GRADING & OFFICE HOURS

Instructor: Zhenghan Wang.
Class: MWF: 11-11.50.
No Hw, no Final, but there will be exercises.
Office hours: By appointments (send an email).

REFERENCES

- Gannon, Terry: The theory of vector-valued modular forms for the modular group, \[\text{arxiv:1310.4458}\]. This will be the main reference in the first half of the course.
- J.P. Serre: Chapter 7 of \[\text{A Course in Arithmetic}\]. We will try to use the notations of this reference.

SOME COMMENTS ON THE NOTES (BY MOJI)

Some of the materials linked to in these notes are to appear in a dissertation (section April 8-10-12th); Please do not share it outside of the class.

I will try to tag as many useful links (Wikipedia mostly!) as I can to important notions. So hover your mouse on those and you might see that it can get you to a website with more explanations. I will try to write what was said during the lectures and add not much to it in their respective section. But sometimes, there will be additional materials (usually a very rough overview of a notion or a proof or some reference) at the end of the notes for each section with hyperlinks provided. I will also try not to add something in advance about what will be explained later in the lectures; for example, I will refrain from adding anything about MTC until after the lectures on MTC.

OVERVIEW OF THE STRUCTURE OF THE COURSE

First week will be on what we will do on the first half of the course and what the subject is about, then next week Colleen Delaney will give the math background on Modular Tensor Categories (MTCs) to get started. First half is on Vector-Valued Modular Forms (vvmf) which will be a kind of infinite-dimensional linear algebra, and it might be the whole quarter! Time permitting, the second part of the course will be on (1+1-)Chiral Conformal Field Theory (χ-CFT) explained in the language of Vertex Operator Algebras (VOAs) and how one can get from a VOA to a vvmf.

1. APRIL 1ST

1.1. Motivations behind the topics. There is something in mathematics called Tannaka-Krein duality. Let us explain the simplest case of this duality:

Suppose \( G \) is a finite group. We have a (linear) representation category \( \text{Rep}(G) \), which objects are homomorphisms \( \rho: G \to U(r) \) with range unitary \( r \times r \) matrices. The range can be replaced by the group \( GL \) but it is a theorem that one can embed it into \( U(r) \). The morphisms between the objects are intertwiners.
Representation category is very concrete because elements are actual matrices instead of the abstract algebra. It is easier to study but not faithful as we might lose some information when you go from $G$ to $\text{Rep}(G)$. So suppose $\text{Rep}(G)$ is known, how can we or can we recover $G$? The answer is almost yes, we can. The answer is that given a symmetric fusion category, we have a corresponding (super)group $(G, \mu)$ where $\mu \in Z(G)$ and $\mu^2 = 1$. This is related to (mathematical version of) the fact that fundamental particles are either fermions or bosons. The proof of this is spread out in two papers by Deligne, and you can find a review here by V. Ostrik (see Additional Material).

In order to generalize this, what we want to do is to replace the representation category by a Modular Tensor Category (MTC) and replace the group by something called a Vertex Operator Algebra (VOA). Of course, we might need some adjectives like rational, etc. similar to the previous case.

There is also a motivation from physics to pursue this. The physical version of above is similar to the correspondence called Ads/CFT; We have a bulk $(2+1)$-TQFT and on the boundary there is a $(1+1)$-CFT. People are looking to establish this correspondence mathematically:

$$(1+1)\text{-CFT Boundary theory (Edge) } \iff (2+1)\text{-TQFT Interior theory (Bulk)}.$$ 

This correspondence is well-established in the case of Fractional Quantum Hall liquid states (see here for an overview) but it is conjectured to be true for any topological phases of matter. We can only say that the correspondence starts from representing TQFT on the inside by an MTC and the chiral CFT by a VOA which representation category is the MTC. More data is needed to get from MTC to a unique VOA (for example, the central charge).

The third motivation is classification of MTCs. Classification of algebraic structures ((super)groups) leads to a classification of category-theoretic structures (symmetric fusion categories). Similarly, a better knowledge of CFTs should shed some light on the structure of MTCs as well.

