In this note, we shall prove that weak commutativity and Jacobi identity are equivalent. The reference is section 3.4-3.5 of this book. We also recommend reading the first pages of this reference as it provides intuition behind the identities.

1. A REVIEW ON FORMAL CALCULUS

First, let us review a couple of notation and conventions in formal algebra calculus; Recall

$$\delta(z) = \sum_{n=-\infty}^{\infty} z^n \tag{1}$$

is the formal delta function, which is the unique Laurent series satisfying

$$f(z)\delta(z) = f(1)\delta(z) \tag{2}$$

for all Laurent **polynomials** $f(z) \in \mathbb{C}[z^{\pm 1}]$. Taking residue, as all coefficients of $\delta(z)$ are one, gives:

$$\mathrm{Res}_z f(z)\delta(z) = \mathrm{Res}_z f(1)\delta(z) = f(1)\mathrm{Res}_z \delta(z) = f(1) \tag{3}$$

which should be looked at as the formal version of the Cauchy integral formula

$$f(1) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-1)} \tag{4}$$

More generally

$$f(z_1)z_2^{-1}\delta(z_1/z_2) = f(z_2)z_2^{-1}\delta(z_1/z_2) \tag{5}$$

In formal calculus any expression $(z+w)^k$ for any integer $k \in \mathbb{Z}$ power can be expanded using the formal binomial expansion. But this expansion has its own conventions, where we always assume that in the binomial expansion, the negative powers only occur for the *first* summand, i.e.

$$(z+w)^k = \sum_{i=0}^{\infty} \binom{k}{i} z^{-k-i} w^i \tag{6}$$

Notice $\binom{k}{0} = 1$ always, and

$$\binom{k}{i} = \frac{k(k-1)\cdots(k-i+1)}{i!} \tag{7}$$

which is well defined even for negative k. Of course, for positive k, we recover the usual finite binomial expansion and, e.g. for $k=-1$, it is the usual expansion of $\frac{1}{z(1-\frac{w}{z})}$ when $|z| > |w|$. In fact, this analytical point of view exactly coincides with the convention. To have negative powers only for the first summand, is to assume that the norm of first summand is greater than that of the second: $|z| > |w|$. With this convention, we can look at the delta function in another way

$$\delta(z) = (1-z)^{-1} + (z-1)^{-1} \tag{8}$$

and this expression, is a reason behind why the delta function is also called *an expansion of zeros*. What is happening is we are taking a rational function $(1-z)$ and expanding this function around its zero (at $z=1$). But around this zero, we find that there are two different possible expansions.

One is where $|z| > 1$ and the other is when $|z| < 1$. *To expand the function around its zero is* to take

$$\iota_+(1-z)^{-1} - \iota_-(1-z)^{-1} \tag{9}$$

where ι_+ is for expansion when $|z| < 1$, which leads to **positive** powers for z in the expansion, and ι_- is for the expansion $|z| > 1$, which leads to a series in z^{-1}, hence **negative** powers. Therefore, the delta function can be viewed as the expansion of $1 - z$ around its zero at $z = 1$.

The notation ι will become useful later on. We can generalize this notation by defining

$$\iota_{1,\ldots,n} f(z_1, \ldots, z_n)$$

to be the formal Laurent series expansion of rational function f when $|z_1| > |z_2| > \ldots > |z_n|$. Notice we are still introducing some analysis to define this notation. A pure formal definition and much more can be found in section 2 of this book.

Formal derivation also works in formal calculus and can be done on formal Laurent series such as the n-th derivative of

$$\delta^{(n)}(z) = n!\Big((1-z)^{-n-1} + (z-1)^{-n-1}\Big) \tag{10}$$

2. Weak Associativity

Recall the weak commutativity axiom, where for any $a, b \in \mathcal{V}$

$$(z_1 - z_2)^k [Y(a, z_1), Y(b, z_2)] = 0 \tag{11}$$

for some nonnegative integer k dependent on a, b. In fact, k can be seen to be exactly the integer for which $a(n)b = 0$ for all $n \geq k$. But we will only use the fact that k is dependent on a, b. We can think of this axiom as the analog of commutativity in algebra where $ab = ba$.

