Riemann-Hilbert's problem of the theory of linear differential equations

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This translation is subject to correction. The numbers in square brackets refer to the literature list of the originial article.

Contents

Introduction

1. Given a system $\frac{dy_i}{dz} = \sum_1^n R_{ij}(z)y_j$, $i = 1, ..., n$, of homogenous differential equations of order 1 with rational coefficients,it is common knowledge that their solutions are in general not unique and meromorph in the Riemann sphere P^1. Though, each solution has just finitely many isolated singularities, which can be either the poles $x_1, ..., x_k$ of the coefficients $R_{ij}(z)$ or maybe the point $x_0 = \infty$. If the solutions $\eta_i = (y_{1i}(z), ..., y_{ni}(z))$, $i = 1, ..., n$ form a fundamental system of solutions, then while circulation around the singular point x_χ the vector η_i turns over into a vector $\sum a_{ij}^{\chi}$. The matrix $\mathfrak{A}^{(\chi)} := (a_{ij}^{(\chi)})$ is not singular and the so called Riemannian Relation $\mathfrak{A}^0 \cdot ... \cdot \mathfrak{A}^{(k)} = 0$ is true. This relation yields that every fundamental system $\eta_1, ...\eta_n$ of solutions induces a homomorphisms from the fundamental group of P^1–$(\{x_0, ..., x_k\})$ into the general linear group $GL(n, \mathbb{C})$ over the field of complex numbers $\mathbb{C}$.

2. In the year 1857 B. Riemann [21] already brought up the question if conversely to every homomorphism from the fundamental group of P^1–$\{x_0, ..., x_k\}$, where $x_0, ..., x_k$ are given randomly, there belongs a system of n linear homogenous ordinary differential equations of order 1 with rational coefficients that has a fundamental system of solutions which induces the given homomorphism. Moreover Riemann required that the system of differential equations has to be of Fuchssian type (compare preliminary remarks). In the same year [20] he proved this problem for $k = n = 2$ by giving explicit fundamental systems of that type. Afterwards among others H. Poincaré [19] and L. Schliesinger [23, 24] dealt with the Riemannian problem, but their proofs are incomplete and not exact from the modern point of view, which was already pointed out by J. Plemelj [18], who provided the first mainly flawless existence proof for arbitrary k and n. In 1900 D. Hilbert includes the Riemannian problem to his mathematical problems [8] (since then it's called Riemann-Hilbert's problem). In 1905 D. Hilbert [10, 11] solved it for $n = 2$ and arbitrary k, which is a special case prior also approached by O. Kellog [12]. Similar to D. Hilbert's and O. Kellog's approaches Plemelj's proof is based on the theory of Fredholm's integral equations. In 1913 G. D. Birkhoff [2] got Plemelj's general result through certain approximations. At the same time he finished a generalisation of Riemann-Hilbert's problem that he claimed earlier. Around 1924 O. Haupt [6-8] worked on a question strongly related to Riemann-Hilbert's problem.

Out of the monographs that deal with Riemann-Hilbert's problem the works of J. A. Lappo-Danilevsky [14] and N. I. Muskhelishvili [15] need to be mentioned particularly. [14] also covers the question of dependence of the fundamental system and the "bifurcation points" $x_0, ..., x_k$ in sufficient generality.

3. In a certain way the result of J. Plemelj led the general theory about systems of linear homogenous differential equations of order 1 and Fuchsian type with rational coefficients to a satisfying end, because one was now able to understand the local and global function theoretical behavior of the solutions completely. Analog studies about systems of linear homogenous differential equations of order 1 whose coefficients are meromorphic functions on an arbitrary compact or non compact Riemannian surface are still missing. Obviously the local theory runs similar to the classical case. Alike on the Riemann sphere one can characterize the set of singularities X' of a solution system. Each fundamental system of solutions again generates a homomorphism from the fundamental group of $X - X'$ in $GL(n, \mathbb{C})$. Therefore it's possible to trans-

fer Riemann-Hilbert's problem and finally ask for dependence of solutions and bifurcation points. The difficulty in treating this type of questions mainly lies in the existence of non-breaking return steps.

4. This thesis proves the general Riemann-Hilbert problem for arbitrary (compact or non compact) Riemannian surfaces. The used methods contrary to the ones used before are of pure function theoretical type. Some fundamental theorems of modern theory of functions in multiple variables and the theory of complex-analytical fiber spaces will be used. The existence problems for differential equations will be translated to statements about existence of complex-analytical sections in complex-analytical fiber spaces (theorem 1 and 2). The presence of such sections will in some cases (theorem 3) be ensured by triviality of the defining cocycle. In the remaining cases that refer to compact Riemannian surfaces the existence will be satisfied either by direct construction or by a theorem of S. Nakano about complex-analytical vector space bundles (theorem 4). It's an easy consideration that with the help of theorem 3 and 4 one can also answer Birkhoff's [2] generalization of Riemann-Hilbert's problem to arbitrary Riemannian surfaces: one only has to replace the cocycle $\xi_{\mathfrak{B}}$ (we won't carry this out in detail). As known theorem 3 and 4 even yield more. Theorem 3 for example contains the Weierstrass factorization theorem for non compact Riemannian surfaces and matrices (instead of functions), which in case of the Riemann sphere was already proved by G. D. Birkhoff [3]. It's known that the Weierstrass factorization theorem doesn't hold for non compact Riemannian surfaces, but Theorem 4 yields: let X be a compact Riemannian surface, $x_1, ..., x_k$ a set of points on X and $f_1(x), ..., f_k(x)$ functions which are meromorphic in a suitable reduced neighborhood of x_i ($i \in \{1, ..., k\}$), then there exists a meromorphic function $f(x)$ on $X - \{x_1, ..., x_k\}$ such that $f(x)f_\kappa^{-1}(x)$, $\kappa = 1, ..., k$, is meromorphic extendable in x_κ. Obviously this is the corresponding formulation of the Weierstrass factorization theorem for compact Riemannian surfaces. Because of the fact, that theorem 4 remains true for algebraic manifolds, one gets a corresponding of the Cousin-II-problem for algebraic manifolds. In contrast to the Cousin-II-problem for holomorphic complete spaces this admits unlimited solutions.

The question of dependence between solutions and bifurcation points will be examined with similar methods as Riemann-Hilbert's problem itself. The main result is: if the bifurcation points vary in simple connected and pairwise distinct areas we can always give solutions $\mathfrak{z}$, depending on the bifurcation points in a meromorphic way, for Riemann-Hilberts problem. Regarding the exact formulation we refer to Theorem II. The methods used in this paper can also successfully be used for a couple of questions concerning the work of O. Teichmueller [25] about varying Riemannian surfaces. Additional it should be indicated that similar questions arise while dealing with certain existence problems in theory of functions in multiple variables.

1 Preliminary notes

Let X be an abstract Riemannian surface, which is always supposed to be connected in this paper. Consider a system of n linear homogenous differential equations of order 1

$$d\eta = \eta\Omega'(x), \tag{1}$$

where $\Omega'(x)$ denotes a matrix of the type (n,n) whose components are meromorphic differential forms on X of degree 1. We say (1) is asystem of differential equations on X. Moreover X' denotes the entirety of poles of the components of $\Omega'(x)$ and we call the union of divisors of the components of $\Omega'(x)$ the divisor of $\Omega'(x)$. As known there always exist non trivial solutions for this system of differential equations. These are vectors $\eta(\tilde{x})$, whose components are meromorph and not all equal to zero on the universal covering space $\widetilde{X-X'}$ of $X-X'$ and who satisfy the equation

$$d\eta(\tilde{x}) = \eta(\tilde{x})\psi^*(\Omega'(x)) \tag{1'}$$

,

where ψ denotes the natural map $\widetilde{X-X'} \to X-X'$ and ψ^* the corresponding monomorphism from the field (ring) of meromorphic functions (differential equations) on $X-X'$ to the field (ring) of meromorphic functions (differential equations) on $\widetilde{X-X'}$. The entirety L of solutions of (1) is a n-dimensional vector space over $\mathbb{C}$.

Let $x_0 \in X-X'$ and consider the fundamental group $\pi_1(X-X', x_0)$. Given $\tilde{x}_0 \in \widetilde{X-X'}$ such that $\psi(\tilde{x}_0) = x_0$, according to the usual construction of the universal covering space the elements $\alpha \in \pi_1(X-X', x_0)$ coincide invertible unique with the points of $\{\psi^{-1}(x_0)\}$. The point belonging to α under this mapping is denoted by $\alpha(\tilde{x}_0)$. Moreover $\eta(\tilde{x})$ denotes the germ of η in $\tilde{x} \in \widetilde{X-X'}$. For $\alpha \in \pi_1(X-X', x_0)$ we define

$$\alpha^* \cdot \eta(\tilde{x}_0) := \eta(\alpha(\tilde{x}_0)).$$

As known α^* is a $\mathbb{C}$-automorphism on L. Because of $(\alpha\beta)^* = \beta^*\alpha^*$ the mapping $\alpha \to \alpha^*$ yields an anti-homomorphism $\mu_0 : \pi_1(X-X', x_0) \to Aut(L)$, where $Aut(L)$ denotes the group of automorphisms on L. Fixing a basis of L, α^* corresponds in a natural way to an element $\mu(\alpha) \in GL(n,\mathbb{C})$. Obviously $\alpha \to \mu(\alpha)$ is a homomorphism. Furthermore it's clear that chosing another basis one needs to change $\alpha \to \mu(\alpha)$ with an equivalent representation. μ_0 is usually called the monodromy of (1) and X' is the set of bifurcation points.

The question arises if there exists a system (1) on X for every representation μ of $\pi_1(X-X', x_0)$ in $GL(n,\mathbb{C})$ such that L yields to μ by the given algorithm. Therefor we assume $X' \subset X$ to have no accumulation points on X and $x_0 \in X-X'$.

Let $\mathfrak{B}(\tilde{x})$ be the matrix whose rows are the vectors of a basis of L. Then for $\alpha \in \pi_1(X-X', x_0)$ we get

$$\alpha^* \cdot \mathfrak{B}(\tilde{x}_0 = \mu(\alpha))\mathfrak{B}(\tilde{x}_0), \tag{2}$$

where α^* is defined analogously for matrices. $\mathfrak{B}(\tilde{x})$ is meromorphic and non singular on $\widetilde{X-X'}$, which means $Det\mathfrak{B}(\tilde{x}) \not\equiv 0$. Conversely, given a meromorphic, non-singular matrix $\mathfrak{B}(\tilde{x})$ of type (n,n) that suffices (2) also $\mathfrak{B}^{-1}(\tilde{x})d\mathfrak{B}(\tilde{x})$ is meromorphic on $\widetilde{X-X'}$. Since

$$\alpha^* \cdot \mathfrak{B}^{-1}(\tilde{x}_0)d\mathfrak{B}(\tilde{x}_0) = \mathfrak{B}^{-1}(\tilde{x}_0)d\mathfrak{B}(\tilde{x}_0)$$

for $\alpha \in \pi_1(X-X', x_0)$, there exists

$$\psi^{*-1}(\mathfrak{B}^{-1}(\tilde{x})d\mathfrak{B}(\tilde{x})) =: \Omega'(x).$$

$\Omega'(x)$ is meromorphic on $X - X'$ and as linear combinations of the rows of $\mathfrak{B}(\tilde{x})$ the solutions of the system of differential equations

$$d\eta = \eta\Omega'(x)$$

on $X - X'$ yield the given representation μ of $\pi_1(X - X', x_0)$. The branches of $\mathfrak{B}(\tilde{x})$ generally have ugly singularities in the points of X' so that one cannot expect $\Omega'(x)$ to expand meromorphic on X.

