Thursday 9th January, 2020

The class will have no homework or exam but we will leave some exercise; Office hours only occasionally Friday 3-4pm or by appointment (only if there is a geometry topology seminar SH 6713).

The topics discussed in this class will be centered around the famous *volume conjecture*.

References that will be used along the way:

- $\frac{1}{3}$ of class : Murakami and Yokota : Volume conjecture for knots. I will not follow the second half of the book because I believe there is a more promising approach.
- $\frac{1}{3}$ of class from two books: The first is by Thurston: geometry and topology of 3 manifolds, which you can find here. I will talk about just one or two chapters. The second book is by Jessica Purcell: hyperbolic knot theory which you can find here. This explains one chapter of the previous book and is much more readable.
- The last third of the class, I will figure it out! We will likely discuss proofs of some special cases for the volume conjecture.

To introduce volume conjecture, it is useful to have a general idea of what worlds this conjecture is trying to connect. There are two worlds of low dimensional topology (dim ≤ 4). There is a quantum world and a classical world. It is of great interest to see how the two worlds are related. Normally people on each side do not talk to each other much. Classical is more or less geometry and topology (homotopy theory, homology, etc.) while quantum is more or less algebraic (Quantum Field Theory, and the stuff you hear about recently).

A very special case of the relation is in the study of knots. We have quantum invariants discovered more recently, and classical topological invariants which are historically older. The volume conj is the most pronounced relationship between these invariants. It relates the quantum invariant which is called the Jones polynomial and the classical invariant which is called the hyperbolic volume of the complement of a hyperbolic knot in S^3. One can generalize this invariant to the Gromov norm, which also works for non-hyperbolic knots, i.e. knots which complement in S^3 is not a hyperbolic manifold.

The history of the volume conjecture starts at roughly 1987, by E. Witten; He wrote a paper on exactly solvable 3d gravity and on the last paragraph he mentioned that if his thinking is correct, then there should be some relation between the Gromov norm and quantum invariants on the knots.

The next important work is that of R. Kashaev, where he defined something called Kashaev invariant of knots. He then formulated a precise volume conjecture which he verified for the figure 8 knot.

This was followed by two works by Murakami. In one the Kashaev invariant depending on N was formulated, which turns out to be the colored Jones polynomial evaluated at some root of unity $q = e^{2\pi i/N}$.

There are few knots for which this conjecture can be verified. Some have been done by numerical computation so you can check it.

If you explore the literature, you will see some superficial connection between the two subjects which might make you think it is an easy conjecture, but this is really not the case!

One difficulty of this conjecture is that while the Gromov norm is easy to calculate (a package called *snap* can be used to calculate it), the colored Jones polynomial can only be computed efficiently on a quantum computer. So perhaps the advent of quantum computers and the numerical simulations that will follow, could help us gain more insight.

Side discussion: It is unknown whether Jones polynomial is a complete invariant or not. In fact we do not know if it can detect even the unknot! There are invariants like Khovanov homology that can detect the unknot (proven in the previous decade).

We want to understand classically what colored Jones polynomial means; more precisely, what classical information can be obtained from the sequence of $N-$colored Jones polynomial of a knot. Any such classical information is a good theorem!

My plan is to explain the colored Jones polynomial in two different ways. I will give today the useless but most elementary definition. Then I will define it using Yang Baxter equation.

Definition 1 *A knot K is a smooth embedding of the circle S^1 into S^3 or $\mathbb{R}^3$ up to isotopy.*

We will always assume a knot is oriented. There are four flavors of orientations. as a knot is in S^3 which itself has the usual $\pm$ orientation, and the knot itself which has also two possible arrows on it. The orientation on S^3 determines the overcrossing or under crossing and the knot arrow helps to compute the sign of the over/undercrossing (used in computing the linking number for example).

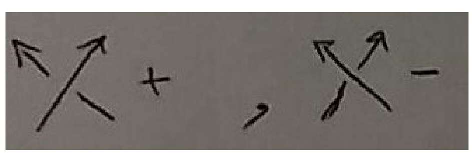


Figure 1: Orientation

The most powerful invariant is of course the complement $S^3\backslash K$. This is actually a deep theorem that this is a complete invariant and determines the knot uniquely.

Like mentioned previously, classical invariants are the ones coming from homology/homotopy of the knot complement. While quantum invariants usually come from quantum physics and partition functions (this is all a rough classification so do not take it too seriously).

We define next the colored Jones polynomial of oriented links L (put an arrow on each component).

Each component of L is associated with a positive integer N. This is the color of the component. N also references the dimension of irreducible representation of $\mathfrak{su}(2)$.

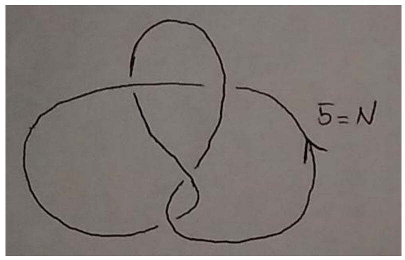


Figure 2: Colored figure eight knot

Normally I would write $L = \cup_i L_i$ with integer c_i attached to L_i. I may not be consistent with my notation throughout the quarter.

Side-discussion: There are speculations on the version of volume conjecture where the colors correspond to the irreps of $\mathfrak{su}(n)$. It is also conjectured instead that by taking can HOMFLY polynomial (a generalization of Jones polynomial) one will get more than the volume on the classical side.

The colored Jones polynomial of colored link (L, c) is a Laurent polynomial $J(L, c; q) \in \mathbb{Z}[q^{\pm 1/2}]$ with variable $q^{\pm 1/2}$ and $q \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We will be interested the most in (K, N), giving the polynomial $J_N(K, q)$. Though we will repeatedly call it a polynomial, note this is not a polynomial.

To make the calculation of Jones polynomial easier, we need to introduce quantum *integers* for $q \in \mathbb{C}^*$:

$$[n]_q := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

There are different conventions and once has to be careful. Sometimes the 1/2 is forgotten.

If we do l'Hospital's rule and take $\to 1$, this gives us n. This corresponds physically to taking the famous Planck constant $\hbar$ to zero as $q = e^{\alpha\hbar}$ where α is some coefficient. So $q \to 1$ corresponds to going from quantum to classical. Mathematically, once can view $[n]_q$ as a $q-$*deformation* of integers.

Show the following as an exercise:

$$[n+1] + [n-1] = [2][n]$$

Note

$$[2] = q^{1/2} + q^{-1/2}$$

Every expression of quantum numbers ultimately becomes a polynomial of $[2]$ and $[1] = 1$, if one uses the above simple identities recursively.

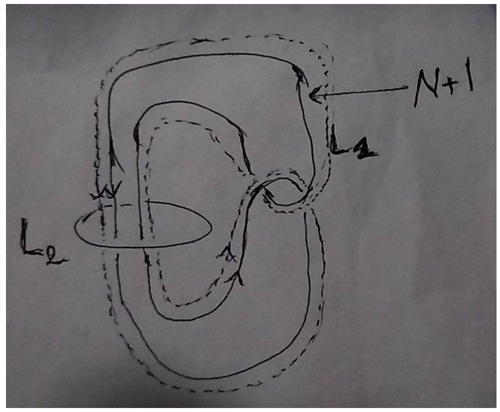


Figure 3: Whitehead link

We would also like to use this relation to define $J(L, c; q)$ recursively.

To have a complete definition we need to first define $J(L, c; q)$ for the base case which corresponds to the coloring by [2] for all components. Of course if any component has color [1], we can safely ignore it:

$$J(L, c_1 \cup \ldots \cup c_n; q) = J(L', c'_1 \cup \ldots \cup c'_n; q)$$

where L' is obtained by dropping all components colored by 1. Physically this corresponds to the vacuum sector which amplitude is always one.

For the nontrivial base case, we define:

$$J(L, 2 \cup \ldots \cup 2; q) := J(L; q)$$

where $J(L; q)$ is the Jones polynomial of L, which will be defined later. Using the exercise above, we can define:

$$J(L = L_1 \cup \ldots, (N+1) \cup \ldots; q) = J() - J(L_1 \cup ..., (N-1) \cup \ldots; q)$$

where $J()$ is corresponding to $[2][N]$:

$$J(L_1^{(2)} \cup \ldots, N \cup 2 \cup \ldots; q)$$

where $L_1^{(2)}$ has two components with color $N, 2$.

The term color goes back to doubling or tripling the knot. So recursively, this says the *colored* Jones polynomial is some linear combination of Jones polynomials of links where we have multiplied the knot N times as shown below for the whitehead link:

Essentially, one considers N parallel running copy of the knots. The way these parallel copies are drawn is by using *0-framing* **push-off** of the knots. The way you produce the **push-off** is by walking along the knot diagram, holding out your right hand, and drawing a parallel knot. The linking number between the knot and this push-off is concentrated at the crossings of the original knot. A framing is a trivialization of the normal vector bundle, up to isotopy. Equivalently, it is a choice of normal vector field along the knot, up to isotopy. The framing is completely characterized by a single integer, the linking number between the knot and a push-off along the chosen normal vector field. There is a special framing (the *0-framing*) given by a Seifert surface of the knot: the neighborhood of the boundary of the surface gives a normal vector field, and the linking number of the push-off with the knot is zero.

Now let me define the Jones polynomial to complete the definition. Using the skein relation:

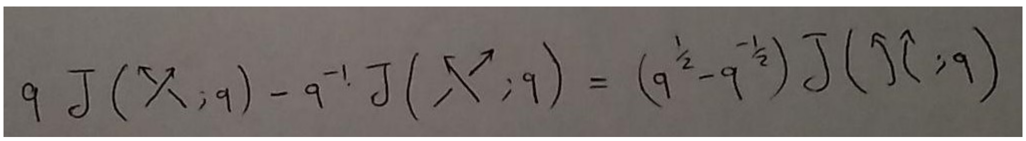


Figure 4: Skein relation

Hence, Jones polynomial of oriented links $J(L;q) \in \mathbb{Z}[q^{\pm 1/2}]$ is defined by

- $J(unknot; q) = [2]$. Sometimes you may have seen the convention that this is one.
- Use skein relation to recursively resolve crossings and get to unknot.

Let us calculate the Jones polynomial of figure eight:

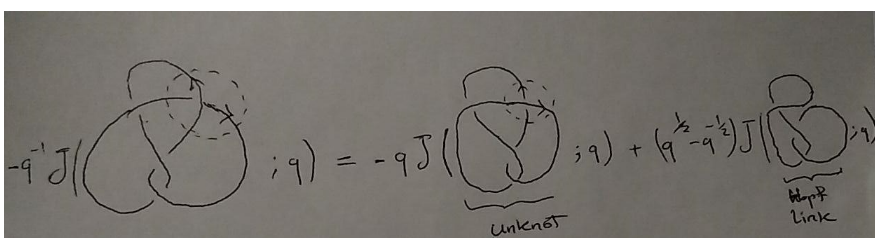


Figure 5: Figure eight Jones polynomial

As an exercise, try to finish the above calculation by computing the Jones polynomial of the Hopf link. You can also find the Jones polynomial of the figure eight knot on its wikipedia page.

Thus taking any crossing of L, which its alternated and resolved version can make the link simpler (there is *always* such a crossing as long as the link is not a collection of unknots), you always get to a place where you have to calculate the Jones polynomial of a simpler knot. But how do we know it is consistent and we get the same answer no matter which crossings we choose to apply the skein relation to? This is actually a (not easy) theorem.

Many significant classes of knots have their closed formula for Jones polynomial found. Now let us discuss the other side of the Volume conjecture which has to do with the Hyperbolic volume. First

Theorem 1 *(Reiley but rediscovered by Thurston) There exists a Riemannian metric on $S^3 \backslash K$ where $K =$ the figure eight, with sectional curvature $= -1$.*

Thurston's idea was to see the noncompact three manifold $S^3 \backslash K$ as a gluing of two tetrahedrons. For a full reference on polyhedral decomposition of any knot, starting with figure eight, we refer to chapter 2 of Jessica Purcell's book in References. More details are also provided in future sections. If one knows that there is a polyhedral decomposition of the complement, it is not hard to see why figure eight gives tetrahedron decomposition, as it divides the plane into 6 regions, number of tetrahedron faces.

The volume conjecture is:

Conjecture 1 *If K is hyperbolic, then*

$$\lim_{N \to \infty} \frac{\log |\frac{J_N(K; e^q)}{[N]_q}|}{N} = \frac{Vol(S^3 - K)}{2\pi} \tag{1}$$

where $q = e^{\frac{2\pi i}{N}}$ and $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

Thursday 23rd Jan 2020

- In previous section, we discussed how to obtain a morphism

$$\varphi : \pi_1(S^3 \setminus K) \longrightarrow PSL(2,\mathbb{C})$$

Recall $PSL(2,\mathbb{C})$ acts on $\mathbb{H}^3 = \{(x,y,t) \mid t>0\}$ by using the quaternion representation. $q \in \mathbb{H}^3 \Rightarrow q = x + iy + jt$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C}) \Rightarrow \gamma(q) = (aq+b)(cq+d)^{-1}$$

Denote $\Gamma = \mathrm{Im}(\varphi)$. Then ρ is discrete if Γ orbits of any $p \in \mathbb{H}^3$ has a discrete Topology, ie $|\Gamma p \cap C| < \infty$ for all compact set $C \subseteq \mathbb{H}^3$.

- For figure eight knot, φ is discrete & faithful ($\ker \varphi = \{id\}$).

- To prove the above, you need an algorithm to get the fundamental domain of φ and then it will be easy to show discreteness. See the reference by Thurston: Three dim manifolds, Kleinian Groups and hyperbolic geometry. (Bulletin AMS)

To see what Thurston did, we will need to decompose the complement to two ideal Tetrahedra.

• Side-discussion on what will come later: Twisted Alexander Poly

• The Alexander Poly is purely classical: Take the fundamental group of the knot and abelianize it, ab: $\pi_1(S^3\setminus K) \to \dfrac{\pi_1(S^3\setminus K)}{[\pi_1(S^3\setminus K), \pi_1(S^3\setminus K)]}$

• The abelianization is first homology which is $\mathbb{Z}$.

• Take the kernel of **ab**, which is a normal subgroup. By standard Topology you can find a space $\tilde{X}$ which is covering space of $S^3\setminus K$ and has $\pi_1(\tilde{X}) = \ker(ab)$. It is called the universal abelian cover.

• Another standard Topology fact states that $\mathbb{Z}$ in the range acts on $\tilde{X}$. Therefore $H_1(\tilde{X};\mathbb{Z})$ is a $\mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t^{\pm 1}]$ module where the first $\mathbb{Z}$ are integers and the second is the abelianization. One then observes that $H_1(\tilde{X}, \mathbb{Z}) \simeq \dfrac{\mathbb{Z}[\mathbb{Z}]}{\langle A(t)\rangle}$ where $A(t)$ is the Alexander Polynomial.

• Let K be hyperbolic. We have two morphisms

$$\begin{tikzcd} \pi_1(S^3\setminus K) \arrow[r, "P_{ab}"] \arrow[dr, "\rho"'] & \dfrac{\pi_1(S^3\setminus K)}{[\pi_1(S^3\setminus K), \pi_1(S^3\setminus K)]} \\ & PSL(2,\mathbb{C}) \end{tikzcd}$$

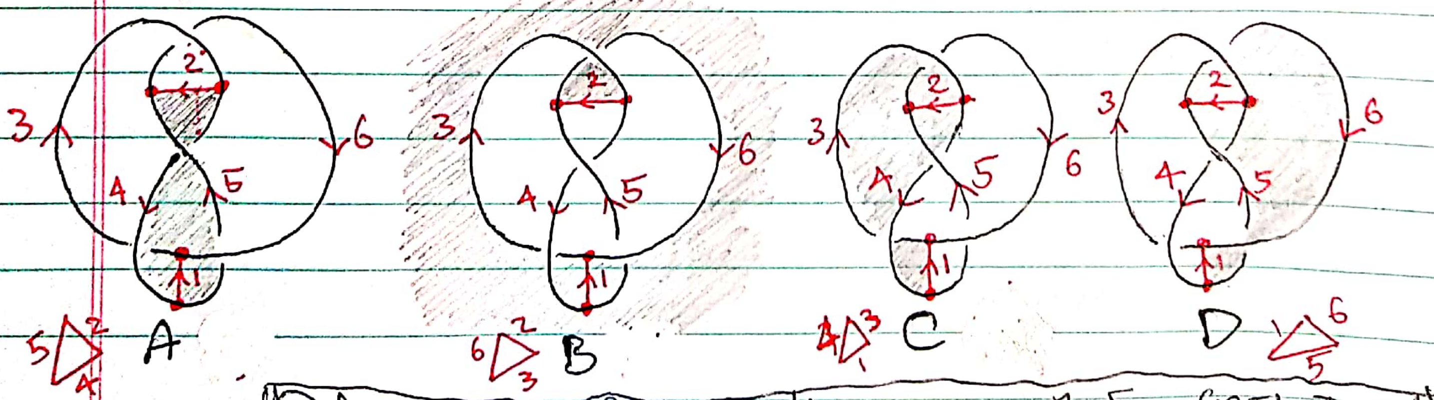
Assume we can lift ρ to $SL(2,\mathbb{C})$.