Another axiom which we shall use is the L_{-1}*-bracket relation*

$$[L_{-1}, Y(a, z)] = \frac{d}{dz} Y(a, z) \tag{12}$$

which after exponentiating, gives the familiar translation formula:

$$e^{wL_{-1}} Y(a, z) e^{wL_{-1}} = Y(a, z + w) \tag{13}$$

Eventually, we want to get the Jacobi identity

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0}) Y(a, z_1) Y(b, z_2) - z_0^{-1}\delta(\frac{-z_2 + z_1}{z_0}) Y(b, z_2) Y(a, z_1) = \tag{14}$$

$$z_2^{-1}\delta(\frac{z_1 - z_0}{z_2}) Y(Y(a, z_0)b, z_2) \tag{15}$$

which can be looked at in many different ways. One way is a compact way to generate infinitely many identities involving infinitely many products given by $a \cdot_n b := a(n)b$ using the power of formal calculus. The other way, is to look at it as the analog of the lie algebra Jacobi identity:

$$\mathrm{ad}_a \mathrm{ad}_b - \mathrm{ad}_b \mathrm{ad}_a = \mathrm{ad}_{\mathrm{ad}_a b} \tag{16}$$

But we want to promote this to a Jacobi identity in a *one complex dimensional lie algebra*. Complex functions satisfy a fundamental identity called the Cauchy residue formula, which states

$$-\mathrm{Res}_{z=\infty} f(z) - \mathrm{Res}_{z=0} f(z) = \mathrm{Res}_{z=z_0} f(z) \tag{17}$$

for any rational function with singularities at $0, \infty, z_0$. Combining this identity with the Jacobi identity from lie algebra, should give us what we want a one-complex dimensional lie algebra to satisfy:

(18)
$$-\mathrm{Res}_{z_1=\infty}(f(z_1)\mathrm{ad}_{a,z_1}\mathrm{ad}_{b,z_2}) - \mathrm{Res}_{z_1=0}(f(z_1)\mathrm{ad}_{b,z_2}\mathrm{ad}_{a,z_1}) = \mathrm{Res}_{z_1=z_2}(f(z_1)\mathrm{ad}_{\mathrm{ad}_{a,z_1-z_2}b,z_2})$$

Notice that in the identity 14, the field operators $Y(a,z)$ have singularities at $z=0,\infty$ and the field operators $Y(Y(a,z_1-z_2)b,z_2)$ has singularity at $(z_1-z_2)=0$, or $z_1=z_2$, which matches the picture we want. It turns out that the above identity is in fact equivalent to the VOA Jacobi identity in 14. Later on, we will see another equivalent formulation of the Jacobi identity using a two-dimensional Cauchy identity.

To derive 14, we will need to first obtain *weak associativity* axiom:

$$(z_0+z_2)^l Y(Y(a,z_0)b,z_2)c = (z_0+z_2)^l Y(a,z_0+z_2)Y(b,z_2)c \tag{19}$$

which should hold for all a,b,c and nonnegative integer l depending only on a,c (notice it is **not** a,b). We can think of this as the analog of associativity in algebra where $(ab)c = a(bc)$.

To derive the above equation, we are going to first assume skew-symmetry, although this is not really an assumption, as it can be derived from the bracket relation and weak commutativity (see section 3.5 for the proof). Skew-symmetry is the analog of the skew symmetry in the lie algebra which is $\mathrm{ad}_a b = -\mathrm{ad}_b a$ or in other words $[a,b]=-[b,a]$

$$Y(a,z)b = e^{zL_{-1}}Y(b,-z)a \tag{20}$$

Another way of thinking about the above equation, is that assuming the field b is inserted at $z=0$, i.e. $Y(b,0)1=b$, then inserting a field a at $z=z$, is the same as inserting a at origin $z=0$ and then inserting b at $-z$, plus a translation by z afterwards.

Now to obtain 19, i.e. $a(bc)=(ab)c$, we follow the following guide:

$$a(bc) = a(cb) = c(ab) = (ab)c \tag{21}$$

where the first and last are due to skew-symmetry and the middle is due to weak commutativity.