1.2. Theory of vector-valued modular forms (vvmf).

**Definition 1.** Throughout the course, there will be three major groups.

- The full (or homogeneous) modular group $SL(2, \mathbb{Z}) = \{(a b, c d) | ad - cb = 1, a, b, c, d \in \mathbb{Z}\}$ denoted by $\Gamma$.
- The modular group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm I\}$ denoted by $\hat{\Gamma}$.
- The 3-strand braid group $B_3 = \langle a, b \mid aba = bab \rangle$, where $a, b$ are the first and second braid generators. Its center is given by $Z(B_3) = \langle (ab)^3 \rangle$ which is the full twist. Taking the quotient $B_3/Z(B_3)$ gives $\hat{\Gamma}$. $B_3$ is in fact the universal central extension of $\hat{\Gamma}$.

The connection of these groups to MTCs is briefly as follows: MTC gives us $(2+1)$-TQFT (Tu-
raev theorem) and it is a theorem that this gives representations of mapping class groups (MCGs) where we will be interested in the case of torus which MCG is $SL(2, \mathbb{Z})$, and in the case of 3-punctured disk which MCG is $B_3$.

In $\hat{\Gamma}$, we find two generators: $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $U = ST^{-1}$ where $U^3 = 1$.

In physics, the convention is to call $\rho(S) = s$, $\rho(T) = t$ for a representation $\rho$, but we will refer to both by the same notation, sometimes $s, t$ and sometimes $S, T$. The important identities are:

$$(ST)^3 = S^2 = -I \implies S^4 = I.$$
Now inside $\Gamma$, we know $-I$ is identity, so $S^2 = I$. As to the concrete relation to $B_3$, there is a way of writing $s, t$ as braid generators. One can also write $\Gamma$ as the free product
\[ \Gamma = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle x, y | x^2 = 1, y^2 = 1 \rangle, \]
where $x = st$ and $y = s$. The representations of $\Gamma$ is therefore easy to get as we need a matrix of order 2, 3 and there are no other relations we need to satisfy.

To be thorough, given any $A, B \in U(r)$, $A^2 = 1, B^3 = 1$, there exists representation of the modular group $\rho$ which sends $ST \to B, S \to A$. So why should we even even care about these reps if they are so trivial? Consider $\rho_{A,B}: \Gamma \to U(r)$ to be that representation. Then we get an important subgroup of $\Gamma$: the kernel of $\rho_{A,B}$. Subgroups of the (full) modular group are of particular interest.

**Definition 2.** A subgroup $H$ of $\Gamma$ is a congruence subgroup (for short, congruent) if $\Gamma(N) \subset H$ for some $N \in \mathbb{N}$, where $\Gamma(N) = \{ X \in \Gamma | X \equiv I \pmod{N} \}$. If $N = 1$, $\Gamma(N) = \Gamma$. $\Gamma(N)$ are called the principal congruence subgroups.

A huge subject in number theory is whether a given subgroup is congruent or not (Zhenghan thinks that there is a theorem which says almost every subgroup is non congruent). If it is congruent, then there is a theory of modular forms attached to it.

Here is the main theorem (roughly) which ties MTC with this notion:

**Theorem 1.1. Ng-Schauenburg Congruence Subgroup Theorem:** Given an MTC $B$, we get a representation $\rho_B: SL(2, \mathbb{Z}) \to U(r)$ where $\ker(\rho_B)$ is congruent.

There are some subtleties and conditions, but this is the gist of it. Let us now define admissible representations, the ones we are interested in, and vvmf:

**Definition 3.** The representation $\rho$ is admissible if $\rho$ is irreducible and $\rho(T)$ is diagonal and of finite order.

Given an admissible $\rho$, define the vector space
\[ M_\rho = \{ f : \mathbb{H} \to \mathbb{C}^r | f(\gamma.\tau) = \rho(\gamma)f(\tau), \forall \gamma \in \Gamma, f \text{ holomorphic on } \mathbb{H}, \text{ meromorphic at } i\infty \}. \]
where $r$ is the same used in $\rho: \Gamma \to U(r)$, and $\mathbb{H} := \{ z \in \mathbb{C} | \text{Im } z > 0 \}$ the upper-half plane, and the action of $\Gamma$ on $\mathbb{H}$ is:
\[ \gamma.\tau = \frac{a\tau + b}{c\tau + d}, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]
It can be checked that this is indeed an action on $\mathbb{H}$ as the imaginary part of $\gamma(\tau)$ is $\frac{\text{Im } \tau}{|c\tau + d|^2}$. Finally, $f = (f_1, \ldots, f_r)$ being holomorphic on $\mathbb{H}$ (meromorphic at cusp $i\infty$) means each $f_i$ is holomorphic on $\mathbb{H}$ (meromorphic at the cusp).