• Now There is a Theorem That (twisted Alex poly)$_{N}$ $\xrightarrow[N \to \infty]{}$ hyperbolic volume.

We need to find the relationship between J_N & (twisted Alex poly)$_N$.

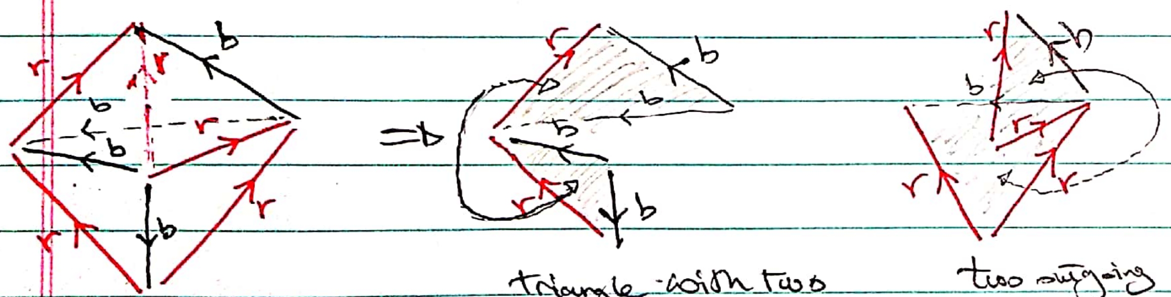
Perhaps finding some quantum version / R matrix from (twisted Alex poly)$_N$ is the way to go.

• We will show how to decompose $S^3 \setminus K$=figure 8 to two tetrahedra. We shall consider K as being on $\mathbb{R}^2$ (except at the crossings) and the two tetrahedra faces meeting each other on the plane and around the crossings. The final picture for how the faces of the top tetrahedra fit is the following.



Ref.: Alexander Gutierrez (Hyp. geom. on the Figure Eight knot compl.

Here, A, B, C, D are the faces of the tetrahedron and the edges are shown by red arrows and numbered from <u>one</u> to <u>six</u>. Notice how <u>two</u> edges are <u>NOT</u> on the strands and instead connect two strands of <u>bigons</u>. This is a general pattern in the decomposition of knot complement to polyhedra, where the edges are the <u>overstrands</u> and the <u>bridge between bigons</u>. The vertices lie on the knot of course. The picture from the bottom for the bottom tetrahedron is very similar. One can track that the identification of edges and faces is similar to the picture below, where one must match faces with the same pattern of edges like the examples shown.

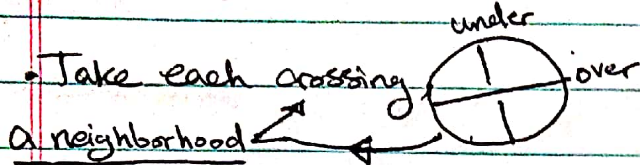


triangle with two <u>outgoing</u> black

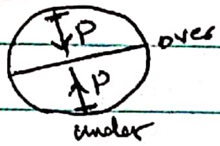
two outgoing red

Now we shall go through the general argument which also works for decomposition to polyhedra for any knot complement.

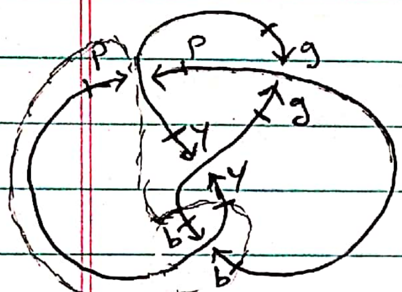
• Take each crossing



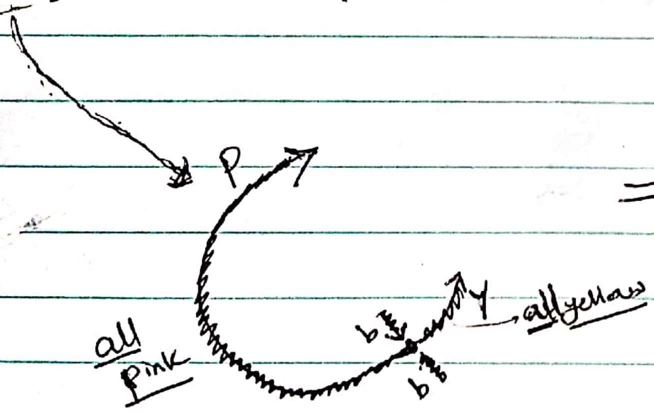
and color it with two opposite arrows



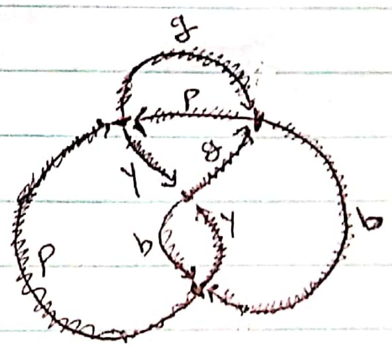
where p : pink. We will use colors y : yellow, g : green, b : blue. We get this picture:



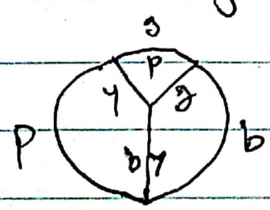
Next you identify the overstrands to a point up to the crossings. This gives:



$\Rightarrow$



The next step is "let bigons be bygones". We shrink the bigons (same as using a bridge in previous pictures). We get the following:



The bottom tetrahedron is similarly built and identified with the top using the colors.

Tuesday 28th Jan 2020

We want to prove the Volume Conjecture for the figure 8 Knot.

Recall Volume Conjecture:

$$\lim_{N\to\infty} \frac{\log\left|\frac{J_N(K;e^q)}{[N]_q}\right|}{N} = \frac{Vol(S^3\setminus K)}{2\pi}$$

where $q = e^{\frac{2\pi i}{N}}$ and $[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$

★ In This section, we will use normalized colored Jones poly.

Therefore Volume Conjecture is: $\lim_{N\to\infty} \frac{\log|J_N|}{N} = \frac{Vol(S^3\setminus K)}{2\pi}$

• We will do The last step of the proof first. We will assume (later prove) That Jones poly for figure 8 Knot K is:

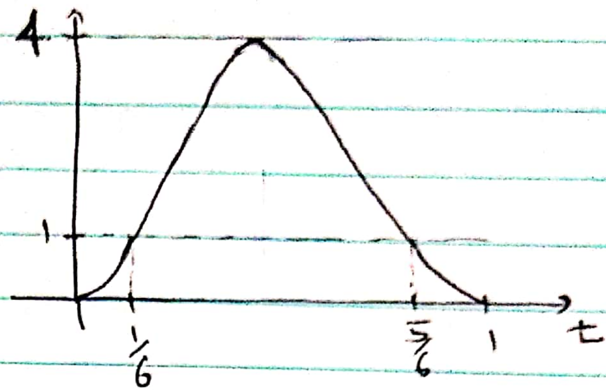
$$J_N(K;q) = \frac{1}{\{N\}} \sum_{j=0}^{N-1} \frac{\{N+j\}!}{\{N-1-j\}!}$$

where $\{n\} = [n]_q(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) = q^{n/2} - q^{-n/2}$, Thus:

$$J_N(K;q) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} (q^{\frac{N-k}{2}} - q^{-\frac{(N-k)}{2}})(q^{\frac{N+k}{2}} - q^{-\frac{(N+k)}{2}})$$

$$\overset{q=e^{\frac{2\pi i}{N}}}{\Longrightarrow} \quad = \sum_{j=0}^{N-1} g_N(j) \text{ for } g_N(j) = \begin{cases} \prod_{k=1}^{j} 4\sin^2\left(\frac{k\pi}{N}\right), & j\neq 0 \\ 1, & j=0 \end{cases}$$

We plot $4\sin^2(t\pi)$:



Therefore for $\begin{cases} 4\sin^2(\frac{k\pi}{N})<1, & 0<k<\frac{N}{6} \\ 4\sin^2(\frac{k\pi}{N})>1, & \frac{N}{6}<k<\frac{5N}{6} \\ 4\sin^2(\frac{k\pi}{N})<1, & k>\frac{5N}{6} \end{cases}$

As $g_N(j)$ are product of $4\sin^2(\frac{k\pi}{N})$, we deduce that $g_N(j)$ is:

① decreasing for $0<j<\frac{N}{6}$

② increasing for $\frac{N}{6}<j<\frac{5N}{6}$

③ decreasing for $\frac{5N}{6}<j<N$

Hence $\max g_N(j)$ occurs at $j=\lfloor\frac{5N}{6}\rfloor$. So

$$g_N(\lfloor\tfrac{5N}{6}\rfloor) < \sum_{j=0}^{N-1} g_N(j) < N\, g_N(\lfloor\tfrac{5N}{6}\rfloor)$$

$$\Rightarrow \frac{\log g_N(\lfloor\frac{5N}{6}\rfloor)}{N} < \frac{\log \sum_0^{N-1} g_N(j)}{N} < \frac{\log N}{N} + \frac{\log g_N(\lfloor\frac{5N}{6}\rfloor)}{N}$$

$$\Rightarrow \lim_{N\to\infty} \frac{\log|J_N|}{N} = \lim_{N\to\infty} \frac{\log g_N(\lfloor\frac{5N}{6}\rfloor)}{N}$$

But $$\lim_{N\to\infty} \frac{\log g_N(\lfloor\frac{5N}{6}\rfloor)}{N} = \lim_{N\to\infty} \sum_{k=1}^{\lfloor\frac{5N}{6}\rfloor} \frac{2\log(2\sin\frac{k\pi}{N})}{N}$$

$$\underset{(*)}{=} \frac{2}{\pi}\int_0^{\frac{5\pi}{6}} \log(2\sin x)\,dx = -\frac{2}{\pi}\Lambda\left(\frac{5\pi}{6}\right)$$

$(*)$ is due to Riemann sum approx, and $\Lambda(\theta) = -\int_0^\theta \log|2\sin x|\,dx$

$\Lambda(\theta)$ is the Lobachevsky function which satisfies:

- Periodic $\Lambda(\theta+\pi) = \Lambda(\theta)$
- odd $\Lambda(-\theta) = -\Lambda(\theta)$
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \frac{\pi}{2})$

All properties can be proved using elementary facts about $\sin(x)$

- $\sin(\theta+\pi) = -\sin(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sin(\frac{\pi}{2}-\theta)$

$\Rightarrow \log|2\sin(2x)| = \log|2\sin x| + \log|2\sin(x+\frac{\pi}{2})|$

Using the above: $\Lambda(\frac{5\pi}{6}) = \Lambda(\pi - \frac{\pi}{6}) = -\Lambda(\frac{\pi}{6}) = \frac{-\Lambda(\frac{\pi}{3})}{2} + \Lambda(\frac{2\pi}{3})$

$$= \frac{-3}{2}\Lambda(\frac{\pi}{3})$$

$$\Rightarrow \lim \frac{\log|J_N|}{N} = \frac{3}{\pi}\Lambda(\frac{\pi}{3}) = \frac{\mathrm{Vol}(S^3\setminus K)}{2\pi}$$

as $\mathrm{Vol}(S^3\setminus K) = 6\Lambda(\frac{\pi}{3})$

(Recall that hyperbolic volume of tetrahedron with angles α, β, γ is $\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$; Also $S^3\setminus K$ is two ideal tetrahedrons with all angles $\frac{\pi}{3}$).

- Next, we prove the formula for colored Jones poly:

$$J_N(K;q) = \frac{1}{\{N\}}\sum_{j=0}^{N-1}\frac{\{N+j\}!}{\{N-1-j\}!}$$

Recall (R,μ,α,β) enhanced R-matrix definition.

Recall for

$$(1)\ R^{ij}_{kl} = \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\,\delta_{k,j-m}\,\frac{\{l\}!\,\{N-1-k\}!}{\{i\}!\,\{m\}!\,\{N-1-j\}!}\,(q^{-1})^{\frac{N-i-1}{2}\cdot\frac{N-j-1}{2}+\frac{m(i-j)}{2}+\frac{m(m+1)}{4}}$$

$$(2)\ (R^{-1})^{ij}_{kl} = \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i-m}\,\delta_{k,j+m}\,\frac{\{k\}!\,\{N-1-l\}!}{\{j\}!\,\{m\}!\,\{N-1-i\}!}\,(q^{-1})^{\frac{i-(N-1)}{2}\cdot\frac{j-(N-1)}{2}+\frac{m(i-j)}{2}-\frac{m(m+1)}{4}}$$

$$(3)\ \mu^i_j = \delta_{i,j}\, q^{\frac{(2i-N+1)}{2}} \qquad (4)\ \alpha = q^{\frac{N^2-1}{4}},\ \beta = 1$$

(R,μ,α,β) is an enhanced R-matrix and

$\forall$ knots K: $\quad J_N(K;q) = \dfrac{\{1\}}{\{N\}}\, T_{(R,\mu,\alpha,\beta)}(K)$

where $T_{(R,\mu,\alpha,\beta)}(K) = \alpha^{-w(K)}\,\beta^{-n}\,\mathrm{Tr}\left(\rho_R(\hat{b})\,\mu^{\otimes n}\right)$

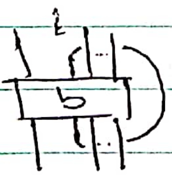
where $K = \hat{b}$ is braid closure of n-strand braid b

and $\rho_R(\hat{b})$ is obtained by placing R or R^{-1} instead of braidings

in b (σ_i or σ_i^{-1}: $\nwarrow\!\!\!\nearrow$ or $\nearrow\!\!\!\nwarrow$). $w(K)$ is writhe index of K.

- Note $J_N(\text{unknot}) = 1$ as $\mathrm{Tr}(\mu) = \sum_{i=0}^{N-1} q^{\frac{2i-N+1}{2}} = \dfrac{\{N\}}{\{1\}}$

- Note $\mathrm{Tr} = \mathrm{Tr}_1 \mathrm{Tr}_2 \cdots \mathrm{Tr}_n$ where Tr_i is obtained by taking

closure on i-th strand only: "  " $= \mathrm{Tr}_i(b)$".

- Every Tr_i gets the average (trace) of the endomorphism from $V^{\otimes n} \to V^{\otimes n}$ on the i-th tensor factor.

- For $K =$ figure eight knot, $b = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$, $w(b) = 0$, b acting on 3 strands. Thus $n = 3 \Rightarrow J_N(K; q) = \frac{\{1\}}{\{N\}} Tr_1 Tr_2 Tr_3 (\varphi_R(\hat{b}) \mu^{\otimes 3})$

- Consider instead $Tr_2 Tr_3 (\varphi_R(\hat{b}) \mu^{\otimes 2})$ where we have gotten rid of two tensor factors. Thus we have an endomorphism of V.

- FACT: The endomorphisms given by R and μ using braids are intertwiners of representations of some quantum group (called $U_q(\mathfrak{sl}_{\mathbb{C}}(2))$). In particular, V is an irreducible representation of that quantum group.

A map $f: V \to W$ is an intertwiner if it is compatible with the representations on V and W, ie

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \phi_V \downarrow & \circlearrowright & \downarrow \phi_W \\ V & \xrightarrow{f} & W \end{array}$$

where $\phi\_$ is the representation action.

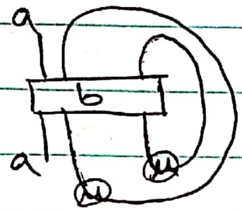
- Hence $Tr_2 Tr_3(\varphi_R(\hat{b}) \mu^{\otimes 2}) \in End(V)$ is an intertwiner of an irreducible representation. Schur's lemma from representation theory applies:

This means $Tr_2 Tr_3(\varphi_R(\hat{b})\mu^{\otimes 2}) = S \times Id_V$ for some $S \in \mathbb{C}$.

Therefore $J_N(K,q) = \frac{\{1\}}{\{N\}} Tr_1(S \times \mu) = \frac{\{1\}}{\{N\}} S\, Tr(\mu)$

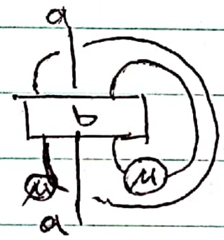
$$= S$$

- So we only need to compute S which is any diagonal element of the matrix $Tr_2 Tr_3(\varphi_R(\hat{b})\mu^{\otimes 2})$ like the following for any index $0 \leq a \leq N-1$:



- It turns out that computing the above is still not easy. Instead, we would like to take Trace on first & third factor instead of second & third.

