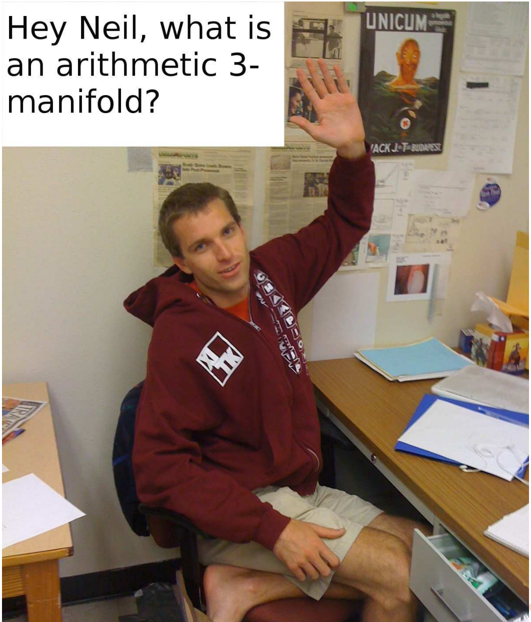

# Dramatic Reenactment

# Kleinian Groups

- The group of orientation preserving isometries of $\mathbb{H}^3$ can be identified with $\mathrm{PSL}_2(\mathbb{C})$.
- A discrete subgroup, $\Gamma$, of $\mathrm{PSL}_2(\mathbb{C})$ is called a *Kleinian group*.
- A *hyperbolic 3-orbifold* is the quotient of $\mathbb{H}^3$ by a Kleinian group, denoted $O \cong \mathbb{H}^3/\Gamma$.
- If $\mathbb{H}^3/\Gamma$ is compact, we say that $\Gamma$ is *cocompact*.
- If $\mathbb{H}^3/\Gamma$ has finite volume we say that $\Gamma$ has *finite covolume*.

# Commensurability

- Two Kleinian groups $\Gamma$ and $\Gamma'$ are *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both $\Gamma$ and $\Gamma'$.
- Two Kleinian groups, $\Gamma$ and $\Gamma'$, are *commensurable in the wide sense* if there exists $g \in \mathrm{PSL}_2(\mathbb{C})$ such that $g^{-1}\Gamma g$ and $\Gamma'$ are commensurable.
- If $O \cong \mathbb{H}^3/\Gamma$ and $O' \cong \mathbb{H}^3/\Gamma'$, then this definition is equivelent to $O$ and $O'$ having a common finite sheeted cover.

# Some Triviality Conditions

Let Γ be a subgroup of $\mathrm{PSL}_2(\mathbb{C})$

- Γ is *reducible* if all the elements of Γ have a common fixed point, otherwise it is *irreducible*.
- The group is *elementary* if it has a finite orbit in its action on $\mathbb{H}^3 \cup \partial\mathbb{H}^3$, otherwise it is *non-elementary*.
- Notice that reducible groups are elementary, but that elementary groups need not be reducible.

## Remark

We will often want to rule out groups in these two categories. Almost all the groups we will consider will be irreducible and non-elementary. For example, finite volume Kleinian groups are irreducible and non-elementary.

# Some Field Theory

Let $k\backslash\mathbb{Q}$ be a finite extension of fields (i.e. $k$ is a field that is a finite dimensional vector space over $\mathbb{Q}$). If $[k:\mathbb{Q}] = n$ then we have the following facts.

- There exists $t \in k$ such the minimal polynomial of $t$ over $\mathbb{Q}$ has degree $n$ and $k \cong \mathbb{Q}(t)$.
- If $t = t_1, t_2, \ldots, t_n$ are the roots of the minimal polynomial of $t$ then the assignments $t \mapsto t_i$ induces an embedding, denoted $\sigma_i$, of $k$ into $\mathbb{C}$.
- Conversely, any embedding of $k$ into $\mathbb{C}$ must map $t$ to one of the $t_i$ and thus there are exactly $n$ embeddings of $k$ into $\mathbb{C}$.
- If $\sigma_i(k) \subset \mathbb{R}$ then the embedding is called a *real place*. If $\sigma_i(k) \not\subset \mathbb{R}$ then there exists $j \neq i$ such that $\sigma_j = \overline{\sigma_i}$, and we refer to the pair $(\sigma_i, \sigma_j)$ as a *complex place*.
- Therefore we arrive at the formula $n = r_1 + 2r_2$ where $r_1$ is the number of real places and $r_2$ is the number of conjugate places of complex embeddings.