**Definition 4.** (Incomplete definition of vvmf) Any $f \in M_\rho$ is called a vector valued modular form.

The trivial case is $r = 1, \rho \equiv 1$ for which $M_\rho$ is given by the famous $j$-function of Klein which is
\[ j(\tau) = q^{-1} + 744 + 196884q + \ldots, \text{ where } q = e^{i2\pi \tau} \]
and the observation leading to the famous moonshine story was that $196884 = 196883 + 1$![Go to Additional Material].
A note on the notations: \( \tau \) has always image \( > 0 \) and \( q = e^{i2\pi \tau} \) and \( z \) is always a general complex number. Notice using the change of variable from \( \tau \) to \( q \), lets us define the function \( f \) on the open punctured disk at origin as \( q = e^{i2\pi \tau} |_{\tau = \infty} = 0 \). The function \( f \) is then meromorphic at the origin in the disk picture.

2. April 3rd

Classroom change from Friday \( \rightarrow \) South Hall 1607 (Mathlab).

Suppose we have an MTC \( B \). We can attach modular forms to this. But ultimately from MTC, we think we should get a CFT. The question is what additional data we should add to get a CFT? Zhenghan thinks (and thinks it will be wrong but) very little data. First, it would be better to know what is the data which characterizes an MTC? The modular data \((S, T)\) is very important but it does not give an MTC uniquely, as this was shown recently (see here for some recent results and overview, and check this website for a nice list of all VOAs and MTCs (I would not say it is necessarily up to date)). Zhenghan thinks that if we add a representation \( \rho : B_3 \rightarrow U(N) \), we should get the whole MTC.

2.1. Vector-Valued Modular Forms. Start with a representation \( \rho \) of \( \Gamma \) to \( U(\mathbb{R}) \). Recall:

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, U = ST^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, U^3 = I
\]

Recall \( S \) has order two and \( T \) has order infinity in \( \bar{\Gamma} \) (One can also see that by looking at their action on the upper-half plane). Recall to be admissible is:

a) \( \rho \) to be irreducible (just for convenience, otherwise it will just split).

b) \( \rho(T) \) to be diagonal (this is easy as it is unitary) and of finite order (this is something like saying want to study rational CFT which is a VOA with finitely many irreducible representations).

One way of generating examples of such representations: Let \( A = n \times n \) be an even (i.e. \( a_{ii} \equiv 0 \mod 2 \)) integral non-singular matrix. So for example

a) \( n = 1 : A = (2) \).

b) \( n = 4 : \) matrix associated to Dynkin diagram \( D_4 \):

\[
D_4 = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}
\]

which has determinant 4.

c) \( E_8 \) which will not be described (has determinant 1 and gives an even unimodular lattice).

Now take \( \mathbb{Z}^n \rightarrow \mathbb{Z}^n \) and define \( L_A = \{ [x] \in \mathbb{Q}^n \mid Ax \in \mathbb{Z}^n, x \sim y \text{ if } x - y \in \mathbb{Z}^n \} \). So in the example a) above it is \( L_A = \{ [x] = 1/2, [x] = 1 \} \). Take a basis of \( L_A \) given by \{\( e_1, \ldots, e_r \)\} (for a), this is \( e_1 = 1, e_2 = \frac{1}{2} \), and define \( S_A = 1/\sqrt{r}(e^{i2\pi \epsilon_i \epsilon_j})_{ij}, T_A = \text{diag}(e^{i\pi \epsilon_i \epsilon_j}), \) both \( r \times r \) matrices. In the example \( A = (2) \), we have

\[
S_A = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, T_A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}
\]

Then \( (S_A T_A^{-1})^3 = e^{i\pi C/4} I, S_A^2 = S_A^3 = I \) where \( C \) is the signature of \( A \) (number of positive eigenvalues minus number of negative eigenvalues); \( C \) is also called the central charge which in
this construction is always an integer (Once we will introduce MTCs, it will become clear this construction is related to the class of abelian theories). \( T_A \) is obviously of finite order. Therefore we obtain a projective representation of \( SL(2, \mathbb{Z}) \); notice if it were linear we must have had \((S_AT_A)^3 = I \) as \((ST)^3 = U^3 = I \). To obtain a linear representation define \( \eta = e^{i\pi C/4} \) and change \( T_A \to \tilde{T}_A = \eta^{-\frac{1}{4}}T_A \); notice there are three different choices. Then \((S_AT_A)\) gives a linear rep of \( SL(2, \mathbb{Z}) \).

