MAPPING CLASS GROUP NOTES

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These notes look to introduce the mapping class groups of surfaces, explicitly compute examples of low complexity, and introduce the basics on projective representations that will be helpful going forward when looking at quantum representations of mapping class groups. The vast majority of the material is modelled off how it is presented in *A Primer on Mapping Class Groups* by Benson Farb and Dan Margalit. Unfortunately many accompanying pictures given in lecture are going to be left out due to laziness.

1. Fixing Notation

Let Σ be a compact connected oriented surface potentially with punctures. Note that when the surface has punctures then it is no longer compact, but it is necessary to start with an un-punctured compact surface first. This is also what is called a surface of finite type.

- g will be the genus of Σ, determined by connect summing g tori to the $2-$sphere.
- b will be the number of boundary components, determined by removing b open disks with disjoint closure
- n will be the number of punctures determiend by removing n points from the interior of the surface

We will denote a surface by $\Sigma^b_{g,n}$, where the indices b and n may be excluded if they are zero. We will also use the notation S^2 for the $2-$sphere, D^2 for the disk, and T^2 for the torus (meaning the genus 1 surface).

The main invariant (up to homeomorphism) of surfaces is the Euler Characteristic, $\chi(\Sigma)$ defined as

$$\chi(\Sigma^b_{g,n}) := 2 - 2g - (b+n).$$

The classification of surfaces tells us that Σ is determined by any 3 of $g, b, n, \chi(\Sigma)$.

2. Mapping Class Group Basics

Definition 1. $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ *is the group of orientation preserving self-homeomorphisms of* Σ *which restrict to the identity on* $\partial\Sigma$.

Definition 2.

$$\mathrm{MCG}(\Sigma) := \pi_0(\mathrm{Homeo}^+(\Sigma, \partial\Sigma)).$$

In words, the mapping class group of Σ *is the group of isotopy classes of* $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$*, where isotopies must fix the boundary pointwise.*

We will always take f to be a mapping class which is the isotopy class of ϕ, meaning $f = [\phi] \in MCG(\Sigma)$.

2.1. **Punctures, Marked Points, and Boundary Components.** If the n puncutes of Σ are instead thought of marked points on the surface, then $\mathrm{MCG}(\Sigma)$ is the group of self-homeomorphisms that leave the set of marked points invariant modulo isotopies which also leave the marked points invariant

When looking at $\Sigma_{g,n}$ it is tempting to use that $\Sigma_{g,n}$ is homeomorphic to the interior of Σ_g^n, but one needs to be careful. In particular, based on our definition we must restrict to the identity on the boundary, and so boundary components cannot be permuted, while punctures need only be fixed set-wise.

3. Alexander's Trick

Theorem 3. $\mathrm{MCG}(D^2)$ *is trivial.*

Reworded: Any oreientation preserving self-homeomorphism ϕ of D^2, that is the identity on ∂D^2, is isotopic to the identity through homeomorphisms that are the identity on ∂D^2.

Proof. Identify D^2 as $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Let $\phi : D^2 \to D^2$ be the identity on ∂D^2. Now define

$$F(x,t) : \begin{cases} (1-t)\phi(\frac{x}{1-t}) & 0 \leq |x| < 1-t \\ x & 1-t \leq |x| \leq 1 \end{cases}$$

This is best seen in Figure 1. $\square$

We also have

Theorem 4. $\mathrm{MCG}(\Sigma_{0,1}^1)$ *is trivial.*

Proof. The same homotopy as above can be used. This is seen in the vertical line rising from the tip of the cone corresponding to the puncture. $\square$

Theorem 5. $\mathrm{MCG}(\Sigma_{0,1})$ *and* $\mathrm{MCG}(S^2)$ *are trivial.*

Proof. Sketch: We can just use stereographic projection (and if we are careful abound a point being fixed in the S^2 case) then we can use the straight line homotopy in $\mathbb{R}^2$. $\square$