# Norm and Trace

- Define the *norm of* $\alpha$ to be $N_{k\backslash\mathbb{Q}}(\alpha) := \prod_{i=1}^{n} \sigma_i(\alpha)$. This is a homomorphism from $k$ to $\mathbb{Q}$.
- Define the *trace of* $\alpha$ to be $Tr_{k\backslash\mathbb{Q}}(\alpha) := \sum_{i=1}^{n} \sigma_i(\alpha)$. This is also a homomorphism from $k$ to $\mathbb{Q}$.
- If $k$ is a quadratic imaginary extesion of $\mathbb{Q}$ then $N_{k\backslash\mathbb{Q}}(\alpha) = |\alpha|^2$ and $Tr_{k\backslash\mathbb{Q}}(\alpha) = 2\mathrm{Re}(\alpha)$.

## Remark

In general, the norm and complex modulus of field element are unrelated.

# Algebraic Integers

- Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$, then an element $\alpha \in \overline{\mathbb{Q}}$ is an *algebraic integer* if its minimal polynomial has entries in $\mathbb{Z}$.
- Given a finite extension, $k$, of $\mathbb{Q}$ let the set $\mathcal{O}_k$ be a set of algebraic integers contained in $k$.
- Since $N_{k\setminus\mathbb{Q}}(\alpha)$ and $Tr_{k\setminus\mathbb{Q}}(\alpha)$ can be thought of as entries of the minimal polynomial of $\alpha$, we see that if $\alpha \in \mathcal{O}_k$ then $N_{k\setminus\mathbb{Q}}(\alpha)$ and $Tr_{k\setminus\mathbb{Q}}(\alpha)$ are in $\mathbb{Z}$. In a degree 2 extension, the converse is also true.

# The Modular Group

- $\mathrm{PSL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C}) \mid \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in \mathbb{Z} \right\}$ is arithmetic
- If $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, then $O = \mathbb{H}^3/\Gamma$ is the 2-orbifold with base space the disk and two cone points of order 2 and 3 respectively.
- As we will see, arithmeticity is a property of a commensurability class, and so any finite sheeted cover of $O$ is arithmetic.
- For example, if $\Gamma' :=$ $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \mid \mathrm{a} \equiv \mathrm{d} \equiv 1 \mod 2, \mathrm{b} \equiv \mathrm{c} \equiv 0 \mod 2 \right\}$. $\mathbb{H}^3/\Gamma'$ is the thrice punctured sphere

# The Bianchi Groups

- Let $k = \mathbb{Q}(\sqrt{-d})$, where $d > 0$, then $\mathrm{PSL}_2(\mathcal{O}_\mathrm{k})$ is a *Bianchi group*. These groups are the 3-dimensional analogue of the modular group.
- The Bianchi groups provide infinitely many commensurability classes of arithmetic, Kleinian groups.
- All 3-dimensional cusped, arithmetic, Kleinian groups are commensurable with a Bianchi group.

# A Non-Example

- Let $k$ be an extension of $\mathbb{Q}$, such that $[k : \mathbb{Q}] > 2$, should we consider $\mathrm{PSL}_2(\mathcal{O}_\mathrm{k})$ to be an arithmetic group?
- No, there exists an element $\gamma \in \mathcal{O}_k$ such that $|\gamma| < 1$.
- Therefore, the sequence $\begin{pmatrix} 1 & \gamma^n \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and so $\mathrm{PSL}_2(\mathcal{O}_\mathrm{k})$ is not even a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$.
- So we see that our examples already have two problems.
  1. We have no closed examples.
  2. Our coefficients are restricted to lie in an extension of $\mathbb{Q}$ of degree at most 2.