**Homework Exercise:** For any abelian MTC classified by \((G, q)\) quadratic form (see Theorem 5.4 in the link), find the corresponding \( A \). This is not a trivial problem! You can check the results in the work done by C.T.C Wall.

Give \( n \) and admissible \( \rho \). Recall the definition of \( M_\rho \) and vvmf. Here, we will revise the definition by adding the last requirement: \( f \in M_\rho \) should be meromorphic across the extended upper-half plane:

**Definition 5.** (\( M_\rho \) and vvmf revised) The vector space \( M_\rho \) is the one defined in [1] with the additional requirement that \( f \in M_\rho \) must be meromorphic not only at cusp but also at the rationals of the real axis. Therefore, \( f \) is called a vvmf if it is covariant with respect to a representation \( \rho \) and holomorphic on \( \mathbb{H} \) and meromorphic throughout \( \mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup i\infty \).

Meromorphicity at the cusp can be expressed through the \( q \)-expansion, which in fact, satisfies a certain kind of periodicity as well, as will be shown shortly.

We know that \( \rho(T) = t \) is supposed to be diagonal, so we can write it as \( e^{2\pi i \Lambda} \) for some non-unique diagonal matrix \( \Lambda = \text{diag}(\lambda_{ii}), \lambda_{ii} \in \mathbb{Q} \) as \( t \) is of finite order.

*Convention throughout the course:* \( z = |z|e^{i\pi \theta} \) with \(-\pi < \theta \leq \pi\).

Now we wish to compute \((q^{-\Lambda}f)\) at \( \tau + 1 \), where recall \( q = e^{i2\pi \tau} \), so \( q^{-\Lambda} = e^{-i2\pi \tau \Lambda} = \text{diag}(e^{-i2\pi \tau \lambda_{ii}}) \). Notice \( \tau + 1 = T(\tau) \), hence due to covariance we have

\[
(q^{-\Lambda}f)(\tau + 1) = e^{-i2\pi (\tau+1)\Lambda} f(T(\tau)) = e^{-i2\pi \tau \Lambda} e^{-i2\pi \Lambda} f(\tau)
\]

But \( t = e^{2\pi \Lambda} \), therefore the above is equal to \((q^{-\Lambda}f)(\tau)\). Hence the function \((q^{-\Lambda}f)\) is periodic. Now \( f \) being meromorphic at cusp means when writing the Laurent series at the origin (recall \( q|_{\tau = i\infty} = 0 \)) gives

\[
f = q^{-\Lambda} \sum_{-\infty}^{\infty} f_{[n]} q^n.
\]

where there must exist \( m \) such that \( f_{[n]} = 0 \) if \( n < m \).

To show that there is anything other than zero in \( M_\rho \) is really hard! But if \( f \in M_\rho \) is nonzero then \( jf \in M_\rho \) so that is why if there is anything, then it is infinite dimensional (it becomes a \( C[j] \)-module where \( C[j] \) is a PID).

**Question:** If \( f \in M_\rho \), then can \( f \) be constant? then you get \( v = \rho(\gamma)v \) but then you have a fixed vector! As we assumed irreducibility, this implies that \( r = 1 \) with a trivial representation.

Here is the first major theorem (maybe with some conditions ...) on \( M_\rho \):

**Theorem 2.1.** *(see Theorem 3.3, p.13, Reference 1)* \( M_\rho \) is a free \( C[j] \)-module of rank \( r \).

As mentioned in Reference 1 this is sort of an analog of Birkhoff Grothendieck Theorem which states that holomorphic bundles over \( \mathbb{P}^1 \) are a direct sum of line bundles.

Here is a *Question-like “theorem”*: define \( P_\Lambda : M_\rho \to \mathbb{C}[q^{-1}] \) which only takes the singular part of \( q^{-\Lambda}f \). When is \( P_\Lambda \) an isomorphism (as a vector space over \( \mathbb{C} \))? This is similar to asking whether CFT determined by the singular part in the OPE [Go to Additional Material].