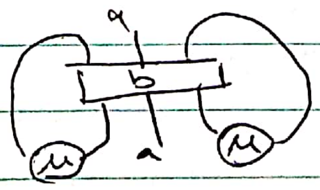
This means the following picture,



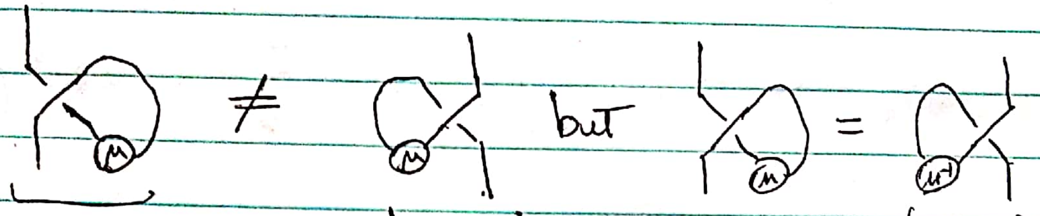
- But due to braidings present at top and bottom (a and a) This becomes still hard to compute.

- We would like to close the first strand from the left to right,

This means:



. It turns out that this is NOT equal to the previous picture. The reason is that to take closure from left to right, one needs to put $\underline{\mu^{-1}}$ instead of $\underline{\underline{\mu}}$. The reason behind this is outside scope of this section but as an exercise you can check that:

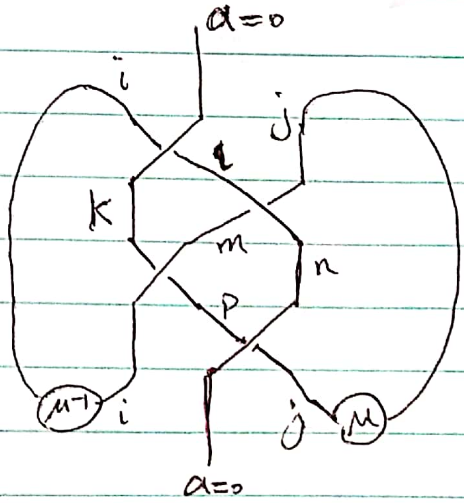


$$\mathrm{Tr}_2(R(\mathrm{Id}\otimes\mu)) = \sum_i R^{ji}_{ji}\,\mu^i_i \qquad \neq \qquad \mathrm{Tr}_1(R(\mu\otimes\mathrm{Id})) = \sum_j R^{ji}_{ji}\,\mu^j_j$$

but

$$\mathrm{Tr}_1(R(\mu^{-1}\otimes\mathrm{Id})) = \sum_j R^{ji}_{ji}\,(\mu^{-1})^j_j$$

Therefore, we need to compute the following, where to make computations easier we choose index $a=0$.

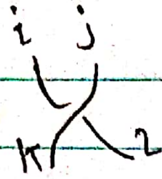


$$\sum_{i,j,k,l,m,n,p} R^{io}_{kl}\,(R^{-1})^{lj}_{mn}\,R^{km}_{ip}\,(R^{-1})^{pn}_{oj}\,(\mu^{-1})^i_i\,\mu^j_j$$

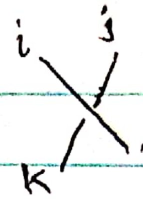
Recall the equations (1) & (2) for R & R^{-1}:

$$R^{ij}_{kl} = \sum \delta_{l,i+m}\,\delta_{k,j-m} \cdots \quad \& \quad (R^{-1})^{ij}_{kl} = \sum \delta_{l,i-m}\,\delta_{k,j+m} \cdots$$

For the terms above to be nonzero, we have the following $(+)$, $(-)$ rules
R $\quad R^{-1}$

R $(+)$ 

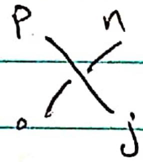
$i+j=k+l$
$l\geq i,\ k\leq j$

R^{-1} $(-)$ 

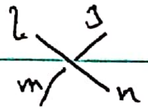
$i+j=k+l$
$l\leq i,\ k\geq j$

Now apply $(+)$ on 

$i+0=k+l$
$l\geq i,\ k\leq 0$ $\Rightarrow$ $k=0$, $l=i$

apply $(-)$ on 

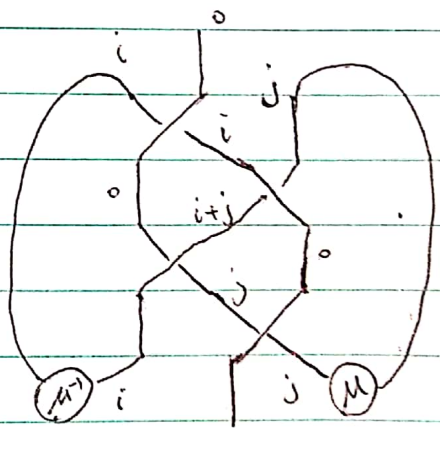
$p+n=0+j$
$n\leq 0\quad p\geq j$ $\Rightarrow$ $n=0$, $p=j$

apply $(-)$ on 

$l+j=m+n$
$n=0$
$l=i$ $\longrightarrow$ $m=i+j$

So we get

$$\sum_{\substack{0\leq i\leq N-1\\ 0\leq j\leq N-1\\ 0\leq i+j\leq N-1}} R^{i0}_{0i}\,(R^{-1})^{ij}_{i+j\,0}\,R^{0\,i+j}_{ij}\,(R^{-1})^{j0}_{0j}\,(\mu^{-1})^{i}_{i}\,\mu^{j}_{j}$$



$$=\sum_{\substack{0\leq i,j\leq N-1\\ 0\leq i+j\leq N-1}} (-1)^{i}\,\frac{\{i+j\}!\,\{N-1\}!}{\{i\}!\,\{j\}!\,\{N-1-j-i\}!}\times q^{-\frac{(N-1)i}{2}+\frac{(N-1)j}{2}-\frac{i^2}{4}+\frac{j^2}{4}-\frac{3i}{4}+\frac{3j}{4}}$$

Put $k=i+j$ $\Rightarrow$ $$\sum_{k=0}^{N-1}\frac{\{N-1\}!}{\{N-1-k\}!}\,q^{\frac{k^2}{4}+\frac{Nk}{2}+\frac{k}{4}}\times\underbrace{\left(\sum_{i=0}^{k}(-1)^{i}\frac{\{k\}!}{\{i\}!\,\{k-i\}!}\,q^{\frac{(-2N-k-1)i}{2}}\right)}$$

For the inner sum, we shall use the following fact.

• Define $T(k,l) = \sum_{i=0}^{k} (-1)^i q^{\frac{li}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q$ where $\begin{bmatrix} k \\ i \end{bmatrix}_q = \dfrac{\{k\}!}{\{i\}!\,\{k-i\}!}$

Then $T(k,l) = \prod_{j=1}^{k} \left(1 - q^{\left(\frac{l+k+1}{2}\right) - j}\right)$

*Proof: exercise. Use Pascal's identity $\begin{bmatrix} k \\ i \end{bmatrix}_q = q^{-\frac{k-i}{2}} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q + q^{\frac{i}{2}} \begin{bmatrix} k-1 \\ i \end{bmatrix}_q$

to get recessive relation $T(k,l) = \left(1 - q^{\frac{l+k+1}{2} - k}\right) T(k-1, l+1)$. ∎

Plugging $l = -2N-k-1$ in above we get by direct calculations

$$T(k,l) = \frac{\{N+k\}!}{\{N\}!} q^{-\frac{k^2}{4} - \frac{Nk}{2} - \frac{k}{4}}$$

This implies

$$J_N(K;q) = \sum_{k=0}^{N-1} \frac{\{N-1\}!}{\{N-1-k\}!} q^{\frac{k^2}{4} + \frac{Nk}{2} + \frac{k}{4}} T(k, -2N-k-1)$$

$$= \sum_{k=0}^{N-1} \frac{\{N-1\}!}{\{N-1-k\}!} \frac{\{N+k\}!}{\{N\}!}$$

$$= \frac{1}{\{N\}} \sum_{k=0}^{N-1} \frac{\{N+k\}!}{\{N-1-k\}!} \qquad \blacksquare$$

Tuesday 4th February 2020

- We can define twisted Alex poly $\Delta_K^{\xi_N}(t) \in \mathbb{C}[t^{\pm1}]$ s.t $\forall \xi \in \mathbb{C}$, $|\xi|=1$, we have (arxiv 1912.12946)

Theorem: $$\lim_{N\to\infty} \frac{\log|\Delta_N^{\xi_N}(\xi)|}{N^2} = \frac{1}{4\pi} \mathrm{Vol}(S^3\setminus K)$$

- We will study the definition of twisted Alex. poly not the proof of the above.

- Goal is to show that Colored Jones Poly and twisted Alex. poly are related.

But note the latter is classical. So either we make the latter more quantum or the former less quantum.

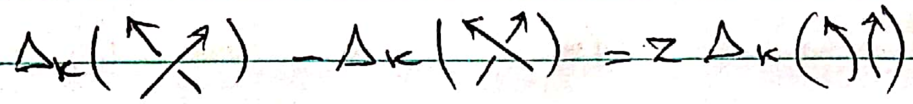
- There is an interpretation by Stephen Bigelow of Jones poly (not colored) in terms of "semi-"classical constructions (using "intersection forms & configuration space, ...).

First, we start with just Alexander polynomial. There are many ways to define this polynomial.

- Let $K \subset S^3$, then the Alexander poly is $\Delta_K(t) \in \mathbb{Z}[t^{\pm1}]$ up to any power t^n $(n \in \mathbb{Z})$. There is also the Alexander-Conway poly

$\Delta_K(z)$ which after change of variable $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ gives $\Delta_K(t)$.

The easiest definition is using The Skein-relation:

- $\Delta_K(z) = 1$ if K = unknot.
- 

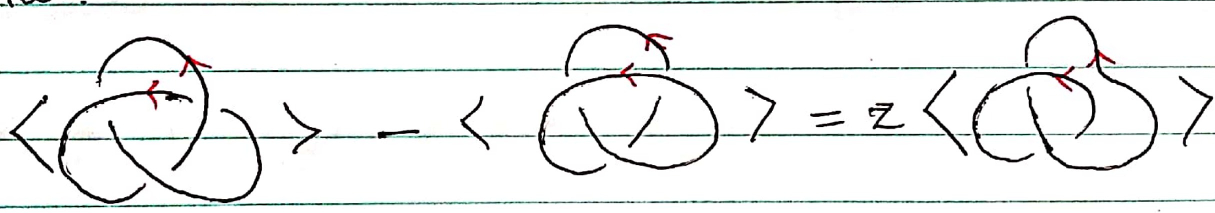
Example: Trefoil knot. We claim $\Delta_K(z) = 1+z^2 = 1+(t^{\frac{1}{2}}-t^{-\frac{1}{2}})^2$

$= t^{-1}-1+t$

"=" $1-t+t^2$

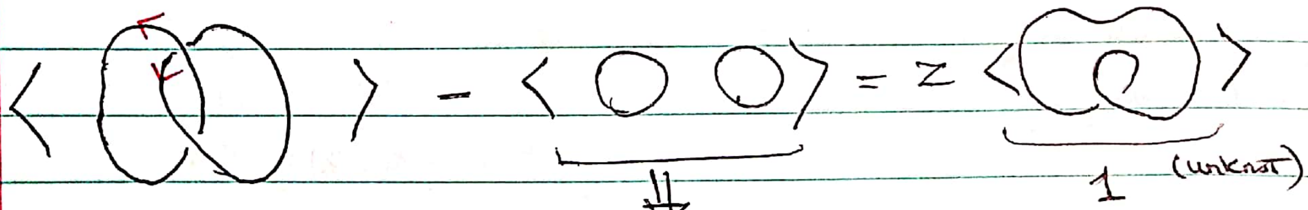
"up to any power t^n"

Proof:

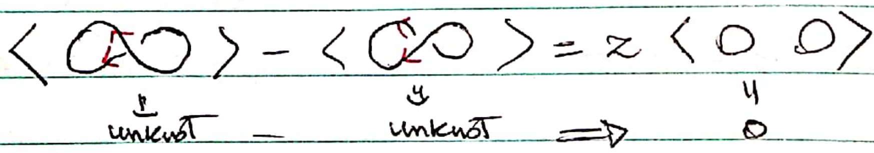


$\Rightarrow \Delta_{\text{trefoil}}(z) = 1 + z\,\Delta_{\text{Hopf link}}(z)$

And for the Hopf link:



$\Downarrow$



Therefore $\Delta_{\text{Hopf link}}(z) = z \Rightarrow \Delta_{\text{Trefoil}}(z) = 1+z^2$

Exercise: Show $\Delta_{\text{figure eight}}(z) = 1 - z^2$.

* But what is the Topological meaning of Alexander poly?

Recall that for $E_K = S^3 \setminus K$ the knot complement (the knot exterior), $H_*(E_K, \mathbb{Z}) \cong H_*(S^1, \mathbb{Z})$ and $\pi_n(E_K, *) = 0$ if $n \neq 1$. This means E_K is a $K(\pi_1(E_K), 1)$ Eilenberg-McLane space.

• Take the abelianization map $\varphi : \pi_1(S^3 \setminus K) \twoheadrightarrow \frac{\pi_1}{[\pi_1, \pi_1]} \cong H_1$

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle \mapsto \langle x_1, \dots, x_n \mid x_i x_j = x_j x_i, r_1, \dots, r_n \rangle$$

but $H_1 \cong \mathbb{Z}$ with generator being the meridian of the knot.

Taking the following short exact sequence

$$1 \to \ker \varphi \to \pi_1(S^3 \setminus K) \to \mathbb{Z} \to 1$$

Then by standard topology, This corresponds to

a covering space $\tilde{E}_K^{ab}$ of E_K with $\pi_1(\tilde{E}_K, *) = \ker \varphi$ $\longmapsto E_K$ such that $\mathbb{Z}$ is the deck transformations of $\tilde{E}_K^{ab}$.

Definitions and standard topology facts:

Let X be any finite CW complex and $\varphi : \pi_1(X) \twoheadrightarrow G$ onto

Then $1 \to \ker \varphi \to \pi_1(X) \twoheadrightarrow G \to 1$

and we can construct a space $\tilde{X}$ such that $\tilde{X}$ is covering space of X with $\pi_1(\tilde{X}) = \ker \varphi$ and all homeomorphisms that commute with the covering map p is G.

homeo

$$\begin{tikzcd} \tilde{X} \arrow[dr, "p"'] & \tilde{X} \arrow[l, "h"'] \arrow[d, "p"] \\ & X \end{tikzcd}$$

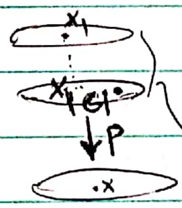
$p \longrightarrow$ a local homeo (e.g. $\begin{array}{c} \mathbb{R} \\ \downarrow p \\ S^1 = \mathbb{R}/\mathbb{Z} \end{array}$ for $\varphi : \mathbb{Z} \to \mathbb{Z}$ identity)

All such h's above give a group which coincides with G. These are called the deck transformations.

In fact $p^{-1}(x) \simeq G$ as a set $\forall x \in X$. Thus we can say that:

pick a basepoint $x_0 \in X$, $\tilde{X} = \{ (y, [\alpha]) \mid y \in X$ and α is a path from x_0 to y and $\alpha \sim \beta$ if $\beta^{-1}\alpha \in \ker\varphi$, in other words α, β are in the same coset of $\pi_1(X)/\ker\varphi$ which is isomorphic to G as φ is surjective. $\}$

Locally, p^{-1} gives decks $\downarrow$ as p is a local homeo



and G elements permute these decks.

$\to |G|$ many decks

Apply the above theorem to $\varphi : \pi_1(\underset{E_K}{\overset{\shortparallel}{S^3 \setminus K}}) \twoheadrightarrow \mathbb{Z}$, then we get

$\tilde{E}_K^{ab}$ (universal abelian cover) with $\pi_1(\tilde{E}_K^{ab}) \simeq \ker\varphi \simeq [\pi_1, \pi_1]$

$\downarrow$

E_K

with deck transformations being $H_1(E_K, \mathbb{Z})$ isomorphic to $\pi_1(E_K)/\ker\varphi$

- Denote $J = \mathbb{Z}$ as a group, let $\Lambda = \mathbb{Z}[J]$ be a group ring which is isomorphic to $\mathbb{Z}[t^{\pm 1}]$.