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# The Invariant Trace Field

- Let Γ be a finite covolume Kleinian group, then the *trace field of* Γ, denoted $\mathbb{Q}(\mathrm{tr}\Gamma)$, is the field obtained by ajoining the traces of all elements of Γ to $\mathbb{Q}$.
- Let $\Gamma^{(2)} = \langle \gamma^2 : \gamma \in \Gamma \rangle$, then $\Gamma^{(2)}$ is a finite index, normal subgroup of Γ, whose quotient is an Abelian 2-group.
- The trace field of $\Gamma^{(2)}$ is called the *invariant trace field*, and is denoted $k\Gamma$.

## Theorem

$\mathbb{Q}(\mathrm{tr}\Gamma)$ *is an finite extension of* $\mathbb{Q}$ *and is a topological invariant of* $\mathbb{H}^3/\Gamma$.

## Theorem

$k\Gamma$ *is an invariant of the commensurability class of* Γ.

# “Small” Algebraic Integers

A useful fact about algebraic integers is that there are only finitely many “small” algebraic integers. More precisely we have.

## Lemma

*There are only finitely many algebraic integers of bounded degree all of whose Galois conjugates are bounded.*

## Proof.

- Let $\alpha$ be such an algebraic integer. The coefficients of the minimal polynomial of $\alpha$ are symmetric functions in $\alpha$ and its conjugates.
- Since all conjugates of $\alpha$ are bounded the entries of the minimal polynomial of $\alpha$ are bounded.
- Since there are only finitely many polynomials of bounded degree with bounded integer coefficients there are only finitely many algebraic integers with the above property.

# The Motivating Theorem

Given an arbitrary group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$, it is typically difficult to decide whether or not it is discrete.
The following theorem shows how we can use arithmetic data to show discreteness.

## Theorem (The Motivating Theorem)

*Let* $\Gamma$ *be a finitely generated subgroup of* $\mathrm{PSL}_2(\mathbb{C})$ *such that*

1. $\Gamma^{(2)}$ *is irreducible.*
2. *tr*($\gamma$) *is an algebraic integer for every* $\gamma \in \Gamma$.
3. *For each embedding,* $\sigma : k\Gamma \to \mathbb{C}$, *other than Id and complex conjugation, the set* $\{\sigma(\mathrm{tr}(\gamma) : \gamma \in \Gamma^{(2)}\}$ *is bounded.*

*Then* $\Gamma$ *is discrete.*

# A First Definition of Arithmeticity

The previous theorem provides us with our first definition of arithmeticity.

- A group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ is *arithmetic* if it satisfies the conditions of the Motivating Theorem and has finite covolume.
- Notice that the Bianchi groups are arithmetic under this definition.
- Since $k\Gamma$ is a commensurability invariant we see that if $\Gamma$ is arithmetic and $\Gamma'$ is commensurable with $\Gamma$, then $\Gamma'$ is arithmetic.
- Unfortunately, these conditions are not packaged in the best way. In particular the third one can be difficult to verify.
- Fortunately, there are better ways to view condition 3.

# Quaternion Algebras and Hilbert Symbols

- A *quaternion algebra* $A$ over a field $F$ of characteristic $\neq 2$ is a four-dimensional algebra over the field $F$, with basis $\{1, i, j, k\}$ and elements $a, b \in F^*$ such that

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

- A quaternion algebra can be described by a (non-unique) *Hilbert symbol*, $\left(\frac{a,b}{F}\right)$, where the entries are squares of a pair of the basis elements other than 1.
- Other possible Hilbert symbols include $\left(\frac{b,a}{F}\right)$ and $\left(\frac{a,-ab}{F}\right)$.
- Another possible Hilbert symbol for $\left(\frac{a,b}{F}\right)$ is $\left(\frac{ax^2,by^2}{F}\right)$, where $x, y \in F^*$, however this involves a basis change to $i' = xi$, $j' = yj$ and $k' = i'j'$.

# Some Examples

Some familiar examples of quaternion albegras include

- The Hamilton quaternions, denoted $\mathcal{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$.
- For any field $F$, $M_2(F) = \left(\frac{1,1}{F}\right)$, where $i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- Since every element of $\mathbb{C}$ is a square we see that every quaternion algebra over $\mathbb{C}$ is $\left(\frac{1,1}{\mathbb{C}}\right) = M_2(\mathbb{C})$.
- Similarly, since every positive real is a square, every quaternion algebra over $\mathbb{R}$ is equal to one of $\left(\frac{1,1}{\mathbb{R}}\right)$, $\left(\frac{1,-1}{\mathbb{R}}\right)$, and $\left(\frac{-1,-1}{\mathbb{R}}\right)$. The first two are $M_2(\mathbb{R})$ and the third is $\mathcal{H}$.