**Definition 6.** \( \Lambda \) is called bijective if \( P_\Lambda \) is an isomorphism.
3. April 5th

One of the motivations behind the topics was to generalize the Tannaka duality and describe what needs to be added to the data of a $(2+1)$-TQFT to obtain a $\chi$-CFT. Mathematically, $(2+1)$-TQFT is and MTC and the other side can be explained using either VOA or sth called local conformal nets (LCN). It is a big project (see here) that VOA $\equiv$ LCN. VOA is formal field point of view and LCN is the Haag-Kastler axiom point of view, i.e. local observables Von Neumann algebras $\mathcal{A}(I)$ defined on each interval $I$ of the circle $S^1$.

We will try a different point of view starting from admissible representations of the modular group. We have described them in previous sections. Here are some other examples:

**Examples:**

1. **(Semion theory)** Take $\rho(S) = s = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. This is the very famous Hadamard matrix, previously encountered in (3), which also gives a discrete $\mathbb{Z}_2$ Fourier transform. Then $s^2 = I$ which means $\rho$ descends to a rep on $PSL(2, \mathbb{Z})$ as intended. Define $\rho(T) = e^{-i\pi/24} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ and it can be checked that we have a rep $SL(2, \mathbb{Z}) \rightarrow U(2)$.

2. **(Fibonacci anyons)** We can also define representation of $PSL(2, \mathbb{Z})$ by using a rep $\rho'$: $B_3 \rightarrow U(2)$. Define $\rho'((\sigma_1 \sigma_2)^3) = \lambda \in \mathbb{C}$. Since $B_3/\langle(\sigma_1 \sigma_2)^3\rangle = PSL(2, \mathbb{Z})$ and the relationship between $\sigma_1, \sigma_2$, i.e. $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$, is homogeneous, a scalar normalization can be safely performed which leads to a rep of $PSL(2, \mathbb{Z})$.

Zhenghan thinks that what we should add to an MTC is $M_{\rho}$ is going to provide the data needed for the space of characters and conformal blocks. What we could hope is that the characters and conformal blocks give *uniquely* a $\chi$-CFT. In brief, we are looking for a representation of CFT using some finite amount of data. Another motivation behind the pursuit of this representation is the simulation of CFT using Quantum Computers. Quantum computers work only with finite data, therefore it is crucial to be able to show CFT has such a representation.

Recall that for a trivial representation $\rho \equiv 1$, $M_{\rho}$ is given by functions where $f(\gamma, \tau) = f(\tau)$, and as briefly mentioned in the first lecture, it is a theorem that

**Theorem 3.1.** $M_{\rho}$ is the polynomial ring over $j$, i.e. $M_{\rho} = \mathbb{C}[j]$.

For a proof of the above theorem, see Proposition 6 in Chapter 7 in Reference 2.

Let us now describe what $j$ is. There are four closely related notions, modular functions, modular forms, elliptic functions and elliptic curves.

**Definition 7.** A modular function is a function $f: \mathbb{H}^* \rightarrow \mathbb{C}$ which is only required to be meromorphic.
Now \( j \)-function is actually modular form (holomorphic on \( \mathbb{H} \)) of weight zero and level 1. Let us define them all. For the notion of principal congruence subgroups, refer to its definition.

**Definition 8.** (modular form) A function \( f : \mathbb{H}^* \to \mathbb{C} \) is a modular form of weight \( N \) and level \( 2k \) for \( k \in \mathbb{Z} \) (though typically a positive integer), if it is:

- holomorphic on \( \mathbb{H} \), meromorphic on \( \mathbb{H}^* \),
- (level \( N \)) invariant under \( \Gamma(N) \), where \( N \) is the lowest such \( N \),
- (weight \( 2k \)) satisfies

\[
 f(\gamma.\tau) = (c\tau + d)^{-2k}f(\tau), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. 
\]

As \( \Gamma(1) = \Gamma \), this tells us the already mentioned fact that \( j \) is invariant under all transformations of \( \Gamma = SL(2, \mathbb{Z}) \).

Notice the derivative of \( \gamma \)'s action with respect to \( \tau \) is \( 1/(c\tau + d)^2 \). Hence, for a level \( 2k \) modular form, considering the differential form \( d\tau^k \), we have:

\[
 f(\tau)d\tau^k = f(\gamma,\tau)d(\gamma,\tau)^k, 
\]

giving the motivation behind the term “form”.