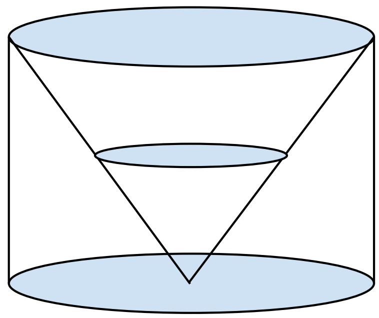

FIGURE 1. The level sets of F as the support of the cone in the cylinder

4. The 3−punctured Sphere

Lemma 6. *Any two essential simple proper arcs with the same end points in $\Sigma_{0,3}$ are isotopic.*

Proof. Let α and β be two arcs connecting the same marked points. In general position α and β can be taken to intersect transversely.

Now the third disjoint point can be used in applying sterographic projection to the plane.

Now if α and β interesect, then there exists an innermost disk bounded by subarcs of α and β. Then α can be isotoped across this disk to reduce the intersection number until α and β and disjoint away from their end points.

Now cut $\Sigma_{0,3}$ along $\alpha \cup \beta$. This gives a disk disjoint union a once punctured disk (this uses the classification of surfaces). Thus we have that α and β originally bounded an embedded disk and so are isotopic. □

Theorem 7. $\mathrm{MCG}(\Sigma_{0,3}) \cong S_3$*, where* S_3 *is the permutation group on* 3 *elements.*

Proof. Define $\psi : \mathrm{MCG}(\Sigma_{0,3}) \to S_3$ which sends ϕ to the permutation induced on the 3 marked points. This is clearly a surjective homomorphism.

Now assume that ϕ fixes the 3 marked points, meaning the induced permutation is the identity. Now let α be an arc connecting two of the marked points. By assumption $\phi(\alpha)$ is an arc between the same two marked points (as they are fixed by ϕ). Thus we have that α and $\phi(\alpha)$ are isotopic. Then using a face from point set topology, we have that ϕ is isotopic to a map which fixes α point wise, and as such we may assume that ϕ fixes α pointwise.

Now cut $\Sigma_{0,3}$ along α to obtain a disk with a marked points (and two marked points on the boundary, but that is not relevant). Then we can apply Alexander's trick to the induced map $\hat{\phi}$ on the disk to get that ϕ is homotopic to the identity. Thus we have that ψ is injective, and so an isomorphism. $\square$

5. The Annulus

Let A denote $\Sigma_{0,0}^2$ the annulus.

Theorem 8. $\mathrm{MCG}(A) \cong \mathbb{Z}$

Proof. The universal cover of A is the infinite strip, $\tilde{A} = \mathbb{R} \times [0,1]$, as seen in figure 2.

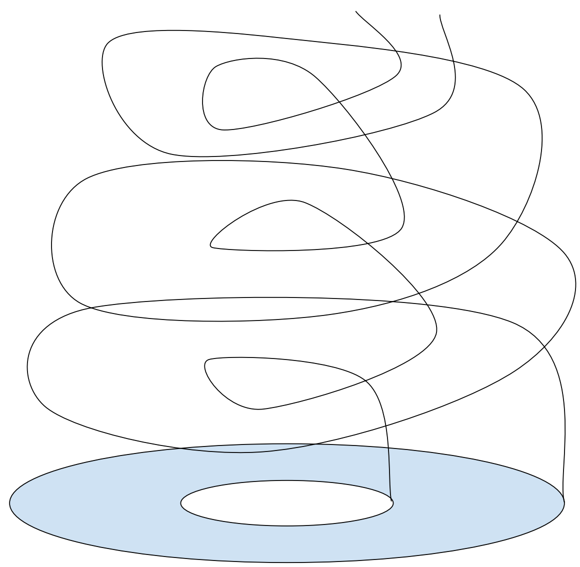

Figure 2. A poorly drawn figure of the universal cover of the annulus

Now any $\phi : A \to A$ has a preferred lift $\tilde{\phi}$ which fixes the origin. Now let $\tilde{\phi}_1 : \mathbb{R} \to \mathbb{R}$ be the restriction to $\mathbb{R} \times \{1\}$. This is a lift to $\mathbb{R}$ of the identity map, and thus an integer translation.