Making it formal, since a,c weakly commute, there is a l dependent on them such that

$$(z_0+z_2)^l[Y(a,z_0),Y(c,-z_2)] = 0 \tag{22}$$

Now we can write in succession the corresponding identities in 21:

$$(z_0+z_2)^l Y(a,z_0+z_2)Y(b,z_2)c = \quad \text{(skew-symmetry)} \tag{23}$$
$$(z_0+z_2)^l Y(a,z_0+z_2)e^{-z_2L_{-1}}Y(c,-z_2)b = \quad \text{(bracket relation)} \tag{24}$$
$$e^{-z_2L_{-1}}(z_0+z_2)^l Y(a,z_2)Y(c,-z_2)b = \quad \text{(weak commutativity)} \tag{25}$$
$$e^{-z_2L_{-1}}(z_0+z_2)^l Y(c,-z_2)Y(a,z_2)b = \quad \text{(skew-symmetry)} \tag{26}$$
$$(z_0+z_2)^l Y(Y(a,z_0)b,z_2)c \tag{27}$$

3. Commutativity and Associativity

The next step is to get commutativity and associativity from their weak versions derived above. To state these axioms, we first introduce the formal dual of the VOA

$$\mathcal{V}' = \oplus V_n' \tag{28}$$

where V_n' are the duals of V_n, which since they are finite-dimensional, can be naturally defined. An element $c' \in \mathcal{V}'$ acts as a linear functional on $\mathcal{V}$ by the notation $\langle c', - \rangle$. Notice that $c' \in \oplus_{n=0}^{N} V_n'$ for some large enough N, which means $\langle c', v\rangle = 0$ for any $v \in \oplus_{n=N+1} V_n$.

The *commutativity axiom* states that the correlation functions

$$\left\langle c', Y(a,z_1)Y(b,z_2)c\right\rangle = \iota_{12} f(z_1,z_2)\ ,\ \left\langle c', Y(b,z_2)Y(a,z_1)c\right\rangle = \iota_{21} f(z_1,z_2) \tag{29}$$

are the different expansions of the same rational function $f(z_1,z_2) = \frac{g(z_1,z_2)}{(z_1-z_2)^k z_1^l z_2^m}$, where $g \in \mathbb{C}[z_1,z_2]$ is a polynomial and k,l,m depend on $(a,b),(a,c),(b,c')$, respectively.

The associativity axiom similarly states

$$\left\langle c', Y(Y(a,z_0)b,z_2)c\right\rangle = \iota_{20} p(z_0,z_2)\ ,\ \left\langle c', Y(a,z_0+z_2)Y(b,z_2)c\right\rangle = \iota_{02} p(z_0,z_2) \tag{30}$$

are the different expansions of the same rational function $p(z_0,z_2) = \frac{q(z_1,z_2)}{(z_0)^k (z_0+z_1)^l z_2^m}$, where $q \in \mathbb{C}[z_0,z_2]$ is a polynomial and k,l,m depend on $(a,b),(a,c),(b,c')$, respectively.

An important observation is that there is a relation between these two axioms. Indeed, it can be easily seen that $\iota_{12} f = \iota_{02} p|_{z_0=z_1-z_2}$. Hence, we have a unique function

$$F(z_0,z_1,z_2) = \frac{g(z_1,z_2)}{z_0^k z_1^l z_2^m} \tag{31}$$

which expansion in different sectors like $|z_1| > |z_2|$ or $|z_2| > |z_0|$ give the different correlation functions.

To derive commutativity (associativity will be similar), We write the following identity from weak commutativity:

$$(z_1-z_2)^k \left\langle c', Y(a,z_1)Y(b,z_2)c\right\rangle = (z_1-z_2)^k \left\langle c', Y(b,z_2)Y(a,z_1)c\right\rangle \tag{32}$$

Notice on the left side, due to the truncation axiom for $b(n)c$, we see only finitely many negative powers of z_2. Also, as observed before $c' \in \oplus_{n=0}^{N} V_n'$ for some large enough N, which implies that we do not need to consider terms coming from $a(-n)$ for arbitrarily large $n > 0$, in fact only need to consider up to $n \sim N$. Therefore, there are only finitely many *positive* powers of z_1 on the left side. On the right side, the opposite happens, as by the same argument, there are only finitely many positive powers of z_2 and negative powers of z_1. Hence the above is a Laurent *polynomial*

$$h(z_1,z_2) = \frac{g(z_1,z_2)}{z_1^l z_2^m} \tag{33}$$

where $g \in \mathbb{C}[z_1,z_2]$ is a polynomial, and as argued, l,m only depend on $(a,c),(b,c')$ respectively. Now, all we need is a division by $(z_1-z_2)^k$ to get function f inside the commutativity axiom.