# Yet Another Norm and Trace

Quaternion algebras are equipped with a natural notion of norm and trace, similar to the complex numbers.

- Let $A_0$ be the subspace of $A$ spanned by $\{i, j, k\}$. The elements of $A_0$ are called *pure quaternions*.
- It turns out that this subspace is independent of the choice of basis.
- Every $x \in A$ has a decomposition $x = a + \alpha$ where $a \in F$ and $\alpha \in A_0$.
- Define the *conjugate* of $x$, denoted $\overline{x}$, by $\overline{x} = x - \alpha$.
- Define the *norm* and *trace* of $x$ by $n(x) = x\overline{x}$ and $tr(x) = x + \overline{x}$, respectively.

# Ramified Vs. Unramified Places

- Let $k$ be a finite extension of $\mathbb{Q}$, let $A = \left(\frac{a,b}{k}\right)$ be a quaternion algebra, and let $\sigma$ be an embedding of $k$ into $\mathbb{C}$.
- If $\sigma$ is real place then

$$A \otimes_{\sigma(k)} \mathbb{R} \cong \left(\frac{a,b}{k}\right) \otimes_{\sigma(k)} \mathbb{R} \cong \left(\frac{\sigma(a),\sigma(b)}{\mathbb{R}}\right).$$

- Let $\iota : A \to A \otimes_{\sigma(k)} \mathbb{R}$ (or $\mathbb{C}$) be the canonical injection, then for $x \in A$, $\sigma(tr(x)) = tr(\iota(x))$ and $\sigma(n(x)) = n(\iota(x))$.
- If $A \otimes_{\sigma(k)} \mathbb{R}$ is isomorphic to $\mathcal{H}$ then $A$ is *ramified at* $\sigma$, and if $A$ is ismorphic to $M_2(\mathbb{R})$ then $A$ is *unramified at* $\sigma$.
- If $\sigma$ is a complex place then $A \otimes_{\sigma(k)} \mathbb{C} \cong M_2(\mathbb{C})$, and so $A$ is unramified at every complex place.

# Norm and Trace in the Ramified Case

In $\mathcal{H}$ there is a strong connection between norm and trace that is absent in $M_2(\mathbb{R})$.

- Let $x \in \mathcal{H}$, then $x = a + bi + cj + dk$.
- Therefore,

$$n(x) = a^2 + b^2 + c^2 + d^2, \text{ and } \mathrm{tr}(\mathrm{x}) = 2\mathrm{a}.$$

- Thus if $n(x) = 1$, then $tr(x) \in [-2, 2]$.
- Conversely, in $M_2(\mathbb{R})$ there are elements with norm 1 whose trace is arbitrarily large.

# The Invariant Quaternion Algebra

Let $\Gamma$ be a finitely generated, non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$, and define the *invariant quaternion algebra*, denoted $A\Gamma$ by,

$$A\Gamma = \{\sum a_i\gamma_i \mid a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)}\}.$$

## Theorem

*$A\Gamma$ is a quaternion algebra over $k\Gamma$.*

## Theorem

*$A\Gamma$ is an invariant of the commensurability class of $\Gamma$.*

# Some Properties of $A\Gamma$

- $\Gamma^{(2)}$ is naturally embedded inside of $A\Gamma$.
- In $A\Gamma$, the trace of $\gamma$ as a matrix agrees with its trace as an element of the quaternion algebra.
- Also, since the determinant of $\gamma \in \Gamma$ is 1 we see that all elements of $\Gamma$ have norm 1.
- We can now rephrase the Motivating Theorem in terms of quaternion algebras.