Now define $\rho : \mathrm{MCG}(A) \to \mathbb{Z}$ by $\rho(f) = \tilde{\phi}_1(0)$, or just $\tilde{\phi}_1$ if we identify $\mathbb{Z}$ with the group of integer translations. This again is clearly a homomorphism.

We need to show that ρ is surjective. Take the linear transformation of $\mathbb{R}^2$ given by

$$M = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

We see that this preserves $\tilde{A}$ and is equivariant. Thus we descend to a homeomorphism $\phi_n : A \to A$, and

$$\rho([\phi_n]) = n$$

by definition.

Now we must show that ρ is injective. Let $[\phi] = f \in \ker(\rho)$, with preferred lift $\tilde{\phi}$. Then as $\rho(f) = 0$ we know that $\tilde{\phi}$ acts as the identiy on $\partial\tilde{A}$. Now we claim that the staright line homotopy form $\tilde{\phi}$ to $Id : \tilde{A} \to \tilde{A}$ is equivariant, meaning $\tilde{\phi}(\tau \cdot x) = \tau \cdot \tilde{\phi}(x)$ for any deck transformation τ and any $x \in \tilde{A}$. We have that

$$\tilde{\phi}(\tau \cdot x) = \phi_*(\tau) \cdot \tilde{\phi}(x).$$

but we have that ϕ fixes ∂A pointwise so ϕ_* is the identity. Thus we have that $\tilde{\phi}$ is homotopic to the identity by the straight line homotopy and this descends to a homotopy of A that fixes A pointwise, making ρ injective, and thus an isomorphism.

□

6. The Torus

Theorem 9. $\mathrm{T}^2 \cong \mathrm{SL}(2, \mathbb{Z})$.

Proof. Define $\sigma : \mathrm{MCG}(T^2) \to \mathrm{SL}(2, \mathbb{Z})$ which sends ϕ to the induced map

$$\phi_* : H_1(T^2, \mathbb{Z}) \to H_1(T^2, \mathbb{Z}).$$

As ϕ is invertible ϕ_* is an automorphism, and $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$. So $\phi \mapsto \phi_*$ induces a map into $Aut(\mathbb{Z} \times \mathbb{Z}) \cong \mathrm{GL}(2, \mathbb{Z})$.

Now we have that algebraic intersection numbers in T^2 correspond to determinants in $\mathrm{GL}(2, \mathbb{Z})$ and these are preserved by orientation preserving homeomorphisms, meaning the map is into $\mathrm{SL}(2, \mathbb{Z})$.

We look to show that σ is surjective. Take $M \in \mathrm{SL}(2, \mathbb{Z})$. This induces an orientation preserving linear homeomorphism of $\mathbb{R}^2$ that is equivariant with respect to deck transformations (meaning $\mathbb{Z} \times \mathbb{Z}$), and thus this descends to a linear homemorphism ϕ_M of $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Now identifiying primitive vectors in $\mathbb{Z} \times \mathbb{Z}$ with homotopy classes of simple closed curves gives us that $\sigma([\phi_M]) = M$.

Now we must show that σ is injective. Suppose that

$$\sigma(f) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

and let α and β be the meridian and longitude of the torus (the $(1, 0)$ and $(0, 1)$ curves). The $\phi(\alpha)$ is isotopic to α and $\phi(\beta)$ is isotopic to β. Up to isotopy ϕ fixes α pointwise, and preserves the two sides of α. Now cut along *alpha* to give an annulus. This induces a homeomorphism $\bar{\phi} : A \to A$. Then β and $\bar{\phi}(\beta)$ are arcs in A. As they are isotopic we knwo that $\rho(\bar{f}) = 0$, using the map from the previous proof. As this map is injective we have that $\bar{f}$ is the identity, and so is $p\bar{h}i$, and thus so is ϕ. This gives us that σ is injective, and thus an isomorphism. $\square$

A similar argument can be used to show that $\mathrm{MCG}(\Sigma_{1,1}) \cong \mathrm{SL}(2,\mathbb{Z})$ just by being careful to say that each linear homeomorphism of $\mathbb{R}^2$ can be taken to fix the origin.