A subtlety of the formal calculus is that multiplication by negative powers such as $(z_1-z_2)^{-k}$ need to be dealt with carefully. If we want this power to cancel that of $(z_1-z_2)^k$, we have to first make sure that the product of $(z_1-z_2)^{-k}$ by the correlation functions is meaningful. For example, it turns out that

$$(z_1-z_2)^{-k} \left\langle c', Y(b,z_2)Y(a,z_1)c\right\rangle \tag{34}$$

does not exist as $(z_1-z_2)^{-k}$ (by formal calculus convention) has infinitely many *negative* powers of z_1, and $\langle c', Y(b,z_2)Y(a,z_1)c\rangle$ has infinitely many *positive* powers of z_1! This means that to compute the coefficient of a monomial $z_1^r z_2^t$, there are infinitely many terms to sum as there are infinitely many ways of getting r, and this is not allowed in the framework of formal calculus. To amend that, we need to take the product

$$(-z_2+z_1)^{-k}\left\langle c', Y(b,z_2)Y(a,z_1)c\right\rangle \tag{35}$$

and there would be no problem. Analytically speaking, this means taking the space $|z_2|>|z_1|$ as it is compatible with the formal calculus expansion of $(-z_2+z_1)^{-k}$. Therefore, the left and right side of 32 need to be multiplied by $(z_1-z_2)^{-k}$ and $(-z_2+z_1)^{-k}$. Hence, we obtain 29.

4. JACOBI IDENTITY

To obtain the Jacobi identity, we use the following identities in formal algebra

$$z_0^{-1}\delta(\frac{z_1-z_2}{z_0}) - z_0^{-1}\delta(\frac{-z_2+z_1}{z_0}) = z_2^{-1}\delta(\frac{z_1-z_0}{z_2}) = z_1^{-1}\delta(\frac{z_2+z_0}{z_1}) \tag{36}$$

and for any rational function F, a more complicated, but very similarly provable version of 5 exists:

$$z_0^{-1}\delta(\frac{z_1-z_2}{z_0})F = z_0^{-1}\delta(\frac{z_1-z_2}{z_0})\iota_{12}F|_{z_0=z_1-z_2} \tag{37}$$

$$z_0^{-1}\delta(\frac{-z_2+z_1}{z_0})F = z_0^{-1}\delta(\frac{-z_2+z_1}{z_0})\iota_{21}F|_{z_0=-z_2+z_1} \tag{38}$$

(39)
$$z_2^{-1}\delta(\frac{z_1-z_0}{z_2})F = z_1^{-1}\delta(\frac{z_2+z_0}{z_1})F = z_1^{-1}\delta(\frac{z_2+z_0}{z_1})\iota_{20}F|_{z_1=z_0+z_2} = z_2^{-1}\delta(\frac{z_1-z_0}{z_2})\iota_{20}F|_{z_1=z_0+z_2}$$

Now using F as the function in 31, we obtain the Jacobi identity in 14.

To get from Jacobi identity to weak commutativity (and similarly to weak associativity), first we multiply by z_0^k, for some $k \geq 0$ that we will later specify, and take a residue of Res_{z_0}:

(40)
$$\mathrm{Res}_{z_0}(z_0^k z_0^{-1}\delta(\frac{z_1-z_2}{z_0})Y(a,z_1)Y(b,z_2)) - \mathrm{Res}_{z_0}(z_0^k z_0^{-1}\delta(\frac{-z_2+z_1}{z_0})Y(b,z_2)Y(a,z_1)) =$$

$$\mathrm{Res}_{z_0}(z_0^k z_2^{-1}\delta(\frac{z_1-z_0}{z_2})Y(Y(a,z_0)b,z_2)) \tag{41}$$

Notice the only contribution of z_0 is from the delta function and $z_0^k z_0^{-1}$ term in

$$\mathrm{Res}_{z_0}(z_0^k z_0^{-1}\delta(\frac{z_1-z_2}{z_0})Y(a,z_1)Y(b,z_2))$$

Therefore, as $\mathrm{Res}_{z_0}(z_0^k z_0^{-1}\delta(\frac{z_1-z_2}{z_0})) = (z_1-z_2)^k$, the result is

$$(z_1-z_2)^k Y(a,z_1)Y(b,z_2) - (z_1-z_2)^k Y(b,z_2)Y(a,z_1) = \tag{42}$$

$$\mathrm{Res}_{z_0}(z_2^{-1}\delta(\frac{z_1-z_0}{z_2})z_0^k Y(Y(a,z_0)b,z_2)) \tag{43}$$

Note that $Y(a,z_0)b$, due to truncation axiom $a(n)b=0$ for n large enough, has only finitely many negative powers of z_0 which after multiplication by z_0^k for a large enough k (only dependent on

a, b) will have no negative powers of z_0. Also the delta function $z_2^{-1}\delta(\frac{z_1 - z_0}{z_2})$ has only positive power contribution for z_0. Hence the right hand side is zero and we recover

(44)
$$(z_1 - z_2)^k[Y(a, z_1), Y(b, z_2)] = (z_1 - z_2)^k Y(a, z_1)Y(b, z_2) - (z_1 - z_2)^k Y(b, z_2)Y(a, z_1) = 0$$

One can similarly derive the weak associativity by multiplying by z_1^l and taking residue of z_1.