Experience has shown that in the theory of linear differential equations in complex space the so called singular points of certainty are "good" singularities. In slight modification of the usual terminology we call $x' \in X'$ a *point of certainty* for the on $\widetilde{X - X'}$ meromorphic vector $\eta(\tilde{x})$ if there exists a neighborhood U of x' in X such that $U \cap X' = \{x'\}$ and for every connected component V_j of $\psi^{-1}(U - \{x'\})$ there exists a matrix $\mathfrak{U}_j$ which sat*of Fuchssian type* if for each element oisfies the existence and meromorphic extension of

$$\psi_j^{*-1}\{exp(\mathfrak{U}_j logt \circ \psi_j(\tilde{x}))\eta(\tilde{x})\}$$

in x' for the local uniformiser $t(x)$ of x' in U with $t(x') = 0$. Here ψ_j denotes the restriction of ψ to V_j. This definition can analogously be made for matrices.

A system of differential equations (1) is called of Fuchsian type if for every element of L all $x' \in X'$ are points of certainty. Given an on $\widetilde{X - X'}$ meromorphic matrix $\mathfrak{B}(\tilde{x})$ such that each point $x' \in X'$ is a point of certainty and 2 is true, it's easy to see that $\Omega'(x)$ is meromorphically extendable on whole X.

Riemann-Hilbert's problem is about constructing a system (1) of Fuchsian type on X for a given $X' \subset X$ and representation μ of $\pi_1(X - X', x_0)$ in $GL(n, C)$, such that the set of solutions L yields μ. Because of the remarks above it suffices to construct an on $\widetilde{X - X'}$ meromorphic, nonsingular matrix $\mathfrak{B}(\tilde{x})$ that suffices (2) and the property that every $x' \in X'$ is a point of certainty. We denote this existence problem by (X, X', μ). $\mathfrak{B}(\tilde{x})$ is called a holomorphically invertible solution of (X, X', μ) if $\mathfrak{B}(\tilde{x})$ is holomorphic on $\widetilde{X - X'}$ and holomorphically invertible.

2 Fiberspaces assigned to Riemann-Hilbert's problem

It's appropriate to solve $(X - X', \emptyset, \mu)$[1] first and remove the points of uncertainty in X' in a second step. In the discussion of $(X - X', \emptyset, \mu)$ we can always assume that $X - X'$ is not compact. Indeed, if X is compact and $X' = \emptyset$, let $X'' \in XX'' := \{x''\}$ and consider $(X - X', \emptyset, \mu^*)$, where $\mu^* = \mu \circ i^*$ and $i^* : \pi_1(X - X'') \to \pi_1(X - X', x_0)$ is the natural homomorphism. Then a solution of $(X - X'', \emptyset, \mu^*)$ is also a solution of $(X, \emptyset, \mu)$ over $X - X''$ which yields there is at most one singularity in x'' left to be removed.

Let X be a non compact Riemannian surface and $\{U_i\}_{i \in I}$ be an open covering of X with connected coordinate neighbourhoods U_i such that $\pi_1(U_i) = 0$ for

[1] $\emptyset$ is the empty set

$i \in I$ and $U_i \cap U_j$ is connected. A $GL(n,\mathbb{C})$-cozycle $\xi_\mu \in H^1(X, GL(n,\mathbb{C})_\omega)$ will be assinged to $(X, \emptyset, \mu)$ in the fallowing way. Choose a point $x_i \in U_i$ for every U_i and connect x_0 and x_i with a curve K_i starting in x_0. For $x \in U_i \cap U_j$ let $D_{ij}(x)$ be a curve from x_i to x in U_i and $D_{ij}(x)$ be a curve from x_j to x in U_j, We denote the homotopy class of a curve K by $\langle K \rangle$. Define

$$g_{ij}(x) := \mu(\langle K_i D_{ij}(x) D_{ji}(x)^{-1} K_j^{-1} \rangle)$$

for $x \in U_i \cap U_j$. Since $g_{ij}(x)$ is constant in each connected component of $U_i \cap U_j$ due to $\pi_1(U_j) = \{0\}$ we obtain $g_{ij}(x)$ is a holomorphic map $U_i \cap U_j \to GL(n,\mathbb{C})$. One calculates easily that $g_{ij}(x)$ suffices $g_{ij}(x) g_{jk}(x) = g_{ik}(x)$ for $x \in U_i \cap U_j \cap U_k$. The cozycle in $H^1(X, GL(n,\mathbb{C})_\omega)$ defined by the $g_{ij}(x)$ is denoted by ξ_μ and doesn't depend on the choice of the points $x_i \in U_i$ and curves K_i. Moreover two of these cozycles are similar, iff there exists an inner automorphism γ of $GL(n,\mathbb{C})$ such that $\mu' = \gamma \circ \mu$.

We want to assign another cozycle $\xi_{\mathfrak{B}}$ to a holomorphic inertible solution $\mathfrak{B}(\tilde{x})$ of $(X - X', \emptyset, \mu)$. Therefor we choose an open covering $\{U_i\}_{i \in I}$ of X such that each U_i has the properties named above and in addition $U_i \cap X'$ is at most one point for every $i \in I$ and every $x' \in X'$ is contained in at most one U_i. For $U_i \cap X' = \emptyset$ define

$$f_i(x) := 1 \in GL(n,\mathbb{C}) \text{ for } x \in U_i.$$

Moreover let $U_j \cap X' = \{x'_j\}$ and let $t_j(x)$ be a local uniformizer in U_j such that $t_j(x'_j) = 0$. Let K_j be a curve from x_0 to $x_j \in U_j - \{x'_j\}$ and D_j a curve in $U_j - \{x'_j\}$ starting and ending in x_j and whose homotopy class generates $\pi_1(U_j - \{x'_j\})$. Then the meromorphic germ

$$exp\left\{-\frac{log\mu(\langle K_j D_j K_j^{-1}\rangle) log t_j \circ \psi(\langle K_j\rangle \tilde{x}_0)}{\langle D_j\rangle log t_j(x_j) - log t_j(x_j)}\right\} \langle K_j\rangle^* \mathfrak{B}(\tilde{x}_0)$$

can be meromorphically extendet to the connected component V_j of $\psi^{-1}(U_j - \{x'_j\})$ to which the point $\langle K_j\rangle \tilde{x}_0$ belongs. This extension is given by

$$exp\left\{-\frac{log\mu(\langle K_j D_j K_j^{-1}\rangle) log t_j \circ \psi(\langle K_j D_j(x)\rangle \tilde{x}_0)}{\langle D_j\rangle log t_j(x_j) - log t_j(x_j)}\right\} \langle K_j D_j(x)\rangle^* \mathfrak{B}(\tilde{x}_0),$$

where $D(x)$ is a curve in $U_j - \{x'_j\}$ starting in x_j and ending in x. One easily sees that

$$f_j(x) := \psi_j^{*-1}(exp\left\{-\frac{log\mu(\langle K_j D_j K_j^{-1}\rangle) log t_j \circ \psi(\langle K_j D_j(x)\rangle \tilde{x}_0)}{\langle D_j\rangle log t_j(x_j) - log t_j(x_j)}\right\} \langle K_j D_j(x)\rangle^* \mathfrak{B}(\tilde{x}_0))$$

exists for $x \in U_j\{-x'_j\}$. The maps

$$g_{ij}(x) := f_i(x) f_j(x)^{-1} \text{ for } x \in U_i \cap U_j$$

are holomorphic and compatible, thus they define a cozycle $\xi_{\mathfrak{B}} \in H^1(X, GL(n,\mathbb{C})_\omega)$. If $GL(n,\mathbb{C})$ operates as a complex automorphismgroup on the complex space Y, we can associate a complex-analytical fiberbundle $(X, \xi_{\mathfrak{B}}, Y)$ to the recently constructed cozycle. Here we just need the cases $Y = GL(n,\mathbb{C})$, which leads to the main-bundle, $Y = \mathbb{P}^n$, where $\mathbb{P}^n$ denotes the n-dimensional complex-projective space, and $Y = \mathbb{P}^{n^2}$. Since $GL(n,\mathbb{C})$ can be naturally understood as a subgroup of $PGL(n+1,\mathbb{C})$ we just need to define how $GL(n,\mathbb{C})$ operates on $\mathbb{P}^{n^2}$. Therefor we fix a $\mathbb{C}^{n^2}$ corresponding to $\mathbb{P}^{n^2}$ and identify it with the set of all matrices over $\mathbb{C}$ with n rows. $GL(n,\mathbb{C})$ acts on the set of all matrices via left-multiplication and thus on $\mathbb{C}^{n^2}$ as a group of linear automorphisms, which as known can be uniquely extended to $\mathbb{P}^{n^2}$.