- Take a deck transformation $T : \tilde{E}_K^{ab} \circlearrowleft$ then $T_* : H_1(\tilde{E}_K^{ab}; \mathbb{Z}) \circlearrowleft$

Therefore $H_1(\tilde{E}_K^{ab}, \mathbb{Z})$ is a $\mathbb{Z}[J]$-module where J are all deck tranformations of $\tilde{E}_K^{ab} \cong \mathbb{Z}$. We have the following Theorem.

- <u>Thm</u>: $H_1(\tilde{E}_K^{ab}, \mathbb{Z})$ as a Λ-module $\cong \Lambda / \langle \Delta_K(t) \rangle$ where $\langle \Delta_K(t) \rangle$ is the ideal generated by $\Delta_K(t)$.

- Notice the analogue ; $\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n$ $A = n \times n$ matrix, $\det A \neq 0$

  e.g $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$

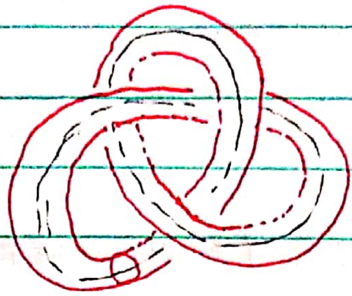
Then we get some abelian subgroup $G_A = \mathbb{Z}^n / \operatorname{Im} A$ which by standard group theory is isomorphic to $\bigoplus_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}$ where order of $G_A = |\det A| = \prod_{i=1}^{n} d_i$.

Of course, in our case, $\Lambda = \mathbb{Z}[J]$ is not a PID but a UFD (unlike $\mathbb{Z}$).

but for our particular case, the analogue does hold.

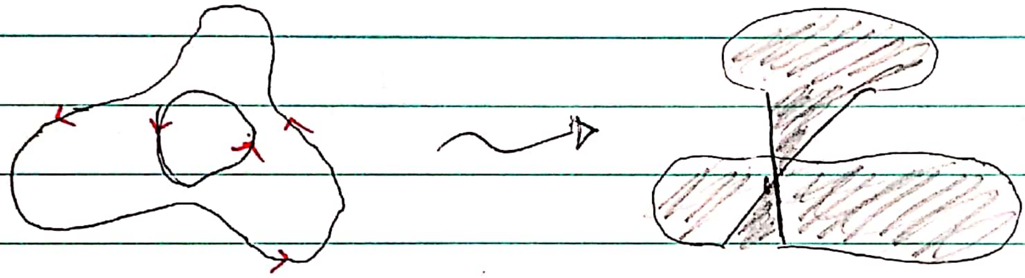
- Next section, we will discuss Fox calculus, which will allow us to compute $\Delta_K(t)$.

• How do we see $\tilde{E}_K^{ab}$?

$\tilde{E}_K^{ab} \longmapsto E_K =$  tubular nghbd of K

First construct a Seifert surface of K, an orientable surface S inside E_K with ∂S = a copy of K in ∂E_K.

How to see the Seifert Surface?



First smooth every crossing $X =)\,($. We get a collection of circles, called Seifert circles. Now let each bound a disk and attach a band at each crossing. Above, we have done it for one crossing.

Note that just taking the disk the knot bounds, could give you an unorientable surface (in this case, the mobius band). This algorithm guarantees orientable.

• Now take the Seifert surface and a tubular nghbd of it, then glue the upper copies to lower copies. This gives you $E_n^{\mathbb{Z}\,ab}$ and

because of orientability you can do that unambiguously.

The deck transformations are taking any copy to another.

Thursday 6th February 2020

- We want to study knots $K \subset S^3$ and their fundamental group

$\pi_1(\underbrace{S^3 \setminus K}_{= E_K}, *) = \Pi_K$ using combinatorial group theory.

($*$: any point in E_K)

- In general, a group with finite presentation has presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$

where r_i are relators e.g. $G_n = \langle x, y \mid \underbrace{xyx = yxy}, \underbrace{x^{n+1} = y^n} \rangle$

is the trivial group.

These are relations

relators: $xyx(yxy)^{-1}$, $x^{n+1}y^{-n}$

- The way we construct G is by taking the free group $F(x_1, \dots, x_n)$

and quotient it by the normal subgroup generated by $r_1, \dots, r_m$: $\dfrac{F(x_1, \dots, x_n)}{N(r_1, \dots, r_m)}$

- Proof for G_n = Trivial:
  - $xyx = yxy \underset{w = xy}{\Longrightarrow} wxw^{-1} = y$ and $x = w^{-1}yw \Rightarrow x^n = w^{-1}y^n w$ $(\star)$

but y^n commutes with w: $y^n w = y^n xy = x^{n+1}xy = xx^{n+1}y =$

$xy^n y = xyy^n = wy^n$

Therefore $(\star) \Rightarrow y^n = x^n$ but $y^n = x^{n+1} \Rightarrow x^n = x^{n+1} \Rightarrow x = 1$

and so $x = y = 1 \Rightarrow G_n$ is trivial.

• In general, There are no algorithms To decide G is Trivial or not.

Now given $G = \langle x_1 \ldots x_n \mid r_1 \ldots r_m \rangle$, construct the group ring $\mathbb{Z}[G]$

defined as : $\{x \mid x = \sum n_g g, n_g \in \mathbb{Z}, g \in G\}$ the finite

finite sum

formal Sums of elements in G, with addition $x+y = \sum (n_g + m_g) g$

and multiplication $xy = \sum_t \left(\sum_{gh=t} n_g m_h\right) t$.

There is a map (a ring map) called augmentation

$$\alpha : \mathbb{Z}G \longrightarrow \mathbb{Z} \quad : \alpha\left(\sum n_g g\right) = \sum n_g$$

There is also the map $F(x_1, \ldots, x_n) \xrightarrow{\gamma} G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$

$\gamma(x_i) = x_i$

which defines a map also called $\gamma : \mathbb{Z}F(x_1, \ldots, x_n) \rightarrow \mathbb{Z}G$

• We want To define The Alexander polynomial as analog of the order of a finite abelian group A;

(Recall discussion in previous section)

$$\mathbb{Z}^n \xrightarrow{A_m} \mathbb{Z}^n \longrightarrow\!\!\!\!\rightarrow A$$

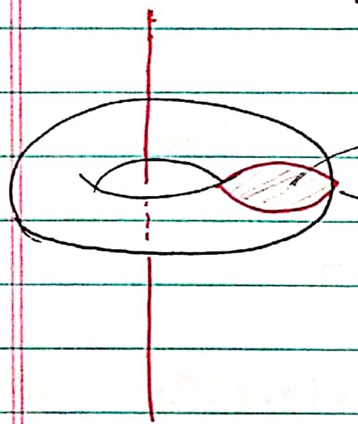
$$A \cong \mathbb{Z}^n / A_m(\mathbb{Z}^n) \cong \bigoplus_{i=1}^{s} \mathbb{Z}/d_i\mathbb{Z} \quad , \; |\det A_m| = \text{order of } A = d_1 \ldots d_s$$

e.g. $\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^2 \to \mathbb{Z}_3$ (A_m is symmetric & even $a_{ii} \overset{2}{\equiv} 0$).

and $|\det A_3| = |\mathbb{Z}_3| = 3$

$H_1(\tilde{E}_K^{ab}, \mathbb{Z})$ as a Λ-module

universal abelian cover of E_K



To visualize $\tilde{E}_K^{ab}$ for K = unknot.

and you glue them. Deck Transformation is shifting by a unit.

In general take a knot on the boundary torus of the tubular neighborhood, then take its Seifert surface and do the same thing we did for unknot.

- There is a theorem that for any finitely presented Λ-module, you can find

a presentation $\Lambda^n \xrightarrow{A_K} \Lambda^n \twoheadrightarrow H_1(\tilde{E}_K^{ab}, \mathbb{Z})$

then $\Delta_K(t) = \det A_K$ up to $t^{\pm m}$.

- A_K is given by Fox Calculus.

R.H. Fox

- Define Fox derivative $D: \mathbb{Z}F_n \to \mathbb{Z}F_n$ be a ring morphism

such that $D(uv) = Du\,\alpha(v) + u\,D(v)$

1) Trivial example $D = 0$.

2) Let us take $g, h \in F_n$:

$$D(gh) = Dg\, \alpha(h) + g\, D(h)$$

$$\Rightarrow D(gh) = Dg + g\, D(h)$$

- This comes from group Cohomology. Let $H^1(G, M)$, for M a G-module, and $\varphi \in C^1(G, M)$ 1-cochain and its coboundary $\delta\varphi \in C^2(G, M)$ is defined by $\delta\varphi(g, h) = g\varphi(h) - \varphi(gh) + \varphi(g)$

Assume $\delta\varphi = 0 \Rightarrow \varphi(gh) = g\varphi(h) + \varphi(g)$

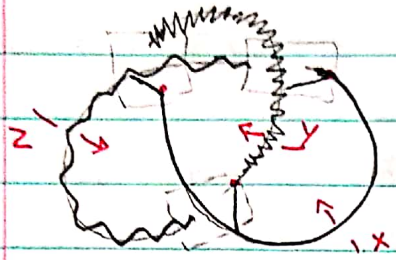
So a Fox derivative is the extension of a 1-cocycle to the group ring $\mathbb{Z}F_n$.

3) $Dg = g - 1$ is a Fox derivative as

$$D(gh) = Dg + g\, D(h) \Leftrightarrow gh - 1 = g - 1 + g(h-1) \checkmark$$

- Going back to Alex. poly. we use Wirtinger presentation for $\pi_1(S^3 \setminus K, *) = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$ (notice number $\prod r_i = 1$ of relators = number of generators)

Recall how we obtained π_1.



for each overstrand, get a generator, and for each crossing a relation.

And for all loops we get trivial element that is $\prod r_i = 1$.

In case of Trefoil,

$$\langle x, y \mid xyx = yxy \rangle$$

$xyxy^{-1}x^{-1}y^{-1}$ ← relator

Now for Trefoil, $F_2 \longrightarrow \pi_{\text{Trefoil}} \xrightarrow[\text{abelianization}]{ab} \mathbb{Z}$

The map $\underline{ab}$ computes the linking number of the representative in π_{Trefoil} with the trefoil knot.

$G \rightarrow G/[G,G]$

Abelianization for trefoil implies $\begin{cases} xy = yx \leftarrow \text{abelianize} \\ xyx = yxy \end{cases} \Rightarrow \underline{x = y}$

So that is why we get $\mathbb{Z}$.

Extend the maps to group rings

$$\mathbb{Z}[F_2] \xrightarrow{\gamma} \mathbb{Z}\pi_{\text{Trefoil}} \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}]$$

Now define the Alexander matrix $A_K = (a_{ij})$

$$a_{ij} = (\alpha \circ \gamma) \frac{\partial r_i}{\partial x_j} \in \mathbb{Z}[t^{\pm}] = \Lambda$$

where $\partial/\partial x_i (x_j) := \delta_{ij}$ and $\partial/\partial x_i : \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n$ is a Fox derivative.

The determinant of A_K will be zero because we deleted one relator (So it is NOT a square matrix). Dropping one column would make it Square and we get The Alexander poly.

- For example for Trefoil: $A_K = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix}$, $\Delta_K = t - 1 + t^{-1}$

where $r = xyxy^{-1}x^{-1}y^{-1}$. We will compute $\frac{\partial r}{\partial x}$.

- Properties of Fox Calculus:

① $D(r_1 r_2^{-1}) = Dr_1 - r_1 r_2^{-1} Dr_2$ for $r_1, r_2 \in F_n$.

proof: $\underbrace{Dr_1 + r_1 Dr_2^{-1}}$ and $D(r_2 r_2^{-1}) = 0 \Rightarrow Dr_2 + r_2 Dr_2^{-1} = 0$

$\Rightarrow Dr_2^{-1} = -r_2^{-1} Dr_2$

which implies $\overset{=}{\longrightarrow}$ $Dr_1 - r_1 r_2^{-1} Dr_2$

But if we take a relator defined as $r = r_1 r_2^{-1}$ then

Since $r = 1$ in the group, $Dr = Dr_1 - Dr_2$.

② $\frac{\partial}{\partial x} x^2 = \frac{\partial}{\partial x} x \cdot 1 + x \cdot 1 = 1 + x$

③ $\frac{\partial}{\partial x}(w_1(x) \cdot w_2(y)) = \frac{\partial}{\partial x} w_1(x) \cdot 1 + 0$

$\frac{\partial}{\partial x}(w_1(x) \cdot w_2(y)) = 0 + w_2(y) \frac{\partial}{\partial x} w_1(x)$

where $w_1(x), w_2(y)$ are words in x, y.

Now let us take: $\frac{\partial}{\partial x}(r_1 r_2^{-1}) = \frac{\partial}{\partial x} r_1 - \frac{\partial}{\partial x} r_2$

$r_1 = xyx$, $r_2 = yxy$

$$= \frac{\partial}{\partial x}(xy)\,1 + xy\,1 - \frac{\partial}{\partial x}(yx).\,1 - 0$$

$$= 1 + 0 + xy - y$$

which after abelianization $\xrightarrow[\substack{x \to t \\ y \to t}]{a} 1 + t^2 - t = \underbrace{t(t-1+t^{-1})}_{\text{Alex. poly up to } t^{\pm 1}}$

And this is det A_K when we drop second column.

$$A_K = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix}$$

You can check that $\frac{\partial r}{\partial y}$ also gives the same thing up to

$$\frac{\partial r_1 r_2^{-1}}{\partial y} = -(1 + yx - x) = -(1 + t^2 - t)$$

a sign (which depends on which column we take out in general).

- $$\begin{tikzcd} \pi_K \arrow[r, "a"] \arrow[dr, "\rho"'] & \mathbb{Z} \\ & GL(n, \mathbb{C}) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathbb{Z}\pi_K \arrow[r, "a"] \arrow[dr, "\rho"'] & \mathbb{Z}[t^{\pm 1}] \\ & M_n(\mathbb{C}) \end{tikzcd}$$

$$\Rightarrow \quad \mathbb{Z}\pi_K \xrightarrow{\rho \otimes a} M_n(\mathbb{C}[t^{\pm 1}])$$

This is what you do for twisted, where the same construction (taking determinant) applies but every $t^{\pm 1}$ is blown up to a matrix.

Tuesday 11th February 2020 (D)

• Alexander Poly. of knots. Recall notations $K \subset S^3$, $E_K = S^3 \setminus K$ knot complement

$\Pi_K = \pi_1(S^3 \setminus K, *)$, abelianization $\Pi_K \xrightarrow{a} \mathbb{Z}$

$x = y = \tau$

e.g. $\Pi_{\text{Trefoil}} = \langle x, y \mid xyx = yxy \rangle = \langle a, b \mid a^3 = b^2 \rangle$ $\{ a = \tau^2, b = \tau^3$ or $\tau = a^{-1}b$

(abelianization $x = y$) (abelianization $ab = ba$)

• Wada's version. Let G be a grp with presentation $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle$
This is called a deficiency $\underline{1}$ presentation
due to $\underline{n-1}$ relators & $\underline{n}$ generators.

$$G \xrightarrow{a} \mathbb{Z} \cong G/[G,G]$$

• Fox Calculus F_n - free grp with $\underline{n}$ gen. $x_1, \ldots, x_n$

Fox derivative: a $D : \mathbb{Z}F_n \longrightarrow \mathbb{Z}F_n$ (group ring) $\mathbb{Z}G = \{ \sum n_g g \mid g \in G, n_g \in \mathbb{Z} \}$

s.t. $\forall x, y \in F_n$ $D(xy) = Dx + x \cdot Dy$

e.g $\partial / \partial x_j$ is a Fox deriv where $\partial / \partial x_j (x_i) = \delta_{ij}$

• Let G be a grp with a presentation $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle$

$$G \xrightarrow{\alpha} \mathbb{Z} = G/[G,G] \quad \text{(the same old } a\text{)}$$

$G = F_n /$ Normal subgrp gen. by $r_1, \ldots, r_{n-1}$

Then $\mathbb{Z}F_n \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] = \Lambda = \mathbb{Z}[\tau^{\pm 1}]$

Def. of Alex. poly of a finitely presented grp G with abelianization $\mathbb{Z}$

$$\text{Alexander matrix} := \left(\alpha\cdot\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right) \text{ of size } (n-1)\times n$$

Delete a column To get $(n-1)\times(n-1)$ matrix $A_G^{(\ell)}$

The ℓ-th column

Then $$\Delta_G(t) = \left[\frac{\det A_G^{(\ell)}}{\alpha(x_\ell)-1}\right](1-t)$$

- It does NOT depend on The presentation (highly nontrivial). Let us check this: $\pi_{\text{trefoil}} = \langle x,y \mid xyxy^{-1}x^{-1}y^{-1}\rangle = \langle a,b \mid a^3b^{-2}\rangle$

- Recall lemma: If you have relator $r = r_1r_2^{-1}$ Then $D(r_1r_2^{-1}) = Dr_1 - Dr_2$ for any fox derivative in $\mathbb{Z}G$. Proof: last section.