7. The 4 Punctured Sphere

7.1. **The Hyperelliptic Involution.** Define the map

$$i : T^2 \to T^2$$

which rotates the square about the center, where the torus is viewed as the square with top and bottom identified and with the left and right edges identified.

This map has 4 fixed points. These are the center of the square, the vertex in the corner, and the midpoint of the two edges. This tells us that T^2/i is $\Sigma_{0,4}$.

Lemma 10. *The hyperelliptic involution induces a bijection between the simple closed curves in T^2 and the simple closed curves in $\Sigma_{0,4}$.*

Proof. In T^2 the simple closed curves are given by (p,q) curves. Let α and β be two simple closed curves in T^2 which intersect once. Then identify α with $(1,0) \in \mathbb{Z} \times \mathbb{Z}$ and β with $(0,1) \in \mathbb{Z} \times \mathbb{Z}$. Let (p,q) be primitive in $\mathbb{Z} \times \mathbb{Z}$. A simple closed curve γ is called a (p,q) curve if $(int(\gamma,\beta), int(\gamma,\alpha)) = \pm(p,q)$. Now take p parallel copies of α and twist by a $\frac{2\pi}{q}$ twist along β. Up to homotopy we may assume i projects α and β to simple closed curves $\bar{\alpha}$ and $\bar{\beta}$ in $\Sigma_{0,4}$ which intersect in not one, but now two points.

Now we can similarly take p copies of $\bar{\alpha}$ and twist along $\bar{\beta}$ by $\frac{\pi}{q}$ to get (p,q) curves on $\Sigma_{0,4}$.

Now let γ be an arbitrary simple closed curve in $\Sigma_{0,4}$. Up to homotopy we may assume γ is in minimal position with respect to $\bar{\alpha}$. Now cut $\Sigma 0,4$ along $\bar{\beta}$ to get two twice punctured disks, with $\bar{\alpha}$ and γ giving collections of disjoint arcs. Now minimal position of γ with resepct to α ensures that each arc is essential. Thus using the argument preceeding the discussion of the three punctued sphere we have that these arcs must be freely homotopic and thus γ is (p,q) curve as constructed.

Finally we have that the pre-image of a (p,q) curve is a $(2p,2q)$ curve on T^2 which is just two copies of a (p,q) curve.

$\square$

7.2. **The 4 Punctured Sphere.**

Theorem 11. $\mathrm{MCG}(\Sigma_{0,4}) \cong \mathrm{PSL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$

Proof. First we define a homomorphism $\bar{\sigma} : \mathrm{MCG}(\Sigma_{0,4}) \to \mathrm{PSL}(2,\mathbb{Z})$, and then we will construct a right inverse which has a kernel isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to prove our claim.

For $[\phi] \in \mathrm{MCG}(\Sigma_{0,4})$ there are two lifts of ϕ to $\mathrm{Homeo}^+(T^2)$, namely $\tilde{\phi}$ and $i\tilde{\phi}$. Now define

$$\bar{\sigma}(f) = \sigma([\tilde{\phi}])$$

.

Then we note that $\sigma(i) = -Id$, and we have our desired homomorphism into $\mathrm{PSL}(2,\mathbb{Z})$.

Now we need a right inverse of $\bar{\sigma}$. An element of $\mathrm{PSL}(2,\mathbb{Z})$ gives an orientation preserving linear homemorphism of T^2 well defined up to hyperelliptic involution. Now each map commutes with i and thus induces an orientation preserving homeomorphism of $\Sigma_{0,4}$, so we have a map

$$\mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{MCG}(\Sigma_{0,4}),$$

and this map is cearly a right inverse so $\bar{\sigma}$.