Theorem 1. *Assume X is not compact. To prove the existence of a holomorphic invertible solution of $(X,\emptyset,\mu)$ it's necessary and sufficient to show that ξ_μ is the trivial cozycle.*

Proof. Let $\mathfrak{B}(x)$ be a holomorphic invertible solution of $(X,\emptyset,\mu)$. Moreover denote by ψ_i the restriction of ψ to the component V_i of $\psi^{-1}(U_i)$ that contains $\langle K_i\rangle\tilde{x}_0$. $\langle K_i\rangle^*\mathfrak{B}(\tilde{x}_0)$ is extendable to $\langle K_iD_i(x)\rangle^*\mathfrak{B}(x_0)$, which is an on V_i holomorphic and holomorphically invertible matrix. Since ψ_i maps V_i topologically to U_i there exists

$$s_i(x) := \psi_i^{*-1}(\langle K_iD_i(x)\rangle^*\mathfrak{B}(\tilde{x}_0)) \text{ for } x\in U_i$$

and is holomorphic and holomorphically invertible. Due to

$$\begin{aligned} s_i &= \psi_i^{*}(\langle K_iD_i(x)D_j^{-1}(x)K_j^{-1}K_jD_j(x)\rangle^*\mathfrak{B}(\tilde{x}_0)) \\ &= \psi_i^{*-1}(\langle K_jD_j(x)\rangle^*\langle K_iD_i(x)D_j^{-1}(x)K_j^{-1}\rangle^*\mathfrak{B}(\tilde{x}_0)) \\ &= g_{ij}(x)\psi_j^{*-1}(\langle K_jD_j(x)\rangle^*\mathfrak{B}(\tilde{x}_0)) \\ &= g_{ij}(x)s_j(x) \end{aligned}$$

the collection of $s_i(x)$ forms a complex-analytical section in the main-bundle corresponding to xi_μ, but that means ξ_μ is trivial. On the other hand let $s(x)$ be a complex-analytical section in the main-bundle corresponding to ξ_μ; such a section obviously exists if ξ_μ is trivial. Let the maps $\phi_i(x,y)$ from $U_i\times GL(n,\mathbb{C})$ to th main-bundle corresponting to ξ_μ be a system of local coordinates of the fiber structure. Define

$$s_i(x) := \phi_{i,x}^{-1}(s(x)) \text{ for } x\in U_i.$$

$s_i(x)$ is holomorphic and holomorphically invertible in U_i and we get $s_i(x)s_j^{-1}(x) = g_{ij}(x)$ in $U_i\cap U_j$. We want to show that $s_0(x)$ can be analytically extendet along any path K starting and ending in a fixed $x_0\in U_0$, i.e. it yields to an on $\widetilde{X-X'}$ meromorphic and non-singular matrix $\mathfrak{B}(\tilde{x})$. Besides we'll obtain that $\mathfrak{B}(\tilde{x})$ is holomorphic and holomorphically invertible and suffices (2). With an appropriate choose of x'_ρ, $\rho = 0,...,r+1$, $x_0 = x'_0 = x'_{r+1}$, we can split K in sub-paths K'_ρ from x'_ρ to $x'_{\rho+1}$, such that $K_\rho\subset U_\rho$ is true for $\rho = 0,...,r$ and appropriate elements U_ρ, $\rho = 0,...,r$, $U_0 = U_r$, of the given covering of X. Moreover let $D_\rho\subset U_\rho$ be a curve that connects x_ρ with x'_ρ. We get

$$K = (K'_0D_1^{-1}K_1^{-1})(K_1D_1K'_1D_2^{-1}K_2^{-1})...(K_{r-1}D_{r-1}K'_{r-1}K'_r)$$

and therefore

$$(\langle K\rangle) = g_{01}(x'_1)...g_{r-1,0}(x'_r). \tag{3}$$

$g_{i,i+1}(x)$ is constant in $U_i \cap U_j$, so $g_{i,i+1}(x'_{i+1})s_{i+1}(x)$ extends $s_i(x)$ to U_{i+1}. With that we can extend $s_0(x)$ analytically along the whole path K which yields an on $\widetilde{X - X'}$ holomorphic and holomorphically invertible matrix $\mathfrak{B}(\tilde{x})$. Because of (3) $\mathfrak{B}(\tilde{x})$ obviously suffices (2). □

Corollary. If X is not compact we get a natural equivalence of holomorphically invertible solutions of $(X, \emptyset, \mu)$ and complex-analytical sections in the main-bundle corresponding to ξ_μ.

The Theorem above yields a criterium for existence of matrices with the desired "ramification behavior". The fallowing Theorem shows which conditions are necessary to remove points of uncertainty that may occur.

Theorem 2. *If $\mathfrak{B}(\tilde{x})$ is a holomorphically invertible solution of $(X - X', \emptyset, \mu)$, the fallowing are true:*

1. *If X is not compact, for the existence of a holomorphically invertible solution of (X, X', μ) it's necessary and sufficient to prove that $\xi_{\mathfrak{B}}$ is the trivial cozycle.*
2. *If X is compact, for the existence of a solution of (X, X', μ) it's necessary and sufficient that the bundle $(X, \xi_{\mathfrak{B}}, \mathbb{P}^{n^2})$ with fiber $\mathbb{P}^{n^2}$ corresponding to $\xi_{\mathfrak{B}}$ admits a complex-analytical section $s(x)$ for which there exists at least one $x \in X$ such that $\phi_{i,x}^{-1}(s(x)) \in GL(n, \mathbb{C})$.*

Proof. 1. If $s(x)$ is a complex-analytical section in the main-bundle corresponding to $\xi_{\mathfrak{B}}$ and

$$s_i(x) := \phi_{i,x}^{-1}(s(x)) \text{ for } x \in U_i$$

we have $f_i^{-1}(x)s_i(x) = f_j^{-1}(x)s_j(x)$ in $U_i \cap U_j$. Therefore the collection of $f_i^{-1}(x)s_i(x)$ defines a matrix $\mathfrak{W}(x)$, which is holomorphic and holomorphically invertible in $X - X'$. Define $\mathcal{B} := \mathfrak{B}(\tilde{x})\psi^*(\mathfrak{W}(x))$, then $\mathcal{B}(x)$ is holomorphic and holomorphically invertible on $\widetilde{X - X'}$ and suffices (2). It remains to prove that for $\mathcal{B}(\tilde{x})$ all points of X' are points of certainty.

$$\begin{aligned}
\langle K_i\rangle^*\mathcal{B}(\tilde{x}_0) &= \langle K_i\rangle^*\mathfrak{B})\tilde{x}_0 \cdot \psi^*(\mathfrak{W}(\langle K_i\rangle\tilde{x}_0)) \\
&= \langle K_i\rangle^*\mathfrak{B}(\tilde{x}_0) \cdot \psi^*(f_i^{-1}s_i(\psi(\langle K_i\rangle\tilde{x}_0))) \\
&= \langle K_i\rangle^*\mathfrak{B}(\tilde{x}_0) \cdot \psi^*(f_i^{-1}(\psi(\langle K_i\rangle\tilde{x}_0))) \cdot \psi^*(s_i(\psi(\langle K_i\rangle\tilde{x}_0))) \\
&= \langle K_i\rangle^*\mathfrak{B}(\tilde{x}_0)\{\langle K_i\rangle^*(\tilde{x}_0)\}^{-1}\times \\
&\times exp\left\{\frac{log\mu(\langle K_iD_iK_i^{-1}\rangle)logt_i \circ \psi(\langle K_i\rangle\tilde{x}_0)}{\langle D_i\rangle logt_i(x_i) - logt_i(x_i)}\right\} \cdot \psi^*(s_i(\psi(\langle K_i\rangle\tilde{x}_0)))
\end{aligned}$$

yields

$$exp\left\{-\frac{log\mu(\langle K_iD_iK_i^{-1}\rangle)^i logt_i \circ \psi_i(\tilde{x})}{\langle D_i\rangle logt_i(x_i) - logt_i(x_i)}\right\}\mathcal{B}(\tilde{x}) = \psi_i^*(s_i \circ \psi(\tilde{x}))$$

for $\tilde{x} \in V$. Thus

$$\psi_i^{*-1}\left(exp\left\{-\frac{log\mu(\langle K_iD_iK_i^{-1}\rangle)logt_i \circ \psi_i(\tilde{x})}{\langle D_i\rangle logt_i(x_i) - logt_i(x_i)}\right\}\right)\mathcal{B}(\tilde{x}) = s_i(x),$$

which means V_i suffices the requirement of certainty. $\alpha^* \mathcal{B}(\tilde{x}_0) = \mu(\alpha)\mathcal{B}(\tilde{x}_0)$ yields the same statement for the remaining components. On the other hand, if $\mathcal{B}(\tilde{x})$ is a holomorphic invertible solution of (X, X', μ) we obtain that $\mathfrak{B}^{-1}(\tilde{x})\mathcal{B}(\tilde{x})$ is holomorphic and holomorphically invertible on $\widetilde{X - X'}$. Since $\alpha^* \cdot \mathfrak{B}^{-1}(\tilde{x})\mathcal{B}(\tilde{x}) = \mathfrak{B}^{-1}(\tilde{x})\mathcal{B}(\tilde{x})$ for $\alpha \in \pi_1(X - X', x_0)$ there exists $\mathfrak{W}(x) := \psi^{*-1}(\mathfrak{B}^{-1}(\tilde{x}\tilde{\S}))$. $\mathfrak{W}$ is holomorphic and holomorphically invertible on $X - X'$. Trivially $s_i^*(x) := f_i(x)\mathfrak{W}(x)$ is meromorphic in U_i and holomorphic and holomorphically invertible in $U_i - U_i \cap X'$. Now we search for a matrix $\mathfrak{B}(x)$, which is meroporphic on X and holomorphic and holomorphically invertible on $X - X'$, such that $s_i^*(x)\mathfrak{B}(x)$ is holomorphic and holomorphically invertible in every U_i. Obviously this is a generalization of the Cousin-II-problem. Analogue to the Cousin-II-problem we can assign a cozycle in $H^1(X, GL(n, \mathbb{C})_\omega)$ to our question. Just like in the ancient case the sections in the associated main-bundle are equivalent to the solutions of the generalized Cousin-II-problem. Due to theorem 3 the defining cozycle is trivial, which means there exists a matrix $\mathfrak{B}(x)$ with the desired properties. Obviously the collection of $s_i(x) := s_i^*(x)\mathfrak{B}(x)$ for $x \in U_i$ is a complex-analytical section in the main-bundle corresponding to $\xi_{\mathfrak{B}}$.

2. The proof of 1) can be transferred to 2).

□

Corollary. For a holomorphically invertible solution $\mathfrak{B}(\tilde{x})$ of $(X - X', \emptyset, \mu)$ we get:

1. If X is not compact the holomorphically invertible solutions of (X, X', μ) are naturally equivalent to the complex-analytical sections in the main-bundle associated to $\xi_{\mathfrak{B}}$.

2. If X is compact the solutions of (X, X', μ) are naturally equivalent to the complex-analytical sections $s(x)$ in the bundle $(X, \xi_{\mathfrak{B}}, \mathbb{P}^{n^2})$ associated to $\xi_{\mathfrak{B}}$, such that there exists $x \in X$ with $\phi_{i,x}^{-1}(s(x)) \in GL(n, \mathbb{C})$.

3 Complex-analytical fiberspaces over non compact Riemannian surfaces

Consider complex-analytical fiber spaces with a basis consisting of a non compact Riemannian surface X and a complex Liegroup G as structure group. We'll proof that these complex-analytical fiber spaces are complex-analytical trivial.

Theorem 3. *Given a non compact Riemannian surface X and a complex Liegroup G, the group $H^1(X, G_\omega)$ consists only of the trivial element.*

Proof. For $\xi \in H^1(X, G)$ we have to prove that there exists a complex-analytical section in the main bundle associated to ξ. First we'll prove the existence of a continuous section in this main-bundle. Due to dimensional reasons there can only be two-dimensional non trivial sections in the main-bundle, so we just need to show that these vanish. This is true, because such a section is an element of $H^2(X, \pi_1(G))$, which is zero due to the universal coefficient theorem and the fact that for a non compact Riemannian surface X the two-dimensional integral homology group $H^2(X,\mathbb{Z})$ consists only of the zero element. Therefore the main-bundle associated to ξ is trivial. Since moreover K. Stein [1] verified that every non compact Riemannian surface is a holomorphically complete space we can use a theorem of H. Grauert [5] to conclude complex-analytical triviality from topological triviality which is the desired statement.