To describe \( j \) we first write its formula and then explain where each term is coming from

\[
 j = \frac{1728g_2^3}{g_2^3 - 27g_3^2} = q^{-1} + 744 + 196884q + \ldots 
\]

Fun fact: \( 1728 = 12^3 \) and 1729 is the smallest number that can be written in two ways of \( a^3 + b^3 \) (10,9 the other combination) and it is here to get a single pole with coefficient 1 at \( q^{-1} \).

### 3.1. Eisenstein Series, examples of modular functions.

We start from a lattice \( L \) in \( \mathbb{C} \) given by \( \mathbb{Z}w_1 + \mathbb{Z}w_2 \) where \( w_1 \in \mathbb{C} \) and are linearly independent over reals, i.e. \( w_1/w_2 \notin \mathbb{R} \). The lattices can be acted upon by \( SL(2, \mathbb{Z}) \). In fact, the thing that determines whether two lattices are the same or not is whether their corresponding ratio \( z = w_1/w_2 \) can be sent to another via an action of a \( \gamma \in \Gamma \) (see Reference 1 Proposition 2).

There is also a corresponding notion of modular forms \( F(w_1, w_2) \) of weight \( 2k \) on lattices, which turns out to be equivalent \( (f(z = w_1/w_2) = w_2^{2k}F(w_1, w_2)) \) to the notion of modular forms on \( \mathbb{H}^* \). So by defining modular forms on lattices, for which we have some examples, we can get to examples of modular forms.

**Eisenstein series:** Given a lattice \( L = (w_1, w_2) \in \mathbb{C} \) define:

\[
 G_k(w_1, w_2) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(mw_1 + nw_2)^{2k}} 
\]

This is called the Eisenstein series of index \( k \).

**Homework Exercise:** Show that the above is absolutely convergent if the integer \( k \geq 2 \). Follow the discussion in Reference 1 Lemma 1 for a hint; It can be bounded by an integral of \( \frac{1}{(x^2+y^2)^k} \) on a punctured plane.

Now one can compute \( G_k(\gamma, (w_1, w_2)) \) and see that it is a modular form on the lattices of weight \( 2k \) (not entirely trivial, see Proposition 4 Reference 1). Further \( G_k \) is in fact holomorphic at the cusp with value \( 2\zeta(2k) \), where \( \zeta \) is the Riemann zeta function. Then, as mentioned previously this gives a modular form \( G_k(\tau) \) of weight \( 2k \). Now let us define

\[
 g_2 = 60G_2, \quad g_3 = 140G_3
\]
For the Eisenstein series of lowest weight \(G_2, G_3\). Computing their values at infinity (i.e. computing \(\zeta(4), \zeta(6)\)) shows that

\[
\Delta = g_2^3 - 27g_3^2
\]

is a \textit{cusp form} of weight 12, i.e. a modular form of weight 12 with value zero at \(i\infty\). Hence the ratio

\[
j = \frac{1728g_2^3}{g_2^3 - 27g_3^2}
\]

is a modular form of weight \(12 - 12 = 0\) and a single pole at infinity.

There is also a relationship here with the elliptic functions and curves. It turns out that any lattice has a \textit{Weierstrass elliptic function} \(\wp(z; w_1, w_2)\) defined in here or Reference 1, p.84, which formula is closely related to \(G_k(w_1, w_2)\). It satisfies the differential equation

\[
y^2 = 4x^3 - g_2(w_1, w_2)x - g_3(w_1, w_2)
\]

where \(y = \frac{d\wp}{dx}, x = \wp\). The cusp form defined above (\(\Delta\)) is in fact, up to a numerical factor, the discriminant of this polynomial, hence the reason behind the notation \(\Delta\). The relations of the form \(y^2 = x^3 + ax + b\) define what is called an \textit{elliptic curve} in the projective plane \(\mathbb{C}P^2\), and they are known to be non-singular. This implies that the discriminant is not identically zero (something we had not checked before making the ratio for defining \(j!)\). In our case, this gives the \textit{elliptic curve} \(\mathbb{C}/L\) for the lattice \(L\) given by \(w_1, w_2\).

Some \textit{aside} discussions were made on the rate of growth of the coefficients in \(j\) which is explained here.

4. April 8-10-12th

Notes can be found in here. Please do not share this link outside of this class.