Now let i_1 and i_2 be the maps which are rotation by π along the axes determined by the equaters seperating the marked ponts, as seen in figure 3. Now that i_1 and i_2 generate a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ subgroup fo $\mathrm{MCG}(\Sigma_{0,4})$.

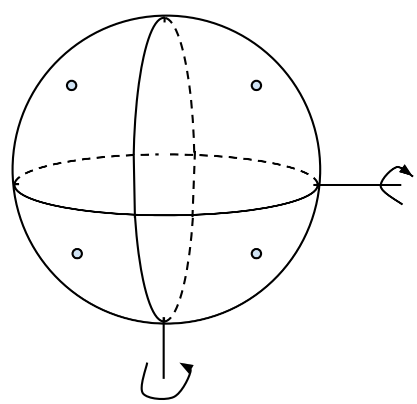

FIGURE 3. The axis of rotation for i_1 and i_2

We have that i_1 and i_2 lift to rotation by π in one of the factors of T^2 when viewed as $S^1 \times S^1$, and are not seen by $\bar{\sigma}$. Now we claim that $\langle i_1, i_2 \rangle = \ker(\bar{\sigma})$.

Take $f \in \ker \bar{\sigma}$. So $\tilde{f} \in \text{Homeo}^+(T^2)$ acts by $\pm Id$ on $H_1(T^2, \mathbb{Z})$, and so acts trivially on the simple closed curves in T^2. Now our lemma gives us that f also acts trivially on homotopy classes of simple closed curves in $\Sigma_{0,4}$.

Thus we have f fixes $\bar{\alpha}$ and $\bar{\beta}$ so we can precompose f with $k \in \langle i_1, i_2 \rangle$ so that fk fixes the 4 punctures. Then say that $fk = [\phi]$ so up to isotopy ϕ fixes $\bar{\alpha}$ and $\bar{\beta}$ pointwise. So we hav ethat ϕ fixes the parked points, and cutting along $\bar{\alpha}$ and $\bar{\beta}$ allows us to use Alexander's trick. □

8. Dehn Twists

For any surface Σ, let α be a simple closed curve. Let N be a regular annular neighborhood of α, with $\phi : A \to N$. Now define $T_\alpha : \Sigma \to \Sigma$ by

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & x \in N \\ x & x \in S - N \end{cases}.$$

Where T is the map $\rho^{-1}(1)$ given in proof of the mapping class group of the annulus.

Theorem 12 (Humphries)**.** *The mapping class group of the closed genus g surface is generated by Dehn twists about the $2g+1$ curves seen in figure 4.*

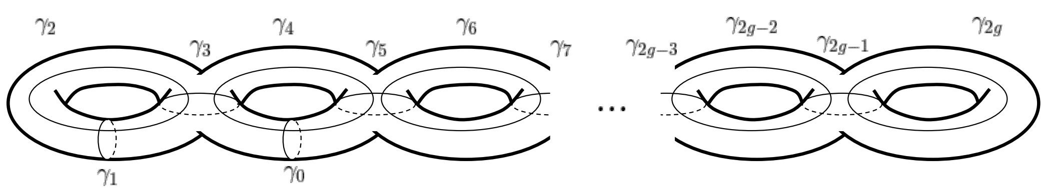

Figure 4. The Humphries Generators

The proof of this is more than we will be able to cover. A more managable proof is that the collection of **all** Dehn twists generates the mapping class group, but we also won't cover that. Dropping down to finitely many Dehn twists is difficult, and Humphries also proved that there are no generating sets of fewer than $2g+1$ Dehn twists (but you can generate with only elements if they don't need to be Dehn twists).

9. The Capping Homomorphism

Let Σ' be the surface obtained from Σ by capping the boundary component β (where β is the curve parallel to the boundary component), with a once punctured disk where the puncture is called p_0. Now let $\text{MCG}(\Sigma, \{p_1, ..p_k\})$ be the subgroup of $\text{MCG}(\Sigma)$ fixing $p_1, ..., p_k$, and similarly for $\text{MCG}(\Sigma', \{p_0, p_1, ..., p_k\})$.