Since there isn't a published proof for the quoted theorem of H. Grauert yet we'll proof theorem 3 again for the special case $G = GL(n,\mathbb{C})$, which is of interest for Riemann-Hilberts problem. Therefor we need a generalization of Runge's famous theorem. H. Grauert [5] uses a similar generalization of the statement needet here to proof his quoted theorem. Let $\mathfrak{A} = (a_{ik})$ be a matrix over the field of complex numbers $\mathbb{C}$ and define $\|\mathfrak{A}\| := \left(\sum_{i,k} |aik|^2\right)^{\frac{1}{2}}$.

If $\mathfrak{A}(x) = (a_{ik}(x))$ is a matrix of holomorphical functions $B \subset X$, define $\|\mathfrak{A}(x)\|_B := sup\{\|\mathfrak{A}(x)\| : x \in B\}$. As known the fallowing rules are true:

1. $\|\mathfrak{A} + \mathfrak{B}\| \leq \|\mathfrak{A}\| + \|\mathfrak{B}\|$
2. $\|\mathfrak{A}\mathfrak{B}\| \leq \|\mathfrak{A}\| \cdot \|\mathfrak{B}\|$
3. $\|\alpha\mathfrak{A}\| = |\alpha|\,\|\mathfrak{A}\|$, for $\alpha \in \mathbb{C}$
4. $\left\|e^{\mathfrak{A}}e^{\mathfrak{B}}e^{\mathfrak{C}} - 1\right\| \leq e^{\|\mathfrak{A}\|+\|\mathfrak{B}\|+\|\mathfrak{C}\|} - (1+\|\mathfrak{A}\|+\|\mathfrak{B}\|+\|\mathfrak{C}\|)$, for $\mathfrak{A}+\mathfrak{B}+\mathfrak{C} = 0$.

We'll use the fallowing generalization of Runge's theorem: Consider $B \subset B' \subset X$, where B is compact in B' and B' is compact in X, B relative B' is simply connected and $\mathfrak{A}(x)$ is a holomorphical map from B to $GL(n,\mathbb{C})$. Then for every $\epsilon > 0$ there exists a holomorphic map $\mathfrak{A}'(x)$ from B' to $GL(n,\mathbb{C})$ such that $\|\mathfrak{A}'(x) - \mathfrak{A}(x)\|_B < \epsilon$.

This means $\mathfrak{A}(x)$ can be approximated by holomorphical maps from B' to $GL(n,\mathbb{C})$ in accordance with the topology of uniform convergence. For $n = 1$ this statement fallows from H. Behnke-K. Stein's [1] proof of a generalization of Runge's theorem and another result of his work, which says that for given periods there always exists an integral of genus 1 on a Riemannian surface. These two theorems also yield that given $x \in B' - B$ for every $\epsilon > 0$ there exists an on B' holomorphic function $h(x)$, whose Divisor on B' is $\{x\}$ and for which $\|h(x) - 1\|_B < \epsilon$ is true. To construct such a function we first choose a holomorphic function $h_1(x)$ on B' with divisor $\{x\}$ and give an integral $f(x)$ of genus 1 on B' which has the same integral periods on B as a certain branch of $logh_1(x)$. Then we approximate the on B holomorphic function $logh_1(x) - f(x)$ by a function $h_2(x)$ which is holomorphic on B', such that $\|logh_1(x) - f(x) - h_2(x)\|_B < \eta$ where $e^{\eta} - 1 < \epsilon$. Finally $h(x) := exp(logh_1(x) - f(x) - h_2(x))$ is the desired function.

Now let $\mathfrak{A}(x)$ be a holomorphic map from B to $GL(n,\mathbb{C})$. If $\mathfrak{A}(x) = (a_{ik}(x))$, referring to H. Behnke-K. Stein [1] we can find functions $a_{ik}^{(0)}(x)$, which are holomorohic on B' and satisfy $\left\|\mathfrak{A}^0(x) - \mathfrak{A}(x)\right\|_B < \frac{\epsilon}{2}$ for $\mathfrak{A}^0(x) := (a_{ik}^0(x))$. For a sufficiently small ϵ every matrix $\mathfrak{B}(x)$, holomorphic on B and satisfying $\|\mathfrak{B}(x) - \mathfrak{A}(x)\| < \epsilon$, is holomorphically invertible on B. In general the divisor $\mathfrak{D}_0$ of $Det\mathfrak{A}^0(x)$ on B' is not zero, but contains no prime divisors of B. Let g denote the total order of the divisor of $Det\mathfrak{A}^0(x)$. If $\{x'\}$ contained in $\mathfrak{D}_0$ and $a_{i_0k}^{(0)}(x') = 0$ for a prime divisor $\{x'\}$ and $k = 1, ..., n$, choose an on B'

meromorphic function $m_1(x)$, whose divisor on B' is equal to $-\{x'\}$ and which satisfies

$$\|m_1(x) - 1\|_B < \frac{\epsilon}{2g} \left\| \mathfrak{A}^{(0)}(x) \right\|_B .$$

Then for $\mathfrak{A}^{(0)}(x) := (a_{ik}^{(1)}(x))$, where $a_{i_0 k}^{(1)}(x) := m_1(x) a_{i_0 k}^{(0)}(x)$ for $k = 1, ..., n$ and $a_{ik}^{(1)}(x) := a_{ik}^{(0)}(x)$ else the fallowing is true:

$$\left\| \mathfrak{A}^{(1)}(x) - \mathfrak{A}(x) \right\|_B < \frac{\epsilon}{2} + \frac{\epsilon}{2g}.$$

The divisor $\mathfrak{D}_1$ of $Det\mathfrak{A}^{(1)}(x)$ is equal to $\mathfrak{D}_0 - \{x'\}$. Similarly we can treat all prime divisors of $\mathfrak{D}_0$ who are a common zero for at least one line. Incomplete induction yields matrix $\mathfrak{A}^{(l)}(x)$, which is holomorphic on B' and whose greatest common divisor of each line in B' is 1. Moreover

$$\left\| \mathfrak{A}^{(l)}(x) - \mathfrak{A}(x) \right\|_B < \frac{\epsilon}{2} + \frac{\epsilon l}{2g}$$

is true and the divisor $\mathfrak{D}_l$ of $Det\mathfrak{A}^{(l)}(x)$ on B' is contained in $\mathfrak{D}_0$ and has total order $g - l$. Let $x' \in \mathfrak{D}_l$. There exist complex numbers λ_i, $i = 1, ..., n$, such that $\sum_{i=1}^{n} \lambda_i a_{ik}(x') = 0$ for $k = 1, ..., n$. For $\lambda_{i_0} \neq 0$ define

$$\mathfrak{A}^{(l+1)}(x) := \begin{pmatrix} 1 & & & & & & & & 0 \\ & . & & & & & & & \\ & & . & & & & & & \\ & 0 & & 1 & & & & & \\ \lambda_1 & . & . & . & \lambda_{i_0} & . & . & . & \lambda_n \\ & & & & & 1 & & 0 & \\ & & 0 & & & & . & & \\ & & & & & & & . & \\ & & & & & & & & 1 \end{pmatrix}^{-1} \cdot$$

$$\begin{pmatrix} 1 & & & & & & & & 0 \\ & . & & & & & & & \\ & & . & & & & & & \\ 0 & & & 1 & & & & & \\ \lambda_1 m_{l+1}(x) & . & . & . & \lambda_{i_0} m_{l+1}(x) & . & . & . & \lambda_n m_{l+1}(x) \\ & & & & & 1 & & 0 & \\ & & & & & & . & & \\ & & & & & & & . & \\ 0 & & & & & & & & 1 \end{pmatrix} \mathfrak{A}^{(l)}(x),$$

where $m_{l+1}(x)$ is a meromorphic function on B' with divisor $\{x'\}$, such that

$$\|m_{l+1}(x)-1\|_B < \frac{\epsilon}{2g\left\|\mathfrak{A}^{(l)}(x)\right\|_B \cdot \|\Lambda\| \sqrt{\|\lambda_1\|^2+...+\|\lambda_n\|^2}},$$

$$\text{for } \Lambda = \begin{pmatrix} 1 & & & & & & & & 0 \\ & . & & & & & & & \\ & 0 & . & & & & & & \\ & & & 1 & & & & & \\ \lambda_1 & . & . & . & \lambda_{i_0} & . & . & . & \lambda_n \\ & & & & & 1 & & 0 & \\ & & & & & & . & & \\ & & 0 & & & & & . & \\ & & & & & & & & 1 \end{pmatrix}^{-1}.$$

$\mathfrak{A}^{(l+1)}(x)$ is holomorphic on B' and the divisor $\mathfrak{D}_{l+1}$ of $Det\mathfrak{A}^{(l+1)}(x)$ satisfies $\mathfrak{D}_{l+1} = \mathfrak{D}_l - \{x'\}$. Moreover $\left\|\mathfrak{A}^{(l+1)}(x) - \mathfrak{A}(x)\right\|_B < \frac{\epsilon}{2} + \frac{\epsilon(l+1)}{2g}$.Incomplete induction finally yields a matrix $\mathfrak{A}'(x) := \mathfrak{A}^{(g)}(x)$ satisfying the desired properties.

To prove the original statement we need the fallowing Lemma, which will also be used in the proof of theorem 4.

Lemma 1. *Let X be a non compact Riemannian surface, $P \subset X$ compact in X and $\{U_1, U_2\}$ an open covering of P. Furthermore let $\{V_1, V_2\}$ be an open covering of P such that V_i is relatively compact in U_i, $i = 1, 2$. Then there exists a positive number δ that only depends on the geometric constellation and has the fallowing property: If $h(x)$ is a holomorphic map from $U_1 \cap U_2$ to $GL(n, \mathbb{C})$ such that $\|h(x) - 1\| < \delta$, then there exist holomorphic maps $g_i(x)$ from V_i, $i = 1, 2$, to $GL(n, \mathbb{C})$ with $g_1(x) = h(x)g_2(x)$ for $x \in V_1 \cap V_2$.*

A Lemma analogous to this one was already proved by H. Cartan [4]. The idea he used in his proof also leads to our destination, because referreing to H. Behnke-K. Stein [1], H. Röhrl [22], respectively H. Tietz [26] we can always give an elementary differential of order 1 in two variables, whose divisor on $(V_1 \cup V_2) \times (V_1 \cup V_2)$ is the analytic set $\{(x, x)\}$. We renounce a detailed proof at this point, because this will later be done by the proof of Lemma 2 and the given modification. With the help of Lemma 1 now we'll show that every restriction of the main bundle associated to ξ to a subset P that is relatively compact in X admits a complex-analytical section. Therefor consider an atlas of the fiber structure of the main bundle. The chart supporter of this atlas are open sets $U_i \times GL(n, \mathbb{C})$. Consider a triangulation of P such that everx simplex is contained in at least one U_i and number the finitely many 2-simplices S_χ that occur in that process. If a complex-analytical section over $\bigcup_{\chi=1}^{k} S_\chi$ is already given, we can get a sufficiently small open neighborhood U of this union as a chart supporter in a new atlas of the fiber structure of the main bundle. If U' is a sufficiently small neighborhood of a 2-simplex S_{k+1} of P, our atlas contains a chart transformation $h(x)$ mapping $U \cap U'$ holomorphically to $GL(n, \mathbb{C})$. Since a suitable choice of U' and an appropriate numbering of the 2-simplices[1] yield U' is simply connected relative to a suitable neigborhood of $U \cup U'$, due to Runge's