# Restatement of the Motivating Theorem

## Theorem

*If $\Gamma$ is a finitely generated subgroup of $\mathrm{PSL}_2(\mathbb{C})$ satisfying conditions 1 and 2 of the Motivating Theorem, then condition 3 is equivalent to the following*

3' *All embeddings, $\sigma$, from $k\Gamma$ to $\mathbb{C}$ other than Id and complex conjugation are real and $A\Gamma$ is ramified at all real places.*

## Proof.

- If $3'$ holds and $\sigma$ is a real embedding other than Id or conjugation, then we can find $\iota : A\Gamma \to \mathcal{H}$ such that for $\gamma \in \Gamma^{(2)}$ we have $\sigma(tr(\gamma)) = tr(\iota(\gamma))$.
- Since $n(\iota(\gamma)) = 1$, we have previously seen that $tr(\iota(\gamma)) \in [-2, 2]$, and thus $\sigma(tr(\gamma)) \in [-2, 2]$.
- Conversely, if 3 is true then it can be shown that $A\Gamma \otimes_{\sigma(k)} \mathbb{R}$ has a Hilbert symbol, $\left(\frac{a,b}{\mathbb{R}}\right)$, where $a, b < 0$, and so $A\Gamma$ is ramified at all real places.

# Orders

Let $\Gamma$ be arithmetic and let

$$A\Gamma \supset \mathcal{O} = \{\sum a_i\gamma_i \mid a_i \in \mathcal{O}_{k\Gamma}, \gamma_i \in \Gamma^{(2)}\} \text{ and } \mathcal{O}^{1} = \{x \in \mathcal{O} \mid n(x) = 1\}$$

- Since $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ we know that $A\Gamma \subset M_2(\mathbb{C})$.
- Let $\sigma$ be the unique complex place of $k\Gamma$, then we have a map $\iota : A\Gamma \to A \otimes_{\sigma(k\Gamma)} \mathbb{C} \cong M_2(\mathbb{C})$.
- By the Skolem-Noether Theorem, there exists $g \in GL_2(\mathbb{C})$ such that $\iota(\alpha) = g\alpha g^{-1}$ for all $\alpha \in A\Gamma$.
- Thus we see that $\Gamma^{(2)} \subset g^{-1}\iota(\mathcal{O}^1)g$ and has finite index.
- We conclude that $\Gamma$ and $\iota(\mathcal{O}^1)$ are commensurable.

# The Real Definition

We can now give the actual definition of arithmeticity and hopefully see how it is the same as our naive definition.

- Let $k$ be a number field with exactly 1 complex place and let $A$ be a quaternion algebra over $k$ which is ramified at all real places. Let $\iota$ be a $k$-embedding of $A$ into $M_2(\mathbb{C})$ and let $\mathcal{O}$ be an order of $A$. Then a subgroup $\Gamma$ of $\mathrm{PSL}_2(\mathbb{C})$ is an *arithmetic Kleinian group* if it is commensurable with some $\iota(\mathcal{O})$.

## Remark

In this case, the field, $k$, ends up being $k\Gamma$ and the algebra, $A$, turns out to be $A\Gamma$.

# Bianchi Groups Revisited

Let $d < 0$, $k = \mathbb{Q}(\sqrt{d})$, and $\Gamma_d = \mathrm{PSL}_2(\mathcal{O}_\mathrm{k})$.

- $k\Gamma_d = k$, which has exactly one complex place.
- $A\Gamma = M_2(k\Gamma_d)$, which is vacuously ramified at all real places of $k\Gamma$.
- $\mathcal{O} = M_2(\mathcal{O}_k)$.
- $\mathcal{O}^1 = \mathrm{PSL}_2(\mathcal{O}_\mathrm{k})$.

# Why Should I Care About All This Arithmetic

You can learn about topology by looking at arithmetic

- Let $\Gamma$ have finite covolume. If $\Gamma$ is not cocompact then $A\Gamma = M_2(k\Gamma)$. If $\Gamma$ is arithmetic then the converse is also true, and in fact $\Gamma$ is commensurable with a Bianchi group.
- (Bass) If there exists $\gamma \in \Gamma$ such that $tr(\gamma)$ is not an algebraic integer, then $\mathbb{H}^3/\Gamma$ is Haken.
- (Lackenby) If $\Gamma$ is arithmetic and torsion free, then $\mathbb{H}^3/\Gamma$ contains an immersed $\pi_1$ injective surface.
- (Margulis) If $\Gamma$ is not arithmetic then the commensurability class of $\Gamma$ contains a unique maximal group.

If you liked this talk please thank Alan Reid.
If you did not like this talk please feel free to blame Eric Staron.