Now define $cap : \mathrm{MCG}(\Sigma, \{p_1, ..., p_k\}) \to \mathrm{MCG}(\Sigma', \{p_0, ..., p_k\})$ by the induced homomorphism. Then the following sequence is exact:

$$1 \to \langle T_\beta \rangle \to \mathrm{MCG}(\Sigma, \{p_1, ..., p_k\}) \to \mathrm{MCG}(\Sigma', \{p_0, ..., p_k\}) \to 1.$$

9.1. **Pair of Pants.**

Theorem 13. $\mathrm{MCG}(\Sigma^3_{0,0}) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

This is just repeated use of the capping homomorphism as well as earlier result on the 3 puncured sphere.

9.2. **The Torus with One Boundary Component.**

Theorem 14. $\mathrm{MCG}(\Sigma^1_1) \cong \tilde{\mathrm{SL}}(2, \mathbb{Z})$

Let $\mathrm{SL}(\tilde{2}, \mathbb{Z}) = \langle a, b : aba = bab \rangle$, and recall that $\mathrm{SL}(2, \mathbb{Z}) = \langle a, b : aba = bab, (ab)^6 = 1 \rangle$. Now using the capping homomorphism we have

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{SL}(\tilde{2}, \mathbb{Z}) & \longrightarrow & \mathrm{SL}(2, \mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \langle T_\beta \rangle & \longrightarrow & \mathrm{MCG}(\Sigma^1_1) & \longrightarrow & \mathrm{MCG}(\Sigma_{1,1}) & \longrightarrow & 1 \end{array}$$

and the $5-$Lemma gives the desired isomorphism.

10. Projective Representations

Let V be a complex vector space, and define

$$\mathrm{PGL}(V) := \mathrm{GL}(V)/\mathbb{C}^* Id.$$

Definition 15. *Let G be a group. A projective representation of G is a homomorphism*

$$\rho : G \to \mathrm{PGL}(V).$$

Put another way this is a collection of operators index by G, meaning an operator $\rho(g)$ for each $g \in G$, and

$$\rho(gh) = c(g, h)\rho(g)\rho(h),$$

where $c(g, h)$ is a constant depending on g and h. This description won't be focused on here, but it tells us that there is group cohomology lurking in the background.

Given a linear representation $r : G \to \mathrm{GL}(V)$ we can always define a projective representation using the quotient map π

$$\begin{tikzcd} G \arrow[r, "r"] \arrow[rr, bend right, "\pi \circ r"'] & \mathrm{GL}(V) \arrow[r, "\pi"] & \mathrm{PGL}(V) \end{tikzcd}$$

Now given a projective representation $\rho : G \to \mathrm{PGL}(V)$, can you find such an r? Put another way, can projective representations be lifted to lienar representations.

10.1. Central Extensions.

A central extension of G is a short exact sequence

$$1 \to A \to E \to G \to 1.$$

Also said as E is an extension of G by A. So if $A \leq Z(E)$ we say this is a central extension.

Now define $H \leq G \times \mathrm{GL}(V)$ as

$$H := \{(g, A) : \pi(A) = \rho(g).$$

Then

$$1 \to \{(e, cId) : c \in \mathbb{C}^*\} \to H \to G \to 1$$

where $\phi : H \to G$ by $(g, A) \mapsto g$. Then we note that $ker(\phi) = \{(e, cId) : c \in \mathbb{C}^*\}$ and so H is a central extension of G.

Now define $\sigma : H \to GL(V)$ by $(g, A) \mapsto A$, then this is a linear representation and $\pi(\sigma(g, A)) = \rho(g) = \rho(\phi(g))$. And so we almost were able to find a lift, but at the cost of passing to a central extension.

11. Quantum Representations

Theorem 16. *A modular tensor category gives rise to a projective representation of the mapping class group for any surface.*

In short

- Associate a vector space $V(\Sigma)$
- Give a projective action of Dehn twists