[1]number the 2-symplices in a way that $\bigcup_{\chi=1}^{k}$ and S_{k+1} never have three 1-simplices in common for $k = 1, 2...$ The possibility of such a choice is pointed out in [1a].

approximation theorem we can find a holomorphic map $h_2(x)$ from $U \cup U'$ to $GL(n,\mathbb{C})$ in a way that

$$\|h_1(x)h_2(x) - 1\|_{V\cap V'} < \delta$$

for relatively compact sets v and V' in U respectively U' which cover the union of simplices we already dealt with respectively the newly added simplex. We can apply Lemma 1 to $h(x) := h_1(x)h_2(x)$ and receive a complex-analytical section in the main bundle over $\bigcup_{\chi=1}^{k+1} S_\chi$. After finitely many steps this yields a complex-analytical section over P. We apply this result to a sequence $((P_n))$ of sets contained in X which have the fallowing properties:

1. P_n is relatively compact in P_{n+1}, $n = 1, 2, ...$
2. There exists a n for every relatively compact set B in X such that $B \subset P_n$.
3. P_n is simply connected relative P_{n+1} for $n = 1, 2, ...$

The existence of these sequences is known (compare H. Behnke-K. Stein [1]). Thus let $s_n(x)$ be a complex-analytical section over P_n in the main bundle associated to ξ. Because of

$$\{\phi_{i,x}^{-1}(s_n(x))\}^{-1}\phi_{i,x}^{-1}(s_{n+1}(x)) = \{\phi_{j,x}^{-1}(s_n(x))\}^{-1}\phi_{j,x}^{-1}(s_{n+1}(x))$$

for $x \in U_i \cap U_j \cap P_n$, where $\{...\}^{-1}$ is taking the inverse in $GL(n,\mathbb{C})$, the collection of $\{\phi_{i,x}^{-1}(s_n(x))\}^{-1}\phi_{i,x}^{-1}(s_{n+1}(x))$ defines a holomorphic map $f_n(x)$ from P_n to $GL(n,\mathbb{C})$. Due to the fact that P_n relative P_{n+1} is simply connected Runge's approximation theorem yields a holomorphic map $h_n(x)$ from P_n to $GL(n,\mathbb{C})$ with $h_1(x) = 1$ such that

$$f(x) := \prod_{n=1}^{\infty}(h_n^{-1}(x)f_n(x)h_{n+1}(x))$$

converges in X in the sense of compact convergence. Thus $f(x)$ is a holomorphic map from X to $GL(n,\mathbb{C})$. Define

$$H_{n+1}(x) := h_{n+1}(x)\prod_{m=1}^{\infty}(h_{n+m}^{-1}(x)f_{n+m}(x)h_{n+m+1}(x))$$

on P_{n+1} then the definition

$$\lambda_i(x) := \phi_{i,x}^{-1}(s_n(x)) \cdot H_n(x), \text{ for } x \in U_i \cap P_n$$

is consistent and the collection of $\lambda_i(x)$ forms a complex-analytical section in the main bundle. □

4 Complex-analytical fiber spaces with a compact Riemannian surface as basis, structue group $GL(n,\mathbb{C})$ and fiber P^n

To examine complex-analytical vector space bundles $(X, \xi, \mathbb{C}^n)$ one should take a look at the cohomology moduls $H^q(X, \Omega(X, \xi, \mathbb{C}^n))$, where $\Omega(X, \xi, \mathbb{C}^n)$ denotes the sheaf of locally complex-analytical sections in $(X,\xi,\mathbb{C}^n)$. If the fiber is the complex projective space $\mathbb{P}^n$ instead of $\mathbb{C}^n$ one usually replaces $\Omega(X,\xi,\mathbb{C}^n)$ by a sheaf $\hat{\Omega}(X,\xi,\mathbb{P}^n)$ defined in the fallowing way: We fix $\mathbb{C}^n$ in $\mathbb{P}^n$ by considering $z'_0,...,z'_n$ with $z'_0 z_\nu = z'_\nu$, $\nu = 1,...,n$, as coordinates in $\mathbb{P}^n$ for given coordinates $z_1,...z_n$ of $\mathbb{C}^n$. Then $\mathfrak{A} \in GL(n,\mathbb{C})$ corresponds to the map in $\mathbb{P}^n$ characterized by $\begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{A} \end{pmatrix}$. Given a complex-analytical section $s(x)$ over $U \subset X$ in the fiber space $(X,\xi,\mathbb{P}^n)$ we can write the representations of $s(x)$ in local coordinates and thereby get z'_ν as holomorphic functions in U_i. Define $s_i(x) := (y'_0(x),...,z'_n(x))$. If $s(x)$ has the property that $z'_0(x)$ doesn't vanish identically in one coordinate representation, this property holds in every coordinate representation. Call a section like this non-degenerated. The set of non-degenerated sections over U naturally has the structure of an $K(X)$-module, where $K(X)$ denotes the field of meromorphic functions on X. Let $\hat{\Omega}(X,\xi,\mathbb{P}^n)$ be the sheaf of non-degenerated locally complex analytical sections. The goal of this paragraph is

Theorem 4. *If X is a compact Riemannian surface, $K(X)$ the field of meromorphic functions on X and $\xi \in H^1(X, GL(n,\mathbb{C})_\omega)$ the fallowing is true:*

$$K(X) - dim H^0(X, \hat{\Omega}(X,\xi,\mathbb{P}^n)) = n.$$

Proof. A) Let $s^{(1)},...,s^{(l)} \in H^0(X, Om\hat{e}ga(X,\xi,\mathbb{P}^n))$. We want to define the rank of these l sections. Therefor choose a U_i and give coordinate representations $s_i^{(\lambda)}(x) = (z'_0(x),...,z_n'^{(\lambda)}(x))$, $\lambda = 1,...,l$, of the given sections. Define the rank so $s^{(1)},...,s^{(l)}$ in U_i to be the rank of

$$\begin{pmatrix} \frac{z_1'^{(1)}}{z'^{(1)}(x)} & , & . & . & . & , & \frac{z_n'^{(1)}(x)}{z_0'^{(1)}(x)} \\ . & . & . & . & . & . & . \\ \frac{z_1'^{(l)}(x)}{z_{0(x)}'^{(l)}} & , & . & . & . & , & \frac{z_n'^{(l)}(x)}{z_0'^{(l)}(x)} \end{pmatrix}$$

It's easy to calculate that this rank is independent of the choose of U_i which makes it possible to speak about a general rank of $s^{(1)},...,s^{(l)}$. A few more calculations lead $s^{(1)},...,s^{(l)}$ are linear independent over $K(X)$ iff the rank of $s^{(1)},...,s^{(l)}$ is smaller than l. Thus $dim(H^0(X,\hat{\Omega}(X,\xi,\mathbb{P}^n))) \leq n$. To prove "=" we use a theorem about complex-analytical vector space bundles with an algebraic manifold as basis of S. Nakano [16]. Since we can think about X as an algebraic manifold in $\mathbb{P}^3$ we can apply S. Nakano's theorem 4 which yields for a sufficiently chosen $\eta \in H^1(X, GL(1,\mathbb{C})_\omega)$ there exist n sections $t^{(1)},...,t^{(n)}$ in $H^0(X,\hat{\Omega}(X,\eta\otimes\xi,\mathbb{P}^n))$ with rank n and thus linear independent over $K(X)$. Since the dimension of $H^0(X,\hat{\Omega}(X,\xi,\mathbb{P}^n))$ is positive for every $\zeta \in H^1(X, GL(1,\mathbb{C})_\omega)$ - for example this fallows from a theorem of K. Kodaira and D. C. Spencer [13] - there exists a section $t \in H^0(X,\hat{\Omega}(X,\eta^{-1},\mathbb{P}^n))$ different to the zero section. Due to $t\otimes t^{(1)},...,t\otimes t^{(n)} \in H^0(X,\hat{\Omega}(X,\xi,\mathbb{P}^n))$ this yields sections in the desired cohomology module which are linear independent over $K(X)$.

B) The given proof uses a lot of tools from algebraic geometry which can be avoided if we consider the special case $X = \mathbb{P}^1$. We want to give a comparative elementary proof for $X = \mathbb{P}^1$. Before we do that we'll proof

Lemma 2. *Let $x_1 \in \mathbb{P}^1$, $\{U_1,U_2\}$ be an open covering of $\mathbb{P}^1$ such that $U_1 \subset \mathbb{P}^1 - \{x_1\}$ and let $\{V_1,V_2\}$ be an open covering of $\mathbb{P}^1$ for V_i relatively compact* *in* U_i, $i = 1, 2$. *Then there exists a positive number* δ *that only depends on the geometric constellation and has the fallowing property:*

> *If* $f(x)$ *is a holomorphic map from* $U_1 \cap U_2$ *to* $GL(n, \mathbb{C})$ *such that* $\|f(x) - 1\| < \delta$, *then there exist holomorphic functions* $g_{i(x)}$ *from* V_i *to* $GL(n, C)$, $i = 1, 2$, *with* $g_1(x) = f(x)g_2(x)$.