Application: $\frac{\partial}{\partial x}(xyx - yxy) = 1 + x\frac{\partial}{\partial x}(yx) - y\frac{\partial}{\partial x}(xy)$

$= 1 + xy - y(1 + x\cdot 0) = 1 + xy - y \xrightarrow{\alpha} 1 - t + t^2 = -\frac{1-t+t^2}{t-1}(1-t)$

- $\frac{\partial}{\partial a}(a^3 - b^2) = 1 + a + a^2$ $\left(\text{general rule: } \frac{\partial}{\partial x}x^n = 1 + x + \cdots + x^{n-1}\right)$

$\xrightarrow{\alpha} \frac{1+t^2+t^4}{t^3-1}(1-t)$ (which is equal to $-\frac{1-t+t^2}{t-1}(t-1)$ up to some power of t)

2 We want To show independence of Alex-Poly with respect To presentation.

Tietze Thm: If $\langle x_1,\dots,x_n \mid r_1,\dots,r_m\rangle$ and $\langle y_1,\dots,y_s \mid R_1,\dots,R_t\rangle$

are presentation of the same group. Then they are related by the following moves. ("you can always rename")

1. Change a relator $r_i \to r_i^{-1}$

2. Change r_i to $r_i, r_i r_j$ for any $i \neq j$
 (add $r_i r_j$ $\forall j \neq i$)

3. add $w r_i w^{-1}$ for any $w \in F_n$

4. add a new generator X and a new relator X.

. Andrew-Curtis Conjecture (Supposed to be wrong!)

★ Given any balanced presentation of the Trivial group
 balanced: number of relators = number of generators

Then it can be reduced to the Trivial representation
$\langle x_1, \dots, x_n \mid x_1, \dots, x_n \rangle$

using ① ③ ④ and ②' ★

it's ② modified where you delete r_i after adding $r_i r_j$ (To keep balance).

e.g. $\langle x y \mid xyx = yxy, x^{n+1} = y^n \rangle$

- For $n \geq 4$ gen. it is unknown, and believed to be wrong.
- There are examples where as a lower bound there needs to be at least $2^{2^{\cdot^{\cdot^{2}}}}\}\log_2 n$ many moves.

- Using Tietze's Thm, we can show the Alex. poly. is indep. of presentation as Δ_G is invariant under those moves.

- Twisted Alex. poly. Wada's version:

Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle \xrightarrow{\alpha} \mathbb{Z}$ and

$\varphi : G \longrightarrow GL(N, \mathbb{C})$

This time define $\mathbb{Z}F_n \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} M_N(\mathbb{Z}[t^{\pm 1}]) = M_N(\Lambda)$

and take $A_{G,\varphi} = \left[(\alpha \otimes \varphi)\, \gamma \frac{\partial r_i}{\partial x_j} \right]$ as the twisted Alexander matrix

Example: $(\alpha \otimes \varphi)(1 + xy - y) \underset{\substack{x \to t \\ y \to t}}{=} 1 + t^2 \varphi(xy) - t\, \varphi(y)$

Then twisted Alex. poly. is obtained by deleting ℓ-th column and taking:

$$\Delta_{G,\varphi}(t) = \frac{\det\left(A_{G,\varphi}^{(\ell)}\right)}{\det\left((\alpha \otimes \varphi)(x_\ell) - 1\right)} (1-t)$$

Example.

- Compute twisted Alex. poly. of figure 8: $\Pi_K = \langle x, y \mid wx = yw \rangle$

$w = xy^{-1}x^{-1}y = [x, y^{-1}]$

where we chose $\varphi : x \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (which is the rep. coming

$y \longrightarrow \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$ from the hyperbolic structure)

where $\omega = e^{\frac{2\pi i}{3}}$.

Recall φ is faithful and discrete and $\Gamma = \varphi(\Pi_K)$ satisfies

$\mathbb{H}^3 / \Gamma \cong S^3 \setminus \infty$.

Need to compute: $\frac{\partial}{\partial x}(wx - yw) = 1 - xy^{-1}x^{-1} + w - y + yxy^{-1}x^{-1}$

$$I - t^{-1}\varphi(xy^{-1}x^{-1}) + \varphi(w) - t\varphi(y) + \varphi(yxy^{-1}x^{-1})$$

$(\alpha\circ\varphi)\gamma$, $x \to T$, $y \to t$

Taking its determinant gives $\dfrac{t^{-2} - 6t^{-1} + 10 - 6t + t^2}{\det((\alpha\circ\varphi)(y) - 1) = (t-1)^2}$ and multiplied $(1-t)$

which should be normalized by

giving $\overset{?}{\pm}\, t^{-2}(t^2 - 4t + 1)$

. Given any hyperbolic knot $K \subset S^3$ with the corresponding

$$\underbrace{\varphi : \pi_1(S^3 \setminus K) \to PSL(2,\mathbb{C})}_{\varphi_N}, \quad \pi_1(S^3\setminus K) \to SL(2,\mathbb{C}) \xrightarrow{Sym^{N-1}} SL(N,\mathbb{C})$$

Then take the twisted Alex. poly. Δ_{K,φ_N}

. The Theorem is : $\forall \xi \in S^1$.

$$\lim_{N\to\infty} \frac{\log|\Delta_{K,\varphi_N}(\xi)|}{N^2} = \frac{1}{4\pi}\,\mathrm{Vol}(S^3\setminus K)$$

How the $\begin{cases} \text{Jones colored } J_N(K;q) \\ \text{twisted Alex. } \Delta_{K,\varphi_N} \end{cases}$ are related ?

① Both are zeta functions, an Ihara zeta function.

② They are both intersection pairing

Thursday February 13th 2020

1) No Class Next Tuesday 18th

- Recall The two families of invariants for knots $K \subset S^3$
  - ★ Colored Jones $\{J_N(K;q)\}_{N=2}^{\infty}$
  - ★ Twisted Alex. Poly. $\varphi : \pi_1(S^3 \setminus K, *) \to SL(2,\mathbb{C})$
    giving $\{\Delta_{K,\varphi}(t)\}$

Volume Conjecture is likely coming from some relation between the two, perhaps for a particular representation φ.

★ If you want to study repr. φ, you can restrict to real / complex parts of $SL(2,\mathbb{C})$, ie $SL(2,\mathbb{R})$ / $SU(2)$. This makes things easier.

★ If K is hyperbolic, then we know there is a canonical (holonomy) map $\varphi : \pi_1(S^3 \setminus K, *) \to PSL(2,\mathbb{C})$ which gets lifted to $SL(2,\mathbb{C})$. Composed with symmetrization map we get a map to $SL(N,\mathbb{C})$

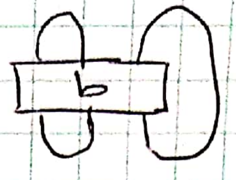
$$\varphi_N : \pi_1(S^3 \setminus K, *) \longrightarrow SL(N,\mathbb{C})$$

how many lifting = how many spin structure

Volume conjecture is:
$$\lim_{N\to\infty} \frac{\log|J_N'(K, e^{\frac{2\pi i}{N}})|}{N} = 2\lim_{N\to\infty} \frac{\log|\Delta_{K,\varphi_N}(\xi)|}{N^2}$$
$\forall |\xi| = 1$

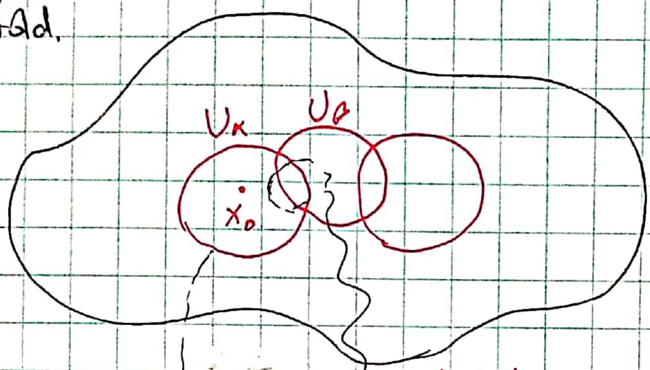
Midterm: (Try to) prove for figure 8.

Goal: Try to prove for all 2-bridge knots.

a 2-bridge knot is of the form  for $b \in B_3$.

It has the fundamental group $\langle x, y \mid wx = yw \rangle$ which is same as that of Figure 8.

① Holonomy representation: Suppose M is a hyperbolic complete manifold.



$U_\alpha \cap U_\beta$

open ball in $\mathbb{H}^3$

Transition function must be an isometry of $\mathbb{H}^3$ (it can be extended to $\overline{\mathbb{H}^3}$)

So an element of $PSL(2, \mathbb{C})$

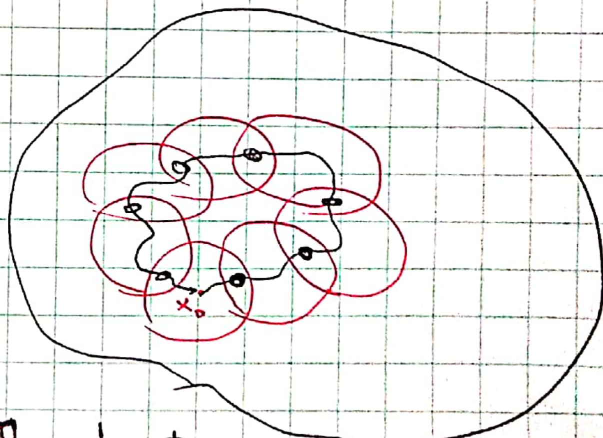
it is a fact that conformal maps in 3-dim extend uniquely.

holonomy means

Take a path and collect those elements

But when you come back to x_0

you may not get back to the same element. This means, as

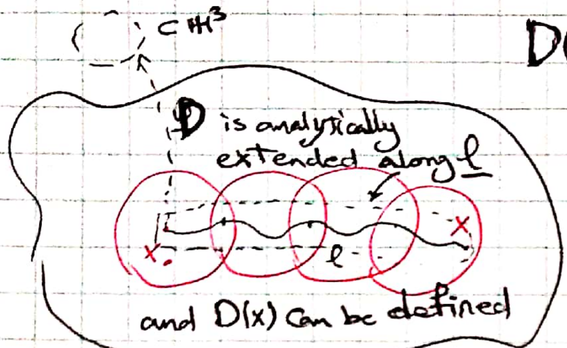


this element is dependent only on the homotopy class of the loop, you get a representation $\varphi : \pi_1 \to PSL(2,\mathbb{C})$

② Developing map: $D : \widetilde{M} \longrightarrow \mathbb{H}^3$

Recall:

$\tilde{M} = \{ (x, [\ell]) \mid \ell$ a path from x_0 to $x \}$ Therefore



$D(x, [\ell])$ depends on x_0 and chart we pick around it only up to a composition by an element of $PSL(2,\mathbb{C})$.

and on x_0 and the chart at x_0 up to an element of $PSL(2,\mathbb{C})$

$D(x)$ is only dependent on $[\ell]$

Facts from hyperbolic geometry:

Have representation $\varphi : \pi_1 M^3 \longrightarrow PSL(2,\mathbb{C})$ and $M^3 \cong \mathbb{H}^3 / \Gamma$

where $\Gamma = \varphi(\pi_1 M^3)$ is discrete & faithful.

φ is projective representation and the scalar $c(g,h)$ in

write associativity rule to see this

$$\varphi(gh) = c(g,h)\, \varphi(g)\, \varphi(h)$$

is a two-cycle. But $H^2(PSL(2,\mathbb{C})) = 0$ so φ can always be lifted to $SL(2,\mathbb{C})$. The different lifting correspond to the different spin structures.

- Recall for figure 8: $\langle x, y \mid wx = yw \rangle \quad w = [x, y^{-1}]$

holonomy $\varphi_2 : \quad x \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \to \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}$

$$\varphi_N : \Pi_K \to SL(2,\mathbb{C}) \xrightarrow{?} SL(N,\mathbb{C})$$

- By repr. Theory, $SL(2,\mathbb{C})$ has a so-called fundamental (defining) representation : $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ = matrix representation on $\mathbb{C}^2$.

Let $V := \mathbb{C}^2$ above, Then we can define representation on $V^{\otimes m}$ by $g \cdot (v_1 \otimes \dots \otimes v_m) \Longrightarrow gv_1 \otimes \dots \otimes gv_m$.

- Now This representation is NOT irreducible. For example:

$$V \otimes V \cong \underbrace{\mathbb{1}}_{\text{"Singlet" representation}} \oplus \underbrace{\mathbb{C}^3}_{\text{"Triplet" representation}}$$

- We can prove by induction That $V^{\otimes m}$ decomposition has always some irreducible representation of dimension $m+1$ called V_{m+1}.

- By using the fact That permutations commute with The representation we can show V_N has a basis $\langle x^{N-1}, x^{N-2}y, \dots, y^{N-1} \rangle$ for which the representation φ_N is:

As an example $N=3$, for figure 8: φ_2: $a \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$b \to \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$

Then to get φ_3 first compute $\varphi(a)^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
$= \begin{pmatrix} x-y \\ y \end{pmatrix}$

Then plug in first coordinate into x and second into y : $\langle x^2, x^1y^1, y^2 \rangle \xrightarrow{\varphi_3} \langle (x-y)^2, (x-y)y, y^2 \rangle$

which means $\varphi_3(a) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$.

Similarly for b we have: $\varphi^{-1}(b)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\\omega x+y\end{pmatrix}$

Thus $\langle x^2, xy, y^2\rangle \xrightarrow{\varphi_3} \langle x^2, x(\omega x+y), (\omega x+y)^2\rangle$

Therefore $\varphi_3(b) = \begin{pmatrix}1 & \omega & \omega^2\\ 0 & 1 & 2\omega\\ 0 & 0 & 1\end{pmatrix}$

We can then compute (using previous section materials):

$$\Delta_{\infty,\varphi_3} = -t^{-3}(t-1)$$

Similarly $\Delta_{\infty,\varphi_4} = t^{-4}(t^2-4t+1)^2$ and more interesting is:

$$\Delta_{\infty,\varphi_5} = -t^{-3}(t-1)(t^4-9t^3+44t^2-9t+1)$$

in Volume conjecture if $t=1$ we get all zero which log is $-\infty$! so need to do normalization for N odd.

For Figure 8 The sequence converges like This:

$$\frac{4\pi \log|\Delta_{\infty,\varphi_N}(1)|}{N^2}$$

N	4	12	24	32
Value	0.54...	1.86...	1.98...	2.006...

The Volume is 2.0298832...

Data from H. Goda paper

- One of the difficulties in matching up the colored Jones sequence and the Twisted Alex. is the specific choice of value $q=e^{\frac{2\pi i}{N}}$

for The colored Jones and The generic choice of ξ, $|\xi|=1$ on The other side.

- We believe There is a version where colored Jones is evaluated at generic values. Some "philosophical" reasons:

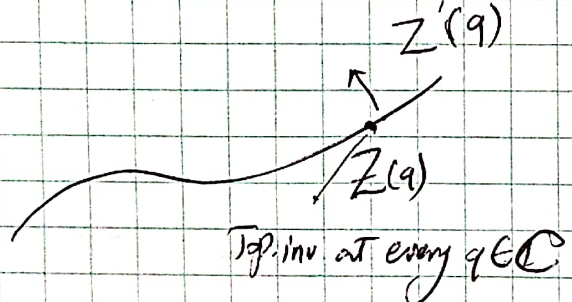
J_N is a top. invariant. A Turaev-Viro inv where one Triangulates The manifold and puts labels on edges, faces, vertices and Take The "state-sum". This is a top. inv. when labels come from a "modular Tensor category".

also called a partition function $Z(q)$ (due To state sum formulation)

Now we ask if for $q \in \mathbb{C}$ we are getting a top. inv

"Then looking at the normal direction of This,

we believe we should get the volume"



Thursday February 20th 2020

1) Where does the V.C. come from? (2+1)-gravity with negative cosmological constant

Witten's papers 1. Exactly...