5. CAUCHY-JACOBI IDENTITY

There are other, more intuitive, version of the Jacobi identity. In fact, there is a Cauchy residue formula for two dimensional complex rational function $H(z_1, z_2)$ with poles at $z_1 = 0, z_2 = 0, z_1 - z_2 = 0$ given by

$$\mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0}H - \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=0}H = \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=z_2}H \tag{45}$$

Notice the order of taking residues is important. For example, $\mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0}H$ is to take a Cauchy integral

$$\frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2} H \tag{46}$$

where the contour C_1 around $z_1 = 0$ has radius **larger** than that of C_2. Similarly

$$\mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=z_2}H = \frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_0} H \tag{47}$$

where C_2 is the *same* C_2 in the previous identity, while C_0 is a circle around the origin in the complex plane $z_1 = z_2$, with radius smaller than that of C_2. Therefore, the Cauchy integral version of 45 is

$$\frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2} H - \frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_1'} H = \frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_0} H \tag{48}$$

where C_1' around $z_1 = 0$, has radius smaller than C_2.

Now taking the function H to be $F|_{z_0=z_1-z_2} \times (z_1 - z_2)^r z_1^s z_2^t$ for any $r, s, t \in \mathbb{Z}$, we obtain infinitely many identities, all of which together are equivalent to the Jacobi identity. The equivalence can be proven by multiplying the Jacobi identity by a the monomial $z_0^r z_1^s z_2^t$, and taking residues $\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}$. Each term in the Jacobi identity will become the corresponding term in the Cauchy-Jacobi identity, e.g.:

(49)
$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\left(x_2^{-1}\delta(\frac{x_1 - x_0}{x_2})F(x_0, x_1, x_2)x_0^r x_1^s x_2^t\right) = \mathrm{Res}_{z_1}\mathrm{Res}_{z_2}\left(F(z_1 - z_2, z_1, z_2)(z_1 - z_2)^r z_1^s z_2^t\right)$$

where we changed the notation from z_i to x_i on the left side to emphasize that side is to be evaluated in the formal calculus way and the right side is meaningful only when $|z_1| > |z_2|$. For more details, see section 2.3, p.41-42, equations (3.1.17-3.1.23) of the reference.

To obtain the Jacobi identity from the Cauchy version, notice as we can vary the powers of the monomials $(z_1 - z_2)^r z_1^s z_2^t$, we can *target* all the monomials inside any of the terms in the Jacobi identity to come out of the residue. Hence we can recover the identity corresponding to that monomial by making a judicious choice of r, s, t.

6. Borcherds Identity

Another version of the Jacobi identity is the Borcherds identity:

(50) $$\mathrm{Res}_{z_1-z_2}\Big(Y(Y(a,z_1-z_2)b,z_2)(z_1-z_2)^m\iota_{z_2,z_1-z_2}((z_1-z_2)+z_2)^n\Big) =$$

(51)
$$\mathrm{Res}_{z_1}\Big(Y(a,z_1)Y(b,z_2)\iota_{z_1,z_2}(z_1-z_2)^m z_1^n\Big) - \mathrm{Res}_{z_1}\Big(Y(b,z_2)Y(a,z_1)\iota_{z_2,z_1}(z_1-z_2)^m z_1^n\Big)$$

for all $m, n \in \mathbb{Z}$. This version is also equivalent to the Jacobi identity.

To see this, consider the correlation function $F(z_0, z_1, z_2)$ formulation of each term of the Jacobi identity. Then, the right side of Jacobi identity is $\iota_{z_1,z_2}F|_{z_0=z_1-z_2}$. A multiplication of the corresponding delta function by $(z_1-z_2)^m z_1^n$ as shown in 39, gives $(z_1-z_2)^m\iota_{z_2,z_1-z_2}((z_1-z_2)+z_2)^n$. The rest is done by taking $\mathrm{Res}_{z_0=z_1-z_2}$.