Proof. Without loss of generality we may assume $t(x)$ maps U_2 conformally to the unit circle and moreover that the boundaries U_i and V_i are smooth curves. Now choose a positive number a with

$$a < Min[Disr(\{t : \|t\| < 1\}, t(RdV_1)), Dist(t(RdU_2), t(RdV_2))],$$

where $Dist(x, y)$ is the euclidean distance between x, $y \in \mathbb{C}$ and $V_{i,k}$, $i = 1, 2$, $k = 1, 2, ...$, denote the domains on X, who are relatively compact in U_i and for which $t(RdV_{i,k})$ is a parallel curve to $t(RdU_i)$ with distance $a - 2^{-k}a$. Moreover choose $x_2 \in X - U_2$. For an elementary differential $dF_{x_1}(x, z)$ in two variables and of first order like in [22] or H. Tietz [26], whose characteriszing divisors only contain the prime divisor $\{x_1\}$, there exists a number K such that

$$\int_{RdV_{1,k}} \left\|dF_{x_1(z,x)}\right\| \leqq \frac{2\pi K}{a} \cdot 2^{k-1}, \text{ for } x \in V_{1,k+1}$$
$$\int_{RdV_{2,k}} \left\|dF_{x_1(z,x)}\right\| \leqq \frac{2\pi K}{a} \cdot 2^{k-1} \text{ for } x \in V_{2,k+1}$$

and for all $k = 1, 2, ...$. It's claimed that $\delta := Min(\frac{1}{2}, \frac{1}{2^5}(1 + \frac{K}{a})^{-2} e^{\frac{-K}{a}})$ suffices the requirements of Lemma 2. To prove that we define $f_0(x) := f(x)$ for $x \in V_{1,0} \cap V_{2,0}$. If $f_k(x)$ is already defined as a holomorphic map from $V_{1,k} \cap V_{2,k}$ to $GL(n, \mathbb{C})$ and $\|f_k(x) - 1\|_{V_{1,k} \cap V_{2,k}} < 1$, we define

$$h_k(x) := -\sum_{m=1}^{\infty} \frac{(-1)^m}{m} (f_k(x) - 1)^m$$
$$h_{1,k}(x) := \frac{1}{2\pi i} \int_{RdV_{1,k}} h_k(z) dF_{x_1}(z, x)$$
$$h_{2,k}(x) := \frac{1}{2\pi i} \int_{-RdV_{2,k}} h_k(z) dF_{x_1}(z, x)$$

in $V_{1,k \cap V2,k}$. Then $h_{1,k}(x) + h_k(x) + h_{2,k}(x) = 0$ and

$$\|h_k(x)\|_{V_{1,k}} \cap V_{2,k} \leqq \frac{\delta}{4^k}$$
$$\|h_{1,k}(x) - 1\|_{V_{1,k+1}} \cap V_{2,k+1} \leqq \frac{K}{a} \cdot \frac{\delta}{2^k}, \|h_{2,k} - 1\|_{V_{1,k}} \cap V_{2,k+1} \leqq \frac{K}{a} \cdot \frac{\delta}{2^k}.$$

Finally define

$$f_{k+1}(x) := exp(h_{1,k}(x)) \cdot f_k(x) \cdot exp(h_{2,k}(x)) \text{ for } x \in V_{1,k+1}V_{2,k+1}$$

which yields

$$\|f_{k+1}(x) - 1\|_{V_{1,k+1} \cap V_{2,k+1}} \leqq \frac{\delta}{4^{k+1}}.$$

Since

$$f(x) = exp(-h_{1,0}(x))...exp(-h_{1,k}(x))f_{k+1}(x) \cdot exp(-h_{2,k}(x))...exp(-h_{2,0}(x))$$

for $x \in V_1 \cap V_2$ and the fact that the sequence of $f_k(x)$ converges uniformly to 1 and moreover the products $\prod_{k=0}^{\infty} exp(-h_{1,k}(x))$ converge uniformly in V_i due to the estimates above, the proof is done. As mentioned earlier this proof fallows a proof of H. Cartan [4]. □

Back to the proof of theorem 4 in the case $X = \mathbb{P}^1$. We fix a system $U_1 \subset X - \{x_1\}, U_2$ of open subsets as covering of X. Let $U_1 \cap U_2$ be of the type of the annulus. By theorem 3 the restriction of the cozycle ξ to U_1 and similar to U_2 is trivial. Therefore we can define $(X, \xi, \mathbb{P}^n)$ by a holomorphic map $g(x)$ from $U_1 \cap U_2$ to $GL(n, \mathbb{C})$. Thus the proof will be done by giving an open covering $\{V_1, V_2\}$ of X with $V_i \subset U_i$, $i = 1, 2$, and non-singular matrices $m_i(x)$, meromorphic in V_i and satisfying $m_1(x) = g(x)m_2(x)$ for $x \in V_1 \cap V_2$. If $g(x)$ satisfies the assumptions of Lemma 2 one has already finished. If not, fix two open, relatively compact subsets W_i in U_i, $i = 1, 2$, such that $W_1 \cup W_2 = X$ and V_i lies relatively compact in V_i. Moreover one can find a matrix $m(x)$ such that $g^{-1}(x) - m(x)$ is meromorphic and meromorphically invertible in $W_1 \cap W_2$ and $\|m(x)\|_{W_1 \cap W_2} < \frac{\epsilon}{\|g(x)\|_{W_1 \cap W_2}}$. Choose ϵ sufficiently small then $f(x) := g(x)(g(x)^{-1} - m(x))$ is a holomorphic map from $W_1 \cap W_2$ satisfying the requirements of Lemma 2. Therefore $m_1 := g_1$, $m_2 := (g(x)^{-1} - m(x))g_2(x)$ are the desired matrices. □

Corollary (Heftungslemma). Let X be a compact Riemannian surface, $\{U_1, U_2\}$ an open covering of X and $h(x)$ a holomorphic map from $U_1 \cap U_2$ to $GL(n, \mathbb{C})$. There exist maromorphic and nonsingular matrices $m_i(x)$, $i = 1, 2$, on U_i such that

$$m_1(x) = m_2(x)h(x) \text{ for } x \in U_1 \cap U_2.$$

One sees easily that this corollary is just another formulation of theorem 4.

Theorem 1-4 together with the remarks at the beginning of the proof of theorem 4 yield

Theorem I. *Let X be a Riemannian surface, $X' \subset X$ a subset without accumulation points in X and μ be a homomorphism from $\pi_1(X - X', x_0)$ to $GL(n, \mathbb{C})$. Then there exists a matrix $\mathfrak{B}(\tilde{x})$, which is meromorphic and nonsingular on $\widetilde{X - X'}$ and only has points of certainty on X'. Moreover for for every $\alpha \in \pi_1(X - X', x_0)$*

$$\alpha \cdot \mathfrak{B}(\tilde{x}_0) = \mu(\alpha)\mathfrak{B}(\tilde{x}_0).$$

5 Dependency of solutions and branching points

In this section we want to examine how the matrices $\mathfrak{B}(\tilde{x})$ from theorem I depend on branching points. More precisely we want to do the fallowing. Consider a system $\{x'_1, ..., x'_k\} \subset X'$ of finitely many branching points. To each x'_χ, $\chi = 1, ..., k$ we assign an open neighborhood U'_χ of x'_χ in X. As far as it makes sense U'_χ is going to be the "space of variability" of x'_χ; the monodromy μ is supposed to be "independent" of the choice of $x_\chi \in U'_\chi$. If two branching points coincide we have to expect certain "degenerations". Therefor we want to restrict to the case where branching points are only allowed to vary in a way that none of them coincide. Therefor we require $U'_\chi \cap (X' - \{x'_1, ..., x'_k\})$ to be empty. Moreover we only consider k-tuples in $U'_1 \times ... \times U'_k - \Delta$, where Δ denotes the set of those k-tuples in $U'_1 \times ... \times U'_k$ that coincide in at least two components. For $(x^*_1, ..., x^*_k) \in U'_1 \times ... \times U'_k - \Delta$ define

$$X'_{x^*_1,...,x^*_k} := (X' - \{x'_1, ..., x'_k\}) \cup \{x^*_1, ..., x^*_k\}$$

and choose a point x_0 in $X - (X' \cup U'_1 \cup ... \cup U'_k)$, then there exists a natural isomorphism $\iota_{x^*_1,...,x^*_k}$ from $\pi_1(X - X'_{x^*_1,...,x^*_k}, x_0)$ to $\pi_1(X - X', x_0)$. Furthermore we'll abbreviate $X \times (U'_1 \times ... \times U'_k - \Delta)$ by $X_\mathfrak{U}$ and $(X' - \{x'_1, ..., x'_k\}) \times (U'_1 \times ... \times U'_k - \Delta)$ by $X'_\mathfrak{U}$, where

$$\Delta' := \bigcup_{\chi=1}^{k} \{\bigcup \{(x^*_\chi, x^*_1, ..., x^*_k) : (x^*_1, ..., x^*_k) \in U'_1 \times ... \times U'_k - \Delta\}\}.$$

Now we assign to each

$$(x^*_1, ..., x^*_k) \in U'_1 \times ... \times U'_k - \Delta$$

the problem $(X, X'_{x^*_1,...,x^*_k}, \iota_{x^*_1,...,x^*_k} \circ \mu)$ and ask for nonsingular matrices $\mathfrak{B}$, which are meromorphic on the universal covering space $\widetilde{X_\mathfrak{U} - X'_\mathfrak{U}}$ of $X_\mathfrak{U} - X'_\mathfrak{U}$ and have the fallowing property:

If χ is the natural map from $X_\mathfrak{U} - X'_\mathfrak{U}$ to $U'_1 \times ... \times U'_k - \Delta$, ψ the natural projection from $\widetilde{X_\mathfrak{U} - X'_\mathfrak{U}}$ to $X_\mathfrak{U} - X'_\mathfrak{U}$ and $\phi_{x^*_1,...,x^*_k}$ the natural projection from $\widetilde{X - X'}_{x^*_1,...,x^*_k}$ to a suitable connected component $Z_{x^*_1,...,x^*_k}$ of $(\chi \circ \psi)^{-1}(x^*_1, ..., x^*_k)$ for $(x^*_1, ..., x^*_k) \in U'_1 \times ... \times U'_k - \Delta$, then the fallowing is true for the restriction $\mathfrak{B}|_{Z_{x^*_1,...,x^*_k}}$: $\phi^*_{x^*_1,...,x^*_k}(\mathfrak{B}|_{Z_{x^*_1,...,x^*_k}})$ is a solution of $(X, X'_{x^*_1,...,x^*_k}, \iota_{x^*_1,...,x^*_k} \circ \mu)$ for every $(x^*_1, ..., x^*_k) \in U'_1 \times ... \times U'_k - \Delta$.

We'll denote this problem by $(X, X', \mathfrak{U}, \mu)$. As earlier a solution $\mathfrak{B}$ of $(X, X', \mathfrak{U}, \mu)$ is called holomorphic if the matrix $\mathfrak{B}$ is holomorphically invertible.

Denote the natural homomorphism from $\pi_1(X - X', x_0)$ to $\pi_1(X_\mathfrak{U} - X'_\mathfrak{U}, (x_0, x'_1, ..., x'_k))$ by j. Obviously

$$j^{-1}(0) \subset \mu^{-1}(0) \tag{4}$$

is necessary for solubility of $(X, X', \mathfrak{U}, \mu)$. The easiest case is $j^{-1} = 0$, because this means for all $(x^*_1, ..., x^*_k) \in U'_1 \times ... U'_k - \Delta$ one - and therefore every - connected component of $(\xi \circ \psi)^{-1}(x^*_1, ..., x^*_k)$ naturally is a universal covering space of $X - X'_{x^*_1,...,x^*_k}$. We have $j^{-1}(0) = 0$ if U'_κ, $\kappa = 1, ..., k$, are simply connected and pairwise disjoint: then $X_\mathfrak{U} - X'_\mathfrak{U}$ is homeomorphic to $(X - X') \times U'_1 \times ... \times U'_k$ and j even is an isomorphism onto $\pi_1(X_\mathfrak{U} - X'_\mathfrak{U}, (x_0, x'_1, ..., x'_k))$.