(2 Revisited)

3. Analytic continuation of CS-theory

. X^3 manifold spacetime, eg X^3 closed oriented

Einstein Hilbert action (EH):

$$I(X^3, g) = \int_{X^3} d^3\mathrm{vol}(R - 2\Lambda)$$

scalar curvature

cosmological constant $\Lambda = -\frac{1}{\ell^2}$

Einstein equation $\rightsquigarrow$ $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0$ $(T_{\alpha\beta})$

There is a spin connecti ω and dreiben (Veilbein)

Fact: If you have a 3 manifold closed oriented then $TX^3 \cong X^3 \times \mathbb{R}^3$ but cannot be trivialized canonically, so there is a 'framing' (dreiben) a choice of the Trivialization $e: TX^3 \xrightarrow{\cong} X^3 \times \mathbb{R}^3$

So instead of g can use (ω, e) Then $I(X^3, g)$ becomes $\rightsquigarrow k_R\, CS(A_+) - k_L\, CS(A_-)$

SO(2,2) gauge field

(two copies of CS theory)

But $SO(2,2) \overset{\text{locally}}{\cong} SO(2,1) \times SO(2,1)$ This was all in Lorentzian signature, we should go

$SO(4) \cong SU(2) \times SU(2)$

into the Euclidean signature by doing a wick rotation. (2)

This makes $SU(2,1) \to SU(2)$ and we end up in Classical CS theory.

Witten said if all these make sense we should get a version of V.C. out of this (as from the integration I we get the volume.)

• What is the V.C. for closed manifolds?

$SU(2)_k$ CS — TQFTs ← Reshetikhin–Turaev, (doubled) Turaev–Viro

$$\frac{\log |Z_k(X^3)|}{N} \xrightarrow{N\to\infty} \mathrm{Vol}(X^3)$$

for $N=k+2$ where k is the level ← Naive generalization to 3mnfld

If Z_k comes from $SU(2)_k$-CS TQFTs then it is known that

$$|Z_k(X^3)| \leq \alpha \cdot D^{2\,\mathrm{genus}(X^3)}$$

α is a constant

where $D^2 = \sum d_i^2$ the total quant. dim. of the TQFT.

(Ref: look at book of Turaev "Quant. invariants of 3mnflds")

Sketch of proof for the bound:

$$X^3 = H_g \cup_f H_g \qquad Z(X^3) = \langle Z(H_g), Z(H_g) \rangle_{V(\Sigma_g)}$$

$f \in \Sigma_g$ (unclear)

in unitary, by Verlinde's formula this is bounded by D^{2g}

But then $\frac{|\log Z_k(X^3)|}{N} \leq \frac{\log(\alpha D^{2g})}{N}$ but $D^2 \sim N^{3/2}$ and so $N\to\infty$ this should go to zero!

So there needs to be a new version of V.C. (by Q Chen and T Yang) for "unitary theories". And the idea (with strong numerical evidence) is to square the q corresponding to $SU(2)_k$ so making the theory non-unitary. Essentially since the Hamiltonian is not hermitian therefore when taking $\mathrm{Tr}\, e^{-itH}$ we can have a rate of growth that is proportional to N.

The second possibility is:

(2) We should not rotate the fibers but just the base so we get CS theory for non compact (unlike $SO(2)$) Lie groups.

(It is in the paper Analytic continuation).

Going back to V.C for hyperbolic knots

$$\lim_{N\to\infty} \frac{\log |V_N(K, e^{2\pi i/N})|}{N} = \frac{1}{2\pi} \mathrm{Vol}(S^3 \setminus K)$$

$$V_N = \frac{J_N}{[N+1]} \quad \text{so that } V_N(\text{unknot}) = 1$$

Now for $SU(2)$ TQFT at $q = e^{2\pi i/N}$ colors are $1, 2, \ldots, k+1 = N-1$ so there is no V_N so V.C. above does not have a $SU(2)$ TQFT interpretation

- But it is likely to have a TQFT interpretation on $SL(2,\mathbb{C})$ or $SL(2,\mathbb{R})$. So we need to do TQFT on non compact theories.

- For $V_N(K,q)$ as long as $q \in \mathbb{C}^{*} = \mathbb{C}\setminus\{0\}$ V_N is well-defined. For $N=2$ this gives Jones poly. Is there a generalization of Jones poly to 3-mnfld?

Witten found generalization only for some roots of unity, $q = e^{\pm \frac{2\pi i}{r}}$ (4)

After many years, we can do almost any root of unity. We claim that the obstruction to the generalization, is the Volume,

of Jones poly to an analytic fcn for 3mflds

- Why are we talking about this?

To relate the Jones colored poly to twisted Alex. poly we want to look at analytic expansion (Taylor series) of J_N and $\Delta_K(t)$ and the series better be similar to each other if N.C. is true.

- Melvin-Morton Conj.

Three proofs:

1) Bar-Natan, Garoufalidis.

2) Lin-W (Zeta function, Lyndon words)

J. Milnor "infinite cyclic covering" discovered that $\frac{1}{\text{Alex.}}$ = zeta-function which is counting some periodic orbits

fixed points of some diffeomorphism

3) "L. Rozansky Expansion" boson fermion correspondence

It's a "physicist" proof

$X \times I$

$(x,0) \sim (f(x),1)$

We will discuss (3).

- In colored Jones q should be interpreted as e^h. The reason is $V_N(k,q) \underset{CS}{=} \int_A \mathcal{D}A\, e^{ikCS(A)}$ (some operator)

$h = \frac{1}{k}$

$= \sum a_n h^n$ Then a_n are called Vassiliev invariants. In other words expanding $V_N(K; e^h) = \sum_{n=0}^{\infty} a_n h^n$ (5)

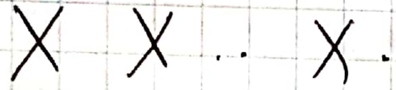
gives invariants a_n.

<u>Thm</u> (XS Lin): a_n are <u>finite type</u> invariants.

ie $\forall n$, $\exists m$ s.t. for any singular link L with m singularities

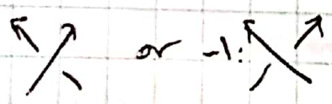
$a_n(L) = 0$ as defined below;

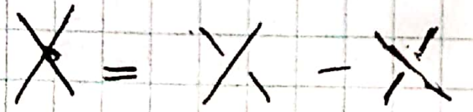
A link L with m crossings that are transversal intersections

like this 

Now define $a_n(L) = \sum_{\epsilon_1, \dots, \epsilon_m} \epsilon_1 \cdots \epsilon_m \, a_n(L_{\epsilon_1 \cdots \epsilon_m})$

where $\epsilon_i \in \{\pm 1\}$ determine how we resolve the crossing

above. +1:  or -1: In other words

 So this is like derivative and

being of finite type of order $\leq m$ means "taking derivative

m times" gives zero.

- <u>Conjecture</u>: Given a finite type invariant V of order m, then $\exists$ a

Universal constant C s.t. $|V(K)| \leq C \, (\#\text{crossings of } K)^m$

Proved by Bar-Norton "poly invariants of poly".

So implication is that a_n should grow polynomially.

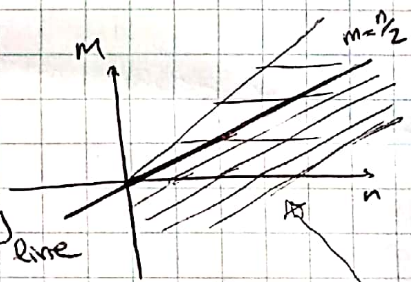
This means V_N is almost like a modular form. At the same time it is a Zeta function. So maybe it is a Mellin Transformation.

- Let us change notation $N \to d$. Note a_n depends on $\underline{d}$.

Consider approx. of $e^h \sim 1+h$. Calculate

$$V_d(k, 1+h) = \sum_{n=0}^{\infty} \underbrace{V_{n,d}(k)}_{\text{finite type invariants}} h^n$$

- Melvin Morton notices that $V_{n,d}(k)$ is a polynomial function of d which can be written as $V_{n,d}(k) = \sum_{0 \le m \le n} D_{m,n}\, d^{2m}$



They also notices $D_{m,n} = 0$ if $m > \frac{n}{2}$

summing over this line

Now take $\sum_{m=0}^{\infty} D_{m,2m}\, a^{2m}$ formal series with $\underline{a}$ formal var.

This is equal to $\dfrac{1}{\Delta(k, e^{\pi i a} - e^{-\pi i a})}$ (This is also a zeta function)

But what if we $\underline{\text{sum other lines}}$ $m = \frac{n}{2} + s$ for any $s \le 0$?

The first line it gives $\dfrac{P_2(z^2)}{\Delta^3}$ $z = e^{\pi i a} - e^{-\pi i a}$

The second line gives $\dfrac{P_3(z^2)}{\Delta^5}$ where P_i are polynomial invariants.

Physical interpretation: boson–fermion are 'inverse' of each other.

boson – colored Jones and Alex poly ~ free fermions (by Saleur, Kauffman)

So that is why Alex poly appears in denominator.

Tuesday February 23th 2020

Paper recommendation: B. Mazur: Knots, primes, and P_0

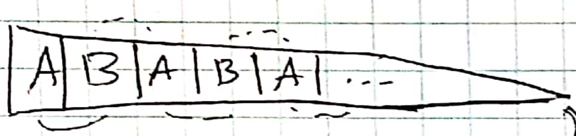
his "propaganda": hyp Knots $K \subset S^3$ $\longleftrightarrow$ primes # $p\mathbb{Z} \subset \mathbb{Z}$

$Vol(S^3 \setminus K) \longleftrightarrow \log p$

① Mazur manifold: compact $M^4 \underset{\text{contractible to}}{\simeq} *$ but it is not D^4.

② He also proved nontrivial knots have no inverse by 'Swindle' argument

$A \# B = \text{unknot} \implies A$ unknot



= unknot

you have a "singularity" here which you can resolve.

③ Twisted Alex polynomial, Colored Jones, Riemann Zeta, Weil Zeta

zeta function

- Weil $\Rightarrow$ Artin - Mazur zeta function

diffeo $f: M^n \to M^n$ then define $\zeta_f(z) = \exp\left(\sum_{n=0}^{\infty} \frac{z^n}{n} (\#\text{fixed pts of } f^n)\right)$

① ζ_f is a rational function

Gamma function

② Functional equation something like "$\zeta(s) = \Gamma(s)\zeta(1-s)$" is a Poincare duality, proven by Grothendieck

③ Product equation $\sum \frac{1}{n^s} = \prod_{\text{primes}} (1-p^{-s})$ proven by Deligne

Milnor: proved why $\zeta_f = \dfrac{1}{\Delta(K,t)}$.

Melvin - Morton Conj: "colored Jones is zeta function"

D. Zagier: (Quant Modular Forms

Melvin - Morton - Rozansky expansion:

Recall given $K \subset S^3$ $J_d(K;q)$: $J_d(\text{unknot}) = [d]$

$$V_d = \frac{J_d}{[d]}, \quad V_d(\text{unknot}) = 1$$

$$V_d(K, q=1+h) = \sum_{n\geq 0} \Big(\sum_{0\leq m\leq 2n} D_{m,n} d^m \Big) h^n$$

(everything in here is in formal calculus).

$$= \sum_{n\geq 0} \sum_{0\leq 2m' \leq 2n} D_{m',n} d^{2m'} h^n$$

$$= \sum_{n\geq 0} \sum_{0\leq m' \leq n} D_{m',n} d^{2m'} h^n$$

$\star$ Thm of MM. $D_{m',n} = 0 \ \forall m' > \frac{n}{2}$ $\star$

$$= \sum_{n=0}^{\infty} \sum_{0\leq m' \leq \frac{n}{2}} D_{m',n} d^{2m'} h^n \qquad (n' = n - 2m')$$

$$= \sum_{n'=0}^{\infty} \Big(\underbrace{\sum_{m'\geq 0} D_{m', 2m'+n'} (dh)^{2m'}}_{} \Big) h^{n'}$$

MMR expansion — Theorem by Rozansky:

$$V_n(K,z) = \frac{P_n(z^2)}{\Delta^{2n+1}(z)}, \qquad z = q^{\frac{d}{2}} - q^{-\frac{d}{2}}, \qquad z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$$

Examples:

① trefoil knot: $V_d(q) = 1 + q^{d-1} \sum_{m=1}^{d-1} q^{md}(1-q^{d-1})\cdots(1-q^{d-m})$

$\Delta = 1+z^2$, $V_1 = 2z^2 + z^4$, $V_2 = 1 - 3z^2 + z^4$, ...

② figure 8: $V_d = 1 + \sum_{m=1}^{d-1} \prod_{j=1}^{m} (q^d + q^{-d} - q^j - q^{-j})$

$\Delta = 1 - z^2$, $P_2 = -1 - z^2$, $P_4 = 4 + 20z^2 + 14z^4 + 2z^6$

We will show outline of the proof in the next few lectures.

$V_d(K, 1+h)$? Defined using R-matrix.

$U_q\mathfrak{sl}(2,\mathbb{C})$: $\exists$ an irrep of each dim V_d $\{f_0, \ldots, f_{d-1}\}$

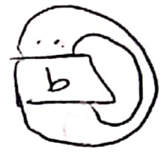
③

- Lie algebra of $Sl(2,\mathbb{C})$ generated by X, Y, H. $X = \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix}$, $Y = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$
- quantum groups : Hopf algebras HA: $U_q sl(2,\mathbb{C})$ $H = \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix}$
  univ enveloping algebra
- $\exists$ universal R-matrix $R \in HA \otimes HA$ (q-deformed) $[H,X] = 2X$, $[H,Y] = 2Y$,

gives $\rightarrow$ (R,μ) enhanced $\longrightarrow$ link invariant $[X,Y] = H$

$b \in B_N$

$J_d(K, q)$ = link invariant from R-matrix of $U_q sl(2,\mathbb{C})$' =



What about V_d ?

leaving first strand open by Schur's lemma gives a scalar $= V_d(K, q)$

- The boson-fermion interpretation : We have a rep on $\mathbb{C}[f_0, \ldots, f_{d-1}] = V_d$

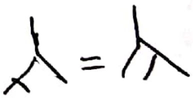
$$X f_m = [m] f_{m-1} \qquad [n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

$$Y f_m = [d-1-m] f_{m+1}$$

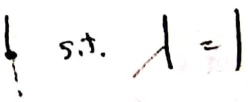
$$H f_m = (d-1-2m) f_m$$

a basis for $V_d^{\otimes N}$ is $\{ f_{m_1} \otimes \cdots \otimes f_{m_N} \}$ which can be mapped to monomials $\mathbb{C}[z_1^{m_1} \cdots z_N^{m_N}]$. Now: $\Delta^{N-1}\left(\frac{1}{2}[N(d-1) - H]\right)$ acts on $V_d^{\otimes N}$ where Δ is the comultiplication for the Hopf algebra $U_q Sl(2,\mathbb{C})$.

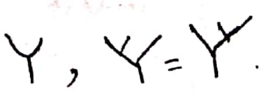
- A quick review on HA: ① μ : multiplication satisfying associativity



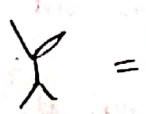
$\exists$ unit



② Δ :



③



example: $\mathbb{C}[G]$ the group algebra is a HA.

S: antipode like taking inverse

④

... and some other compatibility equations.

4

Eigenspace decomp of $V_d^{\otimes N}$ using $\frac{1}{2}[N(d-1)+H]=C$:

$$C(f_{m_1}\otimes\cdots\otimes f_{m_N}) = \underbrace{\left(\sum_{i=1}^{N} m_i\right)}_{n}(f_{m_1}\otimes\cdots\otimes f_{m_N})$$

$$V_d^{\otimes N} = \bigoplus_{n=0}^{\infty} V_N^{(n)} \quad , \quad V_N^{(1)} \cong \mathbb{C}[z_1,\dots,z_N] \text{ as } f_j \to z_j$$

since $\sum m_i = 1 \to \exists!\, j : m_j = 1$ and all others $=0$.

Rozansky's deformation of R-matrix:

$$\check{R}[a;\varepsilon_1,\varepsilon_2,\varepsilon_{12}](f_{m_1}\otimes f_{m_2})$$
$$= \sum_{n\geq 0} \frac{\left(\prod_{m_1-n+1\leq l\leq m_1} l\right)}{n!}\left(e^{\varepsilon_{12}}a\right)^n \left(e^{\varepsilon_1}\right)^{m_1}\left(e^{\varepsilon_2}q^{-d}\right)^{m_2} f_{m_2+n}\, f_{m_1-n}$$

a perturbation of R-matrix using a (spectral parameter) and small numbers $\varepsilon_1,\varepsilon_2,\varepsilon_{12}$

Taking derivative at zero recovers the R-matrix.

Technical Lemma: $\exists$ polynomials $T_{j,k}^{[R,1]}$, $T_{j,k}^{[R,2]}$ s.t.

$$R = q^{\frac{d^2-1}{4}} q^{-\frac{d-1}{2}} \left(1+\sum_{j\geq 1} h^j \partial_a^j \sum_{k\geq 0} h^k\, T_{j,k}^{[R,1]}(\partial\varepsilon_1, \partial\varepsilon_2+\partial\varepsilon_{12})\right)$$
$$\left(1+\sum_{j\geq 1} h^j\, T_j^{[R,2]}(\partial\varepsilon_1,\partial\varepsilon_{12})\right)\left(1+\sum_{j\geq 1} h^j \frac{\prod_{0\leq l\leq j-1}(\partial\varepsilon_2 - l)}{j!}\right)$$

$$\check{R}[a,\varepsilon_1,\varepsilon_2,\varepsilon_{12}]\Big|_{\substack{a=1-q^{-d}\\ \varepsilon_1=\varepsilon_2=\varepsilon_{12}=0}}$$ gives the R matrix.