A main reference for tensor categories can be found here. You can also find a quick review of the axioms of quantum mechanics here.

5. April 15th

Let us introduce two notations:

\[
j = q^{-1} + 744 + \ldots
\]

for the famous \(j\)-function and

\[
J = j - 744
\]

which is the \(j\)-function without the constant term.

Let us start with the case \(r = 1\) and write \(M_\rho\) as a \(\mathbb{C}[j]\) module. Recall \(\rho(S) = s, \rho(T) = t\), and we must have

\[
(st)^3 = s^2 = 1
\]

Since \(r = 1\), so \(s = \pm 1\) which means \(t^3 = \pm 1\). So there will be six solutions. Since by Theorem 2.1 it is a \(\mathbb{C}[j]\) 1-dim free module, we need to know one element that will act as a basis. Let \(\xi_n = e^{2\pi i/n}\). Here is the basis element \(e_r\) for each case:
For the case $r = 2$, we had the Semion theory example where similar to (3), there are three choices for the third root of a parameter $\eta$ to get from a projective representation a linear representation. Once we have chosen $\eta$, we can try to compute $\Lambda$ by $\eta^{-1/3}t = e^{i2\pi\Lambda}$. But this equation has infinitely many solutions as always (adding integers to the diagonal); two of which we want to focus on. Recall that $\Lambda$ is defined to be bijective if $P_\Lambda$ is an isomorphism of $\mathbb{C}$-vector space.

These two solutions are

\[
\Lambda_1 = \frac{1}{24} \text{diag}(17, 11) \quad \text{and} \quad \Lambda_2 = \frac{1}{24} \text{diag}(23, 5)
\]

which have the corresponding basis $e_1, e_2$:

\[
\Lambda_1 : (e_1, e_2) = q^\Lambda (q^{-1} + 133 + 1673q + \ldots, 56 + 968q + \ldots)
\]

where 133 is the dimension of the lie algebra $E_7$, and the above are actually characters of a CFT; characters are the modular invariant functions $\sum c_n q^n$ with coefficients being dimensions of the graded vector space underlying the VOA $\mathcal{V} = \bigoplus_{n=0}^{\infty} \mathcal{V}_n$, where $\dim \mathcal{V}_n = c_n$. The CFT here is the WZW CFT model $(E_7)_1$ and its irreducible module; For a quick algebraic construction of WZW models see here page 16 Ex.1, and 19 Ex.2, where the dimension of $\mathcal{V}_0$ is the dimension of the Lie algebra. The next case is:

\[
\Lambda_2 : (e_1, e_2) = q^\Lambda (q^{-1} + 3 + 4q + 7q^2 + \ldots, 2 + 2q + 6q^2 + \ldots)
\]

where the above comes from $su(2)_1$ and its irreducible module, with 3 being the dimension of $su(2)$.

In the case of $r = 1$, all we have from the MTC point of view is the MTC Vec$_\mathbb{C}$, but we note that it has six CFT in this construction. Moreover, if we try projective representation of $\overline{T}$, we could get even more interesting things such as the monster moonshine VOA.

Recall $P_\Lambda : M_\rho \to \mathbb{C}[q^{-1}]$ where we take the singular part of the expansion of $f \in M_\rho$. We also define $P_{\Lambda_+}$ which is when the zeroth coefficient $f_{[0]}$ of the expansions is taken as well.

We have the following theorem on bijectiveness of $\Lambda$, i.e. $P_\Lambda$ being an isomorphism.

**Theorem 5.1.** *(Theorem 3.2):* If $\Lambda$ is bijective then

\[
\text{Tr}(\Lambda) = \frac{5r}{12} + \frac{\text{Tr}(S)}{4} + \frac{2\sqrt{3}}{9} \text{Re}(e^{-i\pi/6}\text{Tr}(U)).
\]

where $U = ST^{-1}$.

Above is a sufficient condition when $r \leq 5$. The counter example exists for $r = 8$ and is unknown for $r = 6, 7$. 