Now we only want to consider $(X, X', \mathfrak{U}, \mu)$ for the named case. Analog to Riemann-Hilbert's problem we'll divide $(X, X', \mathfrak{U}, \mu)$ into two questions. First we search for a non singular matrix $\mathfrak{B}$ which is meromorphic on $\widetilde{X_{\mathfrak{U}} - X'_{\mathfrak{U}}}$ and has the property:

$$\begin{aligned} &\mathfrak{B}|_{Z_{x_1^*,\dots,x_k^*}} \text{ is a solution of } (X - X'_{x_1^*,\dots,x_k^*}, \emptyset, \iota_{x_1^*,\dots,x_k^*} \circ \mu) \\ &\text{for every } (x_1^*, \dots, x_k^*) \in U'_1 \times \dots \times U'_k. \end{aligned} \tag{5}$$

In another step we'll turn potential points of uncertainty into points of certainty.

As one sees easily the construcion of $\xi_\mu \in H^1(X, GL(n, \mathbb{C})_\omega)$ is independent of the fact that the basis X is a Riemannian surface. Similarly we can do this construction for complex manifolds. Thus the cozycle $\xi_{\mu \circ j^{-1}} \in H^1(X_{\mathfrak{U}} - X'_{\mathfrak{U}}, GL(n, \mathbb{C})_\omega)$ is defined. Since $X_{\mathfrak{U}} - X'_{\mathfrak{U}}$ is homeomorphic to $(X - X') \times U'_1 \times \dots \times U'_k$ we can choose $\xi_{\mu \circ j^{-1}}$ to an open covering of $X_{\mathfrak{U}} - X'_{\mathfrak{U}}$, whose elements are simply connected and whose pairwise intersections are also simply connected and connected. Therefore we can copy the proof of theorem 1. This yields

Theorem 5. *Triviality of $\xi_{\mu \circ j^{-1}}$ is necessary and sufficient for the existence of a matrix $\mathfrak{B}$, which is holomorphic and holomorphically invertible on $\widetilde{X_{\mathfrak{U}} - X'_{\mathfrak{U}}}$ and satisfies (5).*

If we've found a matrix $\mathfrak{B}$ that is holomorphic and holomorphically invertible on $X_{\mathfrak{U}} - X'_{\mathfrak{U}}$ and satisfies (5) we try to construct the cozycle $\xi_{\mathfrak{B}} \in H^1(X_{\mathfrak{U}}, GL(n, \mathbb{C})_\omega)$ in the same way we did before. Therefor we have to define the term "point of certainty" equally to the definition in section 2, but instead of the local uniformizer $t(x)$ we have to choose a function which is holomorphic in a whole neighborhood U of the considered point and whose set of zeros in U coincides with $X'_{\mathfrak{U}}$. To construct $\xi_{\mathfrak{B}}$ we have to start with a suitable open neighborhood of $X_{\mathfrak{U}}$: Therfor we choose an open neighborhood of $X_{\mathfrak{U}} - X'_{\mathfrak{U}}$ and add the sets $U'_\kappa \times U'_1 \times \dots \times U'_k$, $\kappa = 1, 2, \dots$, where U'_κ, $\kappa = 1, 2, \dots$, are open, connected and simply connected coordinate neighborhoods with $X' \subset \bigcup_k U'_k$. Now for this open covering of $X_{\mathfrak{U}}$ we can construct g_{ij} like in section 2, but using a function, which is holomorphic in $U'_\kappa \times U'_1 \times \dots \times U'_k$ and whose set of zeros in $U'_\kappa \times U'_1 \times \dots \times U'_k$ coincides with $X'_{\mathfrak{U}} \cap (U'_\kappa \times U'_1 \times \dots \times U'_k)$, instead of the local uniformizer $t_\kappa(x)$. One shows easily that g_{ij} are holomorphic maps in $GL(n, \mathbb{C})$ and therefore define a cozycle $\xi_{\mathfrak{B}} \in H^1(X_{\mathfrak{U}}, GL(n, \mathbb{C})_\omega)$. As prior one proves

Theorem 6. *For $\mathfrak{B}$ defined as in theorem 5 we have:*

1. *If X is non compact, triviality of the cozycle $\xi_{\mathfrak{B}}$ is necessary and sufficient to prove the existence of a holomorphic solution of $(X, X', \mathfrak{U}, \mu)$.*

2. *If X is compact, to prove the existence of a solution of $(X, X', \mathfrak{U}, \mu)$ it's necessary and sufficient to show that the bundle $(X_{\mathfrak{U}}, \xi_{\mathfrak{B}}, \mathbb{P}^{n^2})$ associated to $\xi_{\mathfrak{B}}$ admits a complex-analytical section $s(u)$, such that the support of the divisors of $s(u)$ contains none of the analytical sets $X \times x_1^* \times \dots \times x_k^*$ for $(x_1^*, \dots, x_k^*) \in U'_1 \times \dots \times U'_k$.*

Thereby the support of the divisors of $s(u)$ denotes the sets of all $\bar{u} \in X_{\mathfrak{U}}$ with $\phi_{i,u}^{-1}(s(u)) \notin GL(n, \mathbb{C})$.

Moreover one confirms easily that also here the corollary to theorem 2 remains true if only X is non compact.

Regarding theorem 5 and 6 the fallowing is of certain interest

Theorem 7. *Let X be a non compact Riemannian surface and G a complex Lie group, then $H^1(X_{\mathfrak{U}}, G_\omega)$ contains only the trivial element. If Y is any Riemannian surface and Y' a subset of Y without accumulation points on Y, then $H^1(Y_{\mathfrak{U}} - Y'_{\mathfrak{U}}, G_\omega)$ only consists of the trivial element.*

Proof. In the given cases $X_{\mathfrak{U}}$ respectively $Y_{\mathfrak{U}} - Y'_{\mathfrak{U}}$ is homeomorphic to a product of a non compact Riemannian surface and $U'_1 \times ... \times U'_k$. Since the integral homology groups of the factors are torsionfree, this is also true for $X_{\mathfrak{U}}$ and $Y'_{\mathfrak{U}} - Y'_{\mathfrak{U}}$ due to a well known theorem. Therefore the integral homology groups can be completely characterized by their Betti numbers. The Künnetsche Formel yields

$$H_q(X_{\mathfrak{U}}) = H_q(Y_{\mathfrak{U}} - Y'_{\mathfrak{U}}) = 0 \text{ for } q > 1.$$

Thus we can transfer the proof of theorem 3. Note that also the second proof of theorem 3 yields our goal if we assume G to be the Lie group $GL(n, \mathbb{C})$. □

By theorem 7 it's already proved that the problem $(X, X', \mathfrak{U}, \mu)$ can always be solved holomorphically for a non compact Riemannian surface X. This means in microcosm we can assume the dependency of solutions of Riemann-Hilbert's problem and branching points to be holomorphic.

Finally we also need:

Theorem 8. *Let X be a compact Riemannian surface and $U'_1, ..., U'_k$ connected, simply connected and pairwise disjoint subdomains of X. If $\xi \in H^1(X \times U'_1 \times ... \times U'_k, GL(n, \mathbb{C})_\omega)$, then the fiber bundle $(X \times U'_1 \times ... \times U'_k, \xi, \mathbb{P}^n)$ associated to ξ has a complex analytical section s such that the support of the divisors of s contains none of the analytical sets $X \times \{x_1\} \times ... \times \{x_k\}$ for $(x_1, ..., x_k) \in U'_1 \times ... \times U'_k$.*

Proof. If X has genus 0, we can apply the idea we used in the proof B) of theorem 4. We leave the details to the reader. In the general case we can do the fallowing (compare S. Nakano [16]). Abbreviate $U'_1 \times ... \times U'_k$ by U, then ξ defines a complex analytical vector space bundle $(X \times U, \xi, \mathbb{C}^n)$. Let $x_* \in X$. $B := \{x_*\} \times U$ is an irreducible, purely k-dimensional analytic set in $X \times U$. The divisor belonging to B naturally defines an element $\beta \in H^1(X \times U, GL(1, \mathbb{C})_\omega)$. We get an exact sequence of sheafs

$$\begin{aligned} 0 \to \Omega(X \times U, \beta^{-1} \otimes \xi, \mathbb{C}^n) &\xrightarrow{\iota} \Omega(X \times U, \xi, \mathbb{C}) \\ &\xrightarrow{\rho} \Omega(\{x_*\} \times U, \xi|_{\{x_*\}} \times U, \mathbb{C}^n) \to 0, \end{aligned}$$

for the injection map ι, the restriction map ρ and the restriction $\xi|_{\{x_*\}} \times U$ of ξ to $\{x_*\} \times U$. Thus also

$$\begin{aligned} H^0(\Omega(X \times U, \xi, \mathbb{C}^n)) &\xrightarrow{\iota} H^0(\Omega(\{x_*\} \times U, \xi|_{\{x_*\}} \times U, \mathbb{C}^n)) \\ &\to H^1(\Omega(X \times U, \beta^{-1} \otimes \xi, \mathbb{C}^n)) \end{aligned}$$

is exact. Chosen ξ such that $H^1(\Omega(X \times U, \beta^{-1} \otimes \xi, \mathbb{C}^n)) = 0$ the map ι is an epimorphism. By theorem 7 respectively H. Grauert [5] the fiber bundle $(\{x_*\} \times U, \xi|_{\{x_*\}} \times U, \mathbb{C}^n)$ is complex analytically trivial. Let $J(U)$ be the integral domain of holomorphic functions in U then $H^0(\Omega(\{x_*\}\times U, \xi|_{\{x_*\}}\times U, \mathbb{C}^n))$ naturally has the structure of an $J(U)$-module. Moreover the elements $(\delta_{j1},...,\delta_{jn})$, $j=1,...,n$, form a $J(U)$-basis of $H^0(\Omega(\{x_*\}\times U, \xi|_{\{x_*\}}\times U, \mathbb{C}^n))$. Therefore there exist n elements in $H^0(\Omega(X\times U, \xi, \mathbb{C}^n))$ whose ranks are equal to n for $(u_1,...,u_k)\in U$.

A few calculations yield

$$H^1(\Omega(X\times U, \beta^{-1}\otimes\phi^*(\eta)\times\xi, \mathbb{C}^n)) = 0,$$

where ϕ denotes the natural projection from $X\times U$ to U and η is the cocycle we already used in section 4. The rest of the proof is similar to the proof in section 4. □

With that we finally get

Theorem II. *Let $U_1',...,U_k'\subset X$ be pairwise disjoint, connected and simply connected open neighborhoods of $x_1',...,x_k'$ for $\{x_1',...,x_k'\}\subset X'$, then $(X, X', \mathfrak{U}, \mu)$ is always solvable. If X is non compact there exist holomorphic solutions of $(X, X', \mathfrak{U}, \mu)$.*

This proves that the solutions of Riemann-Hilbert's problem depend analytically on the branching points.