Define:

- $V_{d,\infty}^{\otimes N} \leftarrow f_0, f_1, \dots, f_{d-1}, f_d, \dots$ where you add the f_j, $j > d-1$
- $V_N^{(1)} \cong \mathbb{C}[z_1,\dots,z_N]$, $\forall i$ consider $\mathbb{C}^2 \cong \mathbb{C}[z_i, z_{i+1}]$ and take the restriction of $\check{R}$ to this $\mathbb{C}^2$, called $\tilde{\check{R}} = \begin{pmatrix} e^{\varepsilon_1+\varepsilon_2}a & e^{\varepsilon_2}q^{-d} \\ e^{\varepsilon_1} & 0 \end{pmatrix}$

If you set $q^{-d} = t$, $a = 1-q$ $\xrightarrow{\varepsilon_1 = \varepsilon_2 = 0}$ $\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$ which is the Burau rep.

Using this linear algebra lemma you get the Alex. poly..

Lemma: O operator on the poly algebra $\mathbb{C}[z_1, \ldots, z_N]$ coming from $\tilde{O}$ on $\mathbb{C}[z_1] \oplus \mathbb{C}[z_2] \oplus \cdots \oplus \mathbb{C}[z_N]$. If spectrum $\lambda \tilde{O} \in \mathring{D}^2$ for some $\lambda \in \mathbb{C}$, Then
$$\sum_{n \geq 0} \lambda^n \operatorname{Tr}_{V_N^{(n)}} O = \frac{1}{\det_{V_N^{(1)}} (1 - \lambda \tilde{O})}.$$

Thursday 27 February 2020

- $\lim_{d\to\infty} \frac{\log|V_d(K; e^{\frac{2\pi i}{d}})|}{d} = \lim_{d\to\infty} \frac{\log|A_d(K,t)|}{d^2}$ ← Zeta function

Two important Zeta function:

Riemann

$\zeta(s) = \sum_1^\infty \frac{1}{n^s}$

$z = q^{-s}$

Weil

X nonsingular projective n-dim variety on $\mathbb{F}_q$

$\zeta_X(z) = \exp\left(\sum_{m=1}^\infty \frac{z^m}{m} N_m\right)$

pts of X defined over $\mathbb{F}_{q^m}$

★ Riemann Zeta Function ★

1. Functional Equation

$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$

2. Euler product: $\sum_{n=1}^\infty n^{-s} = \prod_{\text{primes}} (1-p^{-s})^{-1} = \frac{1}{1-p_1^{-s}} \cdot \frac{1}{1-p_2^{-s}} \cdots$

3. Riemann Hypo: Trivial zeros $s = -2, -4, \cdots$ (you have to do analytic extension to see this)

nontrivial zeros: all have $\mathrm{Re}(s) = \frac{1}{2}$

★ Weil Zeta Function

1. Rationality (done by Dwork) $\zeta_X(z)$ is rational

$\zeta_X(s) = \prod_{i=0}^{2n} P_i(z)^{(-1)^{i+1}}$ and $P_i(z)$ is a poly

$P_0(z) = 1 - z \qquad P_{2n}(z) = 1 - q^n z$

$P_i(z) = \prod_j (1 - \alpha_{ij} z)$ just factorization by the zeros

Deligne proved that α_{ij} satisfy a version of Riemann Hypo

where $|\alpha_{ij}| = q^{\frac{1}{2}}$

2. Functional Equation $\xi_X(q^{-n}z^{-1}) = \pm q^{n\chi/2} z^{\chi} \xi_X(z)$ ③

Euler characteristic

Grothendieck built motif cohomology to prove Riemann Hypo for X but Deligne proved it using p-adic cohomology without Grothendieck motif Cohomology.

• The question we are interested is what is the analog of all these for Colored Jones or the (twisted) Alex. poly. ? Most likely related to Weil instead of Riemann.

	$\frac{1}{\Delta_K(t)}$	$V_d(K;q)$
• rationality	✓	✓ ⟶ as both are rational already
• Functional Equation	?	Probably Zagier's modularity
• Riemann Hypo	?	?

"B. Mazur has some ideas" in "Knots, Primes & P."

• Zagier's modularity conjecture: K knot ⟷ prime p in $\mathbb{Z}$

Given a knot K, define $J_K : \mathbb{Q}/\mathbb{Z} \longrightarrow J_c(K; e^{2\pi i \frac{a}{c}})$

$\{ a/c \mid (a,c)=1 \}$

So every knot gives a function on rationals.

But $SL(2,\mathbb{Z})$ acts on $\mathbb{Q}/\mathbb{Z}$ by $x \in \mathbb{Q}/\mathbb{Z}$: $\gamma(x) = \frac{ax+b}{cx+d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Conjecture: Define $\hbar = \frac{2\pi i}{x + \frac{d}{c}}$, coming from $\varphi : \pi_1(S^3 \setminus K) \to PSL(2,\mathbb{C})$ of the hyp structure, CS invariant

$$J_K(\gamma(x)) = J_K(x) \cdot \left(\frac{2\pi}{\hbar}\right)^{3/2} \cdot e^{i(Vol + i CS)/\hbar} \cdot [\text{const} + \text{h.o.t}]$$

hyp knot

$x = \frac{a_x}{c_x}$ ‖ $J_{c_x}(K; e^{2\pi i x})$

higher order terms in $\hbar$.

Note This implies Volume Conjecture by choosing $x = N-1$, $\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. (3)

- For the rest we will try to argue what functional equ. & R-H. are for the $\frac{1}{\Delta_K(t)}$.

- Recall Milnor's Theorem $\frac{1}{\Delta_K(t)}$ is a Weil zeta function.

$K \subset S^3 \quad E_K = S^3 \setminus K, \quad \tilde{E}_K^{Ab} \leftarrow \pi_1(\tilde{E}^{ab}) = [\pi, \pi]$

$\mathrm{Ker}\, Ab \longrightarrow \pi_1(E_K) \xrightarrow{Ab} \mathbb{Z}$

$\cong [\pi, \pi] \qquad \| \; \pi$

Then $\mathbb{Z}$ acts on $\tilde{E}_K^{Ab}$ as covering transformations

generator $1 \mapsto t : \tilde{E}_K^{Ab} \circlearrowleft$

$\frac{1}{\Delta_K(t)}$ is called the torsion of a knot.

$\tau(\lambda) = f_0(\lambda) f_1(\lambda)^{-1} f_2(\lambda) f_3(\lambda)^{-1}$ where $f_i(\lambda) = \det(\lambda I - t_*^{(i)})$ is the characteristic poly of $t_* : H_i(E_K^{ab}; \mathbb{Q}) \circlearrowleft$

Then we can set up the "Weil" zeta function:

$$\zeta_K(z) = \exp\left(\sum_{n=0}^{\infty} \frac{z^n}{n} L(t^m)\right)$$

where $L(f) = \sum_{i=0}^{\infty} (-1)^i \mathrm{Tr} f_{*i}$

where $f_{*i} : H_i(M; \mathbb{Q}) \circlearrowleft$

for diffeo $f: M \to M$

Milnor's thm: $\tau_K(z)\, \zeta_K(z^{-1}) = z^X$

and $\frac{1}{\Delta_K(\lambda)} = \tau_K(\lambda)$ so $\frac{\zeta_K(z^{-1})}{z^X} = \Delta_K(z)$

- A. Terras (H. Stark) : Garden of zeta functions

Given any finite oriented graph Γ, you can define

$\zeta_\Gamma(z)$: Count primitive loops with weight $e^{-s,\,\mathrm{length}}$

Count primitive loops with weight $e^{-s\,\mathrm{length}}$

Tuesday 3rd March 2020

Zeta function approach to V.C.

	Riemann & Weil	Colored Jones	Twisted Alex
• Rationality	✓	✓	✓
• Functional eq.	✓	Modularity conj	?
• Riemann Hypothesis	✓	?	?

Grothendieck Philosophy, essence of Zeta function?

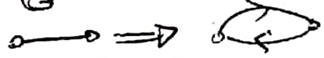
⇓

counting with weights

We will start with Ihara-Selberg zeta function of graphs $\xrightarrow{\text{then}}$ knot diagrams
(J.P. Serre)

Given a finite unoriented graph G
Let G^+ = be the doubled oriented graph

G $\qquad$ G^+



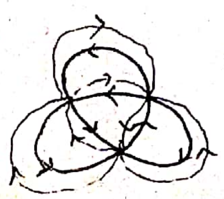
Ihara Zeta function $\zeta_G(u) = \prod_{p \in PG^+} (1 - u^{|p|})$ — u some variable

• $p \in PG^+$ = The set of all primitive, reduced cycles on G^+

$\nexists$ path x s.t. $p = x^n$ $\qquad$ $e_i e_{i+1}$, $e_{i+1} \neq \bar{e}_i$

• $|p|$ = length = # of edges in p

$\zeta(u) = (1-u^3)(1-u^3)$



here you have $|PG^+| = \infty$

So $\zeta_G(u)$ is an infinite product

Bass evaluation: $\zeta_Q(u) = \det(1 - uT)$ where T is a finite matrix! (2)

So The infinite product ultimately gives something finite.

Def. T: Let $V(E) = \mathbb{C}$-span of all edges in G^+

Let $J: V(E) \to V(E)$

$e_{ij} \to \bar{e}_{ij} = e_{ji}$

Succession map $Succ: V(E) \to V(E)$

$e_{ij} \to \sum_{e_{jl} \neq e_{ji}} e_{jl}$

$$\boxed{\text{And then define } T = Succ - J}$$

Example:

1D Ising model: Given lattice S, write $\zeta_S(z,\beta) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(S)\right)$

$Z_n(S)$ — partition of model of size n

Fact: This is rational and equal to $\frac{1}{\det(STH)}$ → determinant of something related to Transfer matrix T.

X. L. Liu and Z. W.

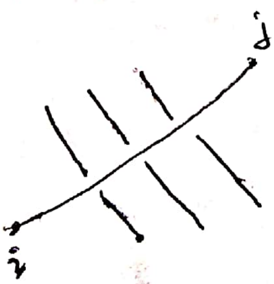
(Random walks on Knot diagrams

⟹ a new proof of The Melvin-Morton Conjecture

Remark by V. Jones: The Burau representation has an interpretation as bowling on braids.

Burau rep $\varphi: B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$

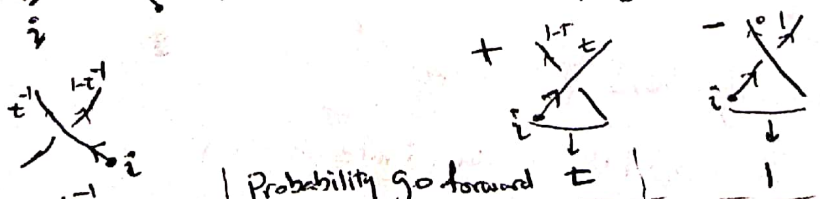
Given $b \in B_n$



$\varphi(b)_{ij} = \sum_{\substack{\text{all paths } p \\ i \to j}} w(p)$, $1 \le i, j \le n$

p goes upwards

$w(p) = \prod$ weights at each crossing



Note $1-t^{-1}$ or $1-t < 0$ but here we are doing formal calculus.

t^{-1}	Probability go forward	t	1
$1-t^{-1}$	" go down	$1-t$	0

R-matrix gives rise to Jones Poly

$$R = q^{\frac{1}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-q^{-1} & q^{-\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = q^{-\frac{1}{4}} \begin{pmatrix} q^{\frac{1}{2}} & & & \\ & q^{\frac{1}{2}}-q^{-\frac{1}{2}} & 1 & \\ & 1 & 0 & \\ & & & q^{\frac{1}{2}} \end{pmatrix}$$

$$\xrightarrow{q \to q^2} q^{-\frac{1}{2}} \begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} \leftarrow \text{Call this } R_J$$

we shall replace q by q^2.

define also $R_A = \begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & -q^{-1} \end{pmatrix}$

R_J gives Jones, R_A gives Alex. Note how close they are.

$R_A : \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft$ write $\mathbb{C}^2 = \mathbb{C}|i\rangle \otimes |j\rangle$ then labels on matrix

Rows & columns are

$$\begin{array}{c|c} & 00 \; \cdots \; 11 \quad (i,j \in \{0,1\}) \\ \hline 00 & \\ \vdots & \begin{pmatrix} & & \\ & & \\ & & -q^{-1} \end{pmatrix} \\ 11 & \end{array}$$

This says interchanging two fermions gives $-q^{-1}$ instead of q. Also note the "phase factor" $q^{\frac{1}{2}}$ for R_J.

Another fact: $R_{J,A} - R_{J,A}^{-1} = (q-q^{-1}).Id$

Get link invariants from enhanced R-matrix:

Both R-matrix give rise to rep of B_n

$\varphi_J : \sigma_i \in B_n \longrightarrow id \otimes \cdots \otimes \underset{i\text{-th place}}{R_J} \otimes \cdots \otimes id$

$\varphi_A :$ similar to above

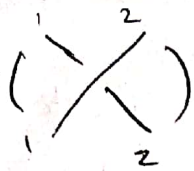
For Jones $J(\hat{b}; q) = \underset{\text{normalization}}{?} \, Tr(\mu^{\otimes n} \varphi_J(b))$

For Alex $\Delta(\hat{b}; q) = ? \; \underset{\text{supertrace}}{Str} (\mu\mu^{+} \varphi_A(b))$

fermion

the odd ones pick up a negative sign in the trace

supertrace $\longrightarrow$ a basis of $(\mathbb{C}^2)^{\otimes n}$ is a length n-bit string $|I\rangle = |i_1 \cdots i_n\rangle$. Sum Σi_j

• For example: $\varphi\Bigg($  $\Bigg) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1-t & t \\ 2 & 1 & 0 \end{array}$

④

• Claim: Burau rep is always reducible. Since $\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ is an eigenvector.
due to preservation of probability $\varphi(b)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$.

So define the reduced Burau by restricting to the space spanned by $\left\{ v_i = \begin{pmatrix}0\\ \vdots\\ -t\\ 1\\ 0\\ \vdots\end{pmatrix} \leftarrow \text{i-th place} \right\}$. This is invariant subspace.

$\tilde{\varphi}(b)$ = reduced Burau, Then define $\Delta(\hat{b}) = \dfrac{\det(1-\tilde{\varphi}(b))}{1+t+\dots+t^{n-1}}$ is the Alex. poly.

if we were to take $\varphi(b)$ then det above would be zero.

• Extend the Burau reps to String links

by using the exact same definition.

String links: n disjoint arcs in $\mathbb{R}^2 \times [0,1]$ relative to $\mathbb{R}^2 \times 0$, $\mathbb{R}^2 \times 1$

it is a semi-group.

examples

FACTS:

1) $\varphi(L) = 1$ for any 1-string link because no matter how the bowling goes, it will end up in the same place as there is only one string, with one end and beginning.

$\varphi(\ ) = \dfrac{1}{2-t}\begin{pmatrix} 1 & 1-t \\ t^{-1}-1 & 3-t-t^{-1}\end{pmatrix}$

2) Each entry in $\varphi(L)$ is a rational function of t.

We want to generalize this idea to Jones poly.
random walk / state-sum models of quant invariants

Amitsur-Levitzki identity: Take any $2n$ matrices $A_1, \dots, A_{2n}$ of size $n\times n$

$$\sum_{\sigma \in S_{2n}} \pi(\sigma)\, A_{\sigma(1)} \cdots A_{\sigma(2n)} = 0$$

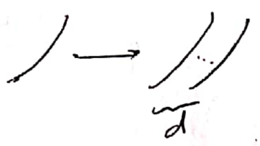
$\pi(\sigma)$: parity of permutation

For $n=1$ → it says complex numbers commute

Zeta function approach to MM Conjecture:

Given a knot K

$$J_d(K; q) \cong J(K^{\otimes d}, q)$$



$$R = q \quad \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & q & & & \\ 01 & & 1-q^{-2} & q & \\ 10 & & q & 0 & \\ 11 & & & & \pm q \end{array}$$

notice how this is like the bowling situation.

Thursday 5^{th} March 2020 ①

Zeta function approach to V.C.