6 Dependency of solutions and the monodromy μ

The methods from the last chapter also allow to examine dependency of solutions of Riemann-Hilbert's problem and monodromy. While talking about dependency of monodromy one considers the fallowing. Fix X and X'. μ is uniquely determined by its values on a canonical generating set of $\pi_1(X-X', x_0)$. Now change the values of μ on finitely many elements of the chosen canonical generating set and ask how the given solution of Riemann-Hilbert's problem changes. We now generalize and specify the question. Let $\alpha_1,...\alpha_k,...$ be the elements of the canonical generating set of $\pi_1(X-X', x_0)$ and $U_1,...,U_k$ open subsets of $GL(n,\mathbb{C})$, where $GL(n,\mathbb{C})$ has the natural complex structure. We want $\mu(\alpha_\kappa)$ to vary in U_κ, $\kappa=1,...,k$. Therefor we assume that $\alpha_1,...,\alpha_k$ don't form a complete canonical generating set of $\pi_1(X-X', x_0)$. If the generating set contains infinitely many elements, we assume $\mu(\alpha_{k+1}),...$ to be given; otherwise we choose k in a way that $\alpha_1,...,\alpha_{k+1}$ is the generating set. In the first case it's obvious that there exists a homomorphism from $\pi_1(X-X', x_0)$ to $GL(n,\mathbb{C})$, which takes on the value $\mu(\alpha_\kappa)$ on α_κ, for every choice of $\mu(\alpha_\kappa)\in U_\kappa$, $\kappa=1,...,k$. In the second case such a homomorphism exists at least if $\alpha_{k+1}X-X'$ splits and $\mu(\alpha_{k+1})$ is chosen suitably, what we want to assume additionally. Denote this homomorphism by $\mu_{\alpha_1,...,\alpha_k}$.

We now define $Y := (X-X')\times U_1\times...\times U_k$ and ask for a matrix $\mathfrak{B}$, which is meromorphic and non singular on the universal covering space $\tilde{Y}$ of Y and has the fallowing property: Let χ be the natural map from Y to $U_1\times...\times U_k$, ψ the natural projection of $\tilde{Y}$ to Y and $\phi_{\alpha_1,...,\alpha_k}$ the natural projection from $\widetilde{X-X'}$ to a suitable connected component $Z_{\mu(\alpha_1),...,\mu(\alpha_k)}$ of $(\chi\circ\psi)^{-1}(\mu(\alpha_1),...,\mu(\alpha_k))$

for $\mu(\alpha_k) \in U_\kappa$, $\kappa = 1, ..., k$. Then

$$\phi^*_{\alpha_1,...,\alpha_k}(\mathfrak{B}|_{Z_{\mu(\alpha_1),...,\mu(\alpha_k)}})$$

is a solution of $(X, X', \mu_{\alpha_1,...,\alpha_k})$ for every $(\mu(\alpha_1), ..., \mu(\alpha_k)) \in U_1 \times ... \times U_k$.

We'll denote this problems of existence by $(X, X', \mu, \mathfrak{U})$. As prior we call a solution holomorphic if $\mathfrak{B}$ is holomorphic and holomorphically invertible.

Theorem IIIa. *If the canonical generating set of $\pi_1(X - X')$ contains infinitely many elements or $\alpha_{k+1}X - X'$ splits,$(X, X', \mu, \mathfrak{U})$ is always solvable if only $U_1, ..., U_k$ are simply connected homology domains contained in $GL(n, \mathbb{C})$ with homology type of the cell. If X is non compact, there exist holomorphic solutions of $(X, X', \mu, \mathfrak{U})$.*

The details needed to prove theorem IIIa are similar to those needed for the proof of theorem II and are left to the reader. Note that for this proof the cocycles ξ_μ respectively $\xi_{\mathfrak{B}}$ are defined in the same way as in 2. If $\{V_i\}_{i\in I}$ is one of the coverings used to construct ξ_μ respectively $\xi_{\mathfrak{B}}$, choose $\{V_i \times U_1 \times ... \times U_k\}_{i\in I}$ to be the covering of $(X - X') \times U_1 \times ... \times U_k$ respectively $X \times U_1 \times ... \times U_k$ needed for the definition. We can choose the matrices $log\mu(\langle K_j D_j K_j^{-1}\rangle)$ appearing in the definition of $\xi_{\mathfrak{B}}$ to be holomorphic in $U_1 \times ... \times U_k$, because each U_κ is simply connected.

The two fallowing cases remain:

1. $\alpha_1, ..., \alpha_k$ form a canonical generating set of $\pi_1(X - X', x_0)$
2. $\alpha_1, ..., \alpha_{k+1}$ form a canonical generating set of $\pi_1(X - X', x_0)$ and α_{k+1} splits, but $X - X'$ does not.

It's clear that in both cases not every choose of $\mu(\alpha_1), ..., \mu(\alpha_k) \in GL(n, \mathbb{C})$ yields a homomorphism $\mu_{\alpha_1,...\alpha_k}$ from $\pi_1(X - X', x_0)$ to $GL(n, \mathbb{C})$ that takes on the given values for α_κ, $\kappa = 1, ..., k$. Riemann's relation is necessary and sufficient for the existence of those homomorphisms:

Let $\alpha_1, ..., \alpha_\eta$ be exactly those elements of the canonical generating set that split in $X - X'$. Then for the first case

$$\prod_{\kappa=1}^{h} \mu(\alpha_\kappa) \prod_{\gamma=1}^{\frac{k-h}{2}} (\mu(\alpha_{h+2\gamma-1})\mu(\alpha_{h+2\gamma})\mu^{-1}(\alpha_{h+2\gamma-1})\mu^{-1}(\alpha_{h+2\gamma})) = 1 \qquad (6)$$

is the Riemann relation if $\alpha_{h+2\gamma-1}$ and $\alpha_{h+2\gamma}$ are so called "Rückkehrschnittpaare" for each $\gamma = 1, ..., \frac{k-h}{2}$. The second case works analogously.

Since the second case can be subsumed to the first one we'll only examine the first one. The question that for certain $U_1, ..., U_k$ was already answered by theorem IIIa will be formulated in the fallowing way:

Let U be a complex space (it'll replace $U_1 \times ... \times U_k$ in theorem IIIa) and $\mu(\alpha_1), ..., \mu(\alpha_k)$ holomorphic maps from U to $GL(n, \mathbb{C})$, which satisfy Riemann's relation (6) and therefore yield a (uniquely determined) homomorphism μ_u from $\pi_1(X - X', x_0)$ to $GL(n, \mathbb{C})$. If there a exists a meromorphic and non singular matrix $\mathfrak{B}$ on the universal covering space $\tilde{Y}$ of $Y := (X - X') \times U$, which - with denotation as above - has the fallowing property:

$$\phi^*_u(\mathfrak{B}|_{\mathfrak{Z}_u}) \text{ is a solution of } (X, X', \mu_u) \text{ for every } u \in U\ . \qquad (7)$$

As above, abbreviating the question of existence by (X, X', μ, U) we get

Theorem IIIb. *Let $\alpha_1, ..., \alpha_k$ be a canonical generating set of $\pi_1(X - X', x_0)$. Moreover let U be a holomorphically complete space of homology type of the cell and $\mu(\alpha_1), ..., \mu(\alpha_k)$ be holomorphic maps from U to $GL(n, \mathbb{C})$, which satisfy Riemann's relation for every $\ulcorner \in U$. Assume suitable branches of $log\mu(\alpha_1), ..., log\mu(\alpha_h)$ to be unique on U. Then there always exist solutions of (X, X', μ, U). If X is non compact, there even exist holomorphic solutions of (X, X', μ, U).*

7 An application to compact Riemannian surfaces

Let X be an unlimited covering spce of $\mathbb{P}^1$ with n petals. Denote the natural projection of X to $\mathbb{P}^1$ by λ and the set of projections of branchingpoints of X in $\mathbb{P}^1$ by $V = \{v_1^{(0)}, ..., v_k^{(0)}\}$. If $p_0 \in \mathbb{P}^1 - V$ and $\{x_1^{(0)}, ..., x_n^{(0)}\} = \lambda^{-1}(p_0)$, every $\alpha \in \pi_1(\mathbb{P}^1 - V, p_0)$ yields a permutation $\pi(\alpha)$ of $\lambda^{-1}(p_0)$ and therefore a permutation matrix $\mu(\alpha)$: α can be pushed up in each point of $\lambda^{-1}(p_0)$; the path pushed up to $x_\nu^{(0)}$ ends in $x_{\pi(\nu)}^{(0)}$ and we have to set $\mu(\alpha) = (\alpha_{\rho\sigma})_{1 \leqq \rho, \sigma \leqq n}$, for $\alpha_{\rho\sigma} := \delta_{\pi(\rho),\sigma}.\alpha \to \mu(\alpha)$ is a representation of $\pi_1(\mathbb{P}^1 - V, p_0)$ of degree n; by a theorem in topology it characterizes X up to trace point faithful automorphisms (i.e. up to renumbering of petals). The elements of a column of a soultion $\mathfrak{B}$ of $(\mathbb{P}^1, V, \mu)$ can be thought of as the branches of a meromorphic function on X: The components of such a column interchange in a way determined by the branching of X over $\mathbb{P}^1$ while circulating a branching point. Moreover they act certainly in this points, which means they at most have one pole as singularity in the local uniformizer. Taking all the functions $y_1, ..., y_n$ defined by the columns of $\mathfrak{B}$ we get a basis of $K(X)$ over $K(\mathbb{P}^1)$.

Now let U_κ, $\kappa = 1, ..., k$, be the connected, simply connected and pairwise disjoint open neigborhoods of x_κ. If $(v_1, ..., v_\kappa) \in U_1 \times ... \times U_k$, we get a natural isomorphism $\phi_{v_1,...,v_k}$ from $\pi_1(\mathbb{P}^1 - \{v_1, ..., v_k\}, p_0)$ to $\pi_1(\mathbb{P}^1 - \{v_1^{(0)}, ..., v_k^{(0)}\}, p_0)$. By a classical result to the representation $\mu \circ \phi_{v_1,...,v_k}$ belongs a unlimited covering space $X_{v_1^{(0)},...,v_k^{(0)}}$ of $\mathbb{P}^1$ with n petals, which yields to this representation by the way described above. We get $X_{v_1,...,v_k}$ out of $X_{v_1^{(0)},...,v_k^{(0)}}$ by "relocating" the branching points. Theorem II yields y_ν, $\nu = 1, ..., n$ depend analytically on v_κ, $\kappa = 1, ..., k$, and form a basis of $K(X_{v_1,...,v_k})$ over $K(^1)$ for every fixed $(v_1, ..., v_k) \in U_1 \times ... \times U_k$. The set

$$T := \{X_{v_1,...,v_k} : (v_1, ..., v_k) \in U_1 \times ... \times U_k\}$$

naturallay has a complex structure and therefore can be considered as an "analytical family of Riemannian surfaces" with "paramater manifolf" $U_1 \times ... \times U_k$ in the sense of O. Teichmüller [25]. It shows that the functions $y_1, ..., y_n$ are meromorphic on T.

Theorem 9. *There exist n meromorphic functions on T that form a basis of $K(X_{v_1,...,v_k})$ over $K(\mathbb{P}^1)$ for each fixed $(v_1, ..., v_k) \in U_1 \times ... \times U_k$.*