	Colored Jones	Twisted Alex Poly
• Rationality	✓	✓
• Functional eq.	Zagier's modularity Conjecture	Symmetric
R.H.	?	? ← R.H. for Ihara Zeta function

Recall Ihara zeta function

G finite connected unoriented graph

G^{+} = doubled G 

Zeta function: $\xi_G(u) = \prod_{\substack{\text{primitive} \\ \text{reduced} \\ \text{cycles } P}} (1-u^{|P|}) \overset{\text{Bass}}{=\!=} \det(I-uT)$

$|P|$: length P

G regular d-graph:  d-edges like $d=4$ for knot diagrams

R.H.: $\xi_G(u)$ has a R.H. (ie every zero with $0 < Re(s) < 1$, then $Re(s) = \frac{1}{2}$) if G is Ramanujan.

Ramanujan: Connected d-regular for which $-d \leq$ spectrum $\leq d$ with $\lambda_1 = d$. Further $\max_{i=2,\dots,n}(|\lambda_i|) \leq 2\sqrt{d-1}$

(Correspondence is $u = q^{-s}$, $q = d-1$)

Zeta function approach to Melvin-Morton Conjecture (X.S. Liu and Z.W)

Side remark. $A: V \to V$, $\dim V < \infty$. Then ① $\det(e^{A}) = e^{trA}$

② $\det(I-A) = \sum_{i \geq 0} (-1)^i Tr(\Lambda^i A)$

extend to exterior product $\Lambda^i A : \Lambda^i V \to \Lambda^i V$

Claim $\dfrac{1}{\det(I-A)}$ should be regarded as a zeta function (we will define A later)

define zeta function of A. $\zeta_A(z) = \exp\left(\sum_{n\geq 1}^{\infty} \frac{z^n}{n} \operatorname{tr}(A^n)\right)$

Note that

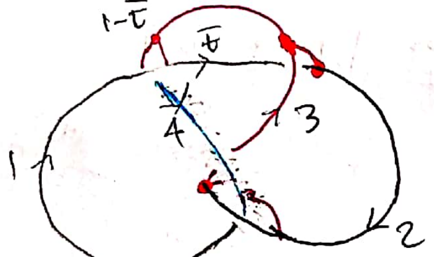
why? $-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \rightsquigarrow \frac{1}{1-x} = \exp \sum \frac{x^n}{n}$

$\Rightarrow$ So for diagonal matrices it is true that ~~det~~ $\dfrac{1}{\det(1-zA)} = \exp\left(\sum \frac{z^n}{n} \operatorname{Tr}(A^n)\right)$

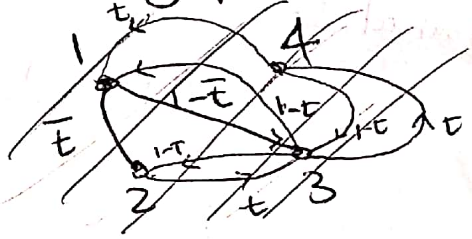
and by Jordan decomposition the rest follows.

Recall Wirtinger presentation for $\pi_1(S^3 \setminus K)$

define a walk graph, vertices = arcs colored

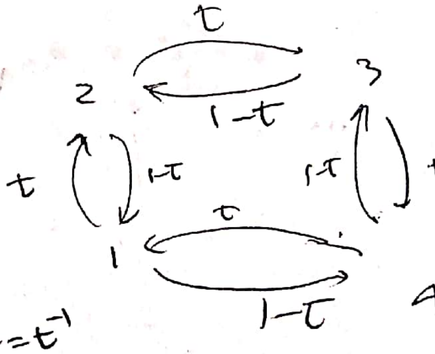


W_D



$\tilde{B} =$

	1	2	3	4
1		$\bar{t}$		$1-\bar{t}$
2	$1-t$		t	
3		$1-\bar{t}$		$\bar{t}$
4	t		$1-t$	



?

where $\bar{t} = t^{-1}$

We claim determinant of $I - B_{(ij)}$ = Alex poly where $B_{(ij)}$ is delete i-th row, j-th column of B. For example $I - B_{11} = \begin{pmatrix} 1 & -t & 0 \\ \bar{t}-1 & 1 & -\bar{t} \\ 0 & t-1 & 1 \end{pmatrix}$

gives determinant $= 3 - t - \bar{t} = 1 - z^2$ for $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$

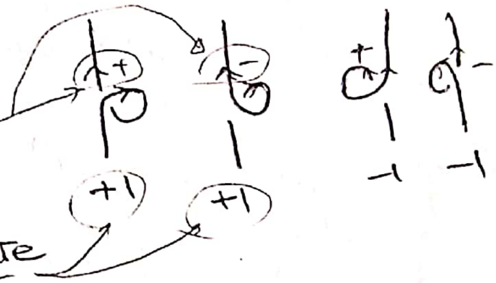
What does it count? They should count "closed orbits" ← count closed geodesics.

delete one row and one column

which corresponds to This →

Knot ⟷ 1-string link

Side remark: For Types of kinks in The plane



we associate +, - based on the crossing. We also associate a rotation number

So we have labeling ++, -+, +-, --

We choose the 1 string using ++ Kink To connect and we do some random walk that do not touch the red part

++

arXiv 98.12039
Foata, Zeilberger: Combinatorial proof of Bass's evaluation

Thm. Given a 1 string T as above

① $\frac{J(K_T)}{q+q^{-1}} = t^{wr(T)-rot(T)}\left(1+\sum_{k=1}^{\infty}\sum_{C=(C_1,\dots,C_k)\in Q_S^k} t^{rot(C)-B_1(C)}\, w(C_1)\cdots w(C_k)\right)$

$t=q^{2}$

② $\lim_{d\to\infty} \frac{J_{d+1}(K, e^{h/d})}{[d+1]} = t^{\frac{rot(T)}{2}}\left(1+\sum_{k=1}^{\infty}\sum_{(C_1,\dots,C_k)\in Q^k} w(C_1)\cdots w(C_k)\right)$

$= t^{-\frac{rot(T)}{2}} / \det(I-B)$ ← Alex. poly

• The nontrivial part in the proof is why $\frac{1}{\det(I-B)} = 1 + \sum_{k} \sum_{(c_1 \ldots c_k) \in Q^k} w(c_1) \ldots w(c_k)$

Need to define Lyndon words: Let A be finite alphabet set totally ordered $A = \{0, 1, 2, \ldots, n\}$. Let A^* = all words including $\emptyset$.

Case we are interested in is $A = \{0, 1\}$.

<u>Def.</u> a Lyndon word is a non-empty word which is
① not a power ② <u>minimal</u> in its cyclic class
$\hookrightarrow$ in lexicographic order

e.g. $0, 1, \cancel{00}, 01, \cancel{10}, \cancel{11}, \cancel{000}, 001, \cancel{010}, \cancel{100}, 011, \cancel{101}, \cancel{110}, \cancel{111}, \ldots$

Theorem: $\beta(\Lambda) = \det(1 - B)$ for any finite matrix B.

where $B = (b_{ij})_{i,j=0}^{n-1}$ and b_{ij} as a set of commuting variable. Let $A = \{0, 1, \ldots, n-1\}$ with L_A = all Lyndon words in $A^* = \coprod_{n=0}^{\infty} A^n$ and $\Lambda = \prod_{\ell \in L_A} (1 - [\ell])$ is a formal commuting variable.

$[\ell]$ ↑ variable labeled by $[\ell]$

Finally $\beta(\underbrace{a_1 \ldots a_n}_{\text{a Lyndon word}}) = b_{a_1 a_2} b_{a_2 a_3} \cdots b_{a_n a_1}$

So $\beta : \mathbb{Z}[\{[\ell]\}_{\ell \in L_A}] \longrightarrow \mathbb{Z}[\{b_{ij}\}]$

↑ infinite generated algebra

e.g. $B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$ $\qquad \det(I-B) = (1-b_{00})(1-b_{11}) - b_{01}b_{10}$

$$\beta(\Lambda) = (1-b_{00})(1-b_{11})(1-b_{01}b_{10})(\ldots$$

Turns out everything else cancels!

$\beta(0) = b_{00}$

$\beta(1) = b_{11}$

$\beta(01) = b_{01}b_{10}$

• Going back to the Theorem $\quad T:$

② For Alex. poly
Look for finite simple cycles $= Q$

then $(c_1, \ldots, c_k) = Q^k$

and Q^k's $\subset Q^k$ are the ones that do not share any edges. Then apply $\beta(\Lambda) = \det(I-B)$.

① For Jones poly: R-matrix $q^* \begin{pmatrix} q & & & \\ & q-\bar{q} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} = q^* q \begin{pmatrix} 1 & & & \\ & \overset{1-t}{1-q^{-2}} & \overset{t}{qq^{-2}} & \\ & qq^{-2} & 0 & \\ & & & 1 \end{pmatrix}$

Tuesday 10th March 2020

- No lecture on Thursday

1. 4-dim Topology (3+1)-D TQFT

2- VC: Configuration approach

Top interpretation of Jones poly

Braid groups: fundamental group of n points in D^2

n distinct pts, unordered

- define Config. space $C^n(X) = (D^2)^n \setminus \Delta$

where $\Delta = \{ (x_i) \mid x_i = x_j, \exists i \neq j \}$

it is well-known $\pi_1(C^n(D^2), *) \cong B_n$

- πM of a manifold M for $n=1,2,3,4$ is a homotopy invariant of M

So not that useful.

- Therefore we replace πM by $C^2(M)$ which is not a topological invariant. In general $X_1 \overset{hom}{\simeq} X_2 \not\to C^n(X_1) \overset{hom}{\simeq} C^n(X_2)$

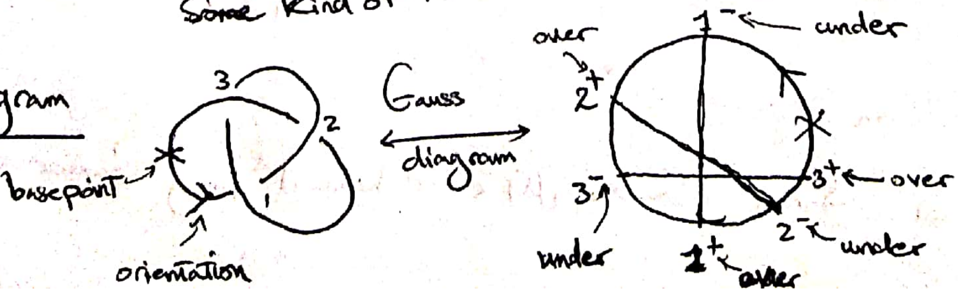
$pt \simeq D^2$ and $C^n(pt) \not\simeq C^n(D^2)$
$\qquad\qquad\qquad\quad \overset{\shortparallel}{pt}$

Volume Conjecture: For a hyperbolic knot $K \subset S^3$, Colored Jones Poly $J_d'(K;q)$ (normalized J_d)

then
$$\lim_{d \to \infty} \frac{\log |J_d'(K; e^{2\pi i/d})|}{d} = \frac{1}{2\pi} \mathrm{Vol}(S^3 \setminus K)$$

Some kind of zeta functions of graph

Knot diagram

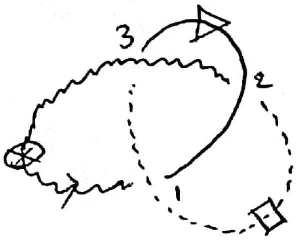


Two graphs: 1) U_D :  The Universe graph

2) Walk graphs $W_D^{\pm}$: + choose all + pts in Gauss diagram

− " " − " " ,

Then turn each over arc into a vertex



From each vertex v two edges.

The reason to introduce these: statistical mechanics on knot diagrams/

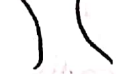
State-sum construction.

① Kauffman ⟶ a state on a knot diagram is an assignment of A and B to each crossing ⤳ $2^{\#\text{ every states}}$

② Turaer

For every states, define $\langle S\rangle = \sum A^{\#\text{ of A's in S}} B^{\#\text{ of B in the states}}$ $\left(d^{\#\text{edges}}\right)$

Given a crossing resolve all crossing

$\xrightarrow[\text{rotate counter clockwise sweep out 2 of 4 regions}]{\text{start with over crossing}}$ 

Jones poly

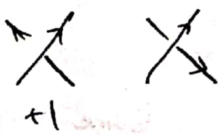
define $J(K_D, A) = (-A)^{-3w(K_D)} \sum_s \langle s\rangle$

writhe index

replacing

Similarly resolve (clockwise) 

⟶ 



$B = A^{-1}$

$d = -A^2 - A^{-2}$

$q = A^{-4}$ (or A^{-2})

② Turaer State sum

to get quant invariant find (R, α, β, μ) ← enhanced "Yang-Baxter operator"

$R =$

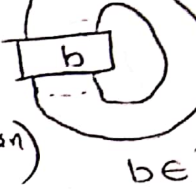
	00	01	10	11
00	$-q$			
01		$\bar{q}-q$		
10		1	1	
11				$-q$

$: (\mathbb{C}^2)^{\otimes 2} \circlearrowleft$

basis $|i\rangle|j\rangle$ $i,j \in \{0,1\}$

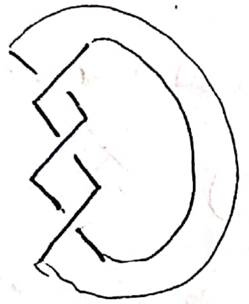
$\bar{q} = q^{-1}$, $\alpha = -q^2$, $\beta = 1$, $\mu = \begin{pmatrix} \bar{q} & 0 \\ 0 & q \end{pmatrix}$

<u>Theorem</u>: $J(K,q) = \alpha^{-w(D)} \beta^{-n} \operatorname{Tr}(\varphi_R(b) \cdot \mu^{\otimes n})$ $K =$



$= (-q^2)^{-w(D)} \operatorname{Tr}(\varphi_R(b) \mu^{\otimes n})$ $b \in B_n$

Trefoil :



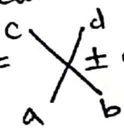
$b \in B_2$
$\shortparallel$
σ_1^3

$\Rightarrow$ <u>Always leads to a state sum</u>

Take any knot diagram, a state on D is an assignment of $\underline{0}$ or $\underline{1}$ to each edge of U_D

① Weight of a state $\pi(s) = \prod_{\text{all vertices}} (R^{\pm})^{c,d}_{a,b}$

$v =$

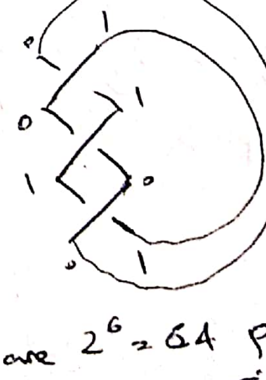


$\pm$ crossing

② a state is admissible if $\pi(s) \neq 0$

<u>Thm</u> : $J(K,q) = (-q^2)^{-w(K_D)} \sum_{\substack{\text{all admissible} \\ \text{states } S}} q^{\operatorname{rot}(\bar{s}) - \operatorname{rot}(s)} \pi(s)$, defined later

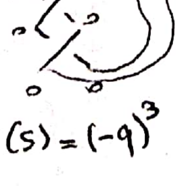
$\pi(s) = (\bar{q}-q)^3$



$\pi(s) = (\bar{q}-q)^2 \cdot 1$

looking at nonzero R entries to see admissability, <u>Examples:</u>

These two must be the same



$\pi(s) = (-q)^3$

Note There are $2^6 = 64$ possibilities but admissability dramatically drops this number.

Rotation number rot(S) invented by whitney

draw the immersed curve which is in general position (not having any ✱)

The Tangent vector

$\xrightarrow[\text{a circle}]{\text{Map it To}}$ S^1 and look how many times you loop around circle Called the rotation number

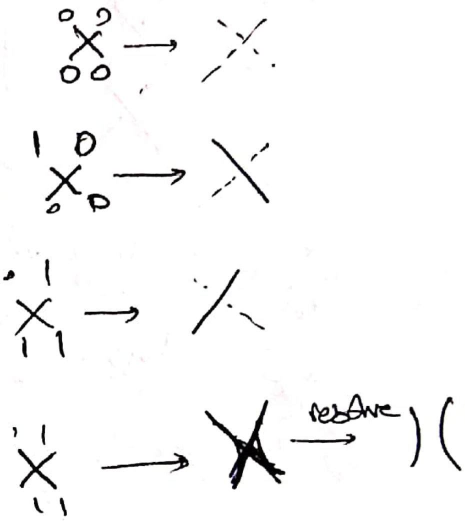
$q^{rot(\bar{S}) - rot(S)}$

Complement of S: $0 \leftrightarrow 1$, $1 \leftrightarrow 0$

Now Take S and replace 0 labeled arcs by ⋱ and 1 labeled ones by \, e.g

Then you follow only \ (after resolving) arcs along the Knot. If

- it's clockwise -1
- .. .. anti .. $+1$



⟹ Jones poly counts disjoint cycles of random walks which essentially implies The MM conjecture.