<table>
<thead>
<tr>
<th>$s$</th>
<th>$t$</th>
<th>$e_r$</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>1</td>
<td>$\xi_6^4$</td>
<td>$j^{1/2}$</td>
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<tr>
<td>1</td>
<td>$\xi_6^2$</td>
<td>$j^{2/3}$</td>
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<td>-1</td>
<td>-1</td>
<td>$(j - 1728)^{1/2}$</td>
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<tr>
<td>-1</td>
<td>$\xi_6^6$</td>
<td>$j^{1/3}(j - 1728)^{1/2} - 1$</td>
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<tr>
<td>-1</td>
<td>$\xi_6^5$</td>
<td>$j^{2/3}(j - 1728)^{1/2} - 1$</td>
</tr>
</tbody>
</table>
Now assume $\Lambda$ is bijective. Then let us pull back the standard basis $e_{n,a}$ of $\mathbb{C}[q^{-1}]$ to obtain a new basis, called the *canonical basis*:

$$X^{a,n} = P^{-1}_\Lambda \begin{pmatrix} 0 \\ \vdots \\ q^{-n} \\ \vdots \\ 0 \end{pmatrix}$$

where $q^{-n}$ is the $a$-th entry where $a = 1, \ldots, r$ and $n = 1, 2, \ldots$.

What we will aim to establish is a certain quantum mechanical version of position and momentum, using the $j$-function and $d/dj$, respectively. We will see later how these two are defined and interact with the canonical basis.

The following matrix will turn out to be *fundamental* in our understanding of the theory:

**Definition 9.** The *fundamental matrix* associated to $\Lambda$ is defined as

$$\Xi = \begin{pmatrix} X^{1,1} & X^{2,1} & \ldots & X^{r,1} \end{pmatrix}$$

i.e. the matrix formed by the column vectors given by the canonical elements $X^{a,1}$ for $1 \leq a \leq r$.

Notice $\Xi$ is a function of $\tau$.

The constant part of this matrix, given by the constant coefficients of each canonical element is called $\chi$, the *character matrix*.

In order to study a CFT, Zhenghan believes that the MTC $B$, $\Lambda$, and $\chi$ are the most important parts. Where does $\Xi$ come up? In fact, as we will see later, $\Xi$ satisfies a (Fuchsian) differential equation through which it can be recovered using $\Lambda, \chi$.

### Exercise/Project:

1. Find the best constants $m_1, m_2$ such that:

$$\ell(\gamma) \leq m_1 \mu(\gamma) + m_2$$

where $\ell$ is the Eichler length of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\mu(\gamma) = a^2 + b^2 + c^2 + d^2$

2. Are there any moonshine theory at $\tau = i$ and $\xi_6 = (\rho =) e^{2\pi i/6}$? (Zhenghan’s conjecture is yes)

$$q_\infty = e^{2\pi i \tau}$$

$$q_2 = \pi \sqrt{E_4(i)} \frac{\tau - i}{\tau + i}$$

$$q_3 = \pi \sqrt{3E_6(\xi_0)^{1/3}} \frac{\tau - \xi_6}{\tau + \xi_6}$$
Now write \( j \) function as

\[
j = \sum_{n} C^{(k)}(n) \frac{n!}{q^n}
\]

then \( C^{(k)}(n) \) has moonshine for both \( k = 2, 3 \).

About the exercises:

For the first exercise, notice that \( S, T \) generate any \( \gamma \). We can look at the minimum length for writing \( \gamma \) as

\[
\gamma = S^{m_1}T^{m_2} \ldots S^{m_n}T^{m_n}
\]

and \( \ell(\gamma) \) is defined as that minimum length. Eichler gives upper bound on that minimum length.

For the second exercise: Recall \( \Gamma \) acts on \( \mathbb{H} \). Since we have an action we can study the quotient \( \mathbb{H}/\Gamma \). First, as it is shown in figure 1 we can obtain the fundamental domain of the action (see here p.78 for proof). Notice the boundaries are identified as they can be shifted by \( T \), and because of \( S \) sending \( \tau \) to \( -1/\tau \), we can identify the 1/6th of the circle to the right of \( i \) with the one on the left. Then, the quotient \( \mathbb{H}/\Gamma \) can be seen to be a sphere with three singularities at \( (\infty, i, \rho) \). What are the stabilizers of these singularities? For \( i \), it is \( S.i = i \) and stabilizer of \( \rho \) is \( ST \). So one has order two and the other has order three. As for infinity, it has order infinity stabilizer which is \( T \) as \( T.\infty = \infty + 1 = \infty \). So the whole subject can be restricted to functions on a Riemann sphere which has three singularities. Now recall the expansion of \( j = q^{-1} + 744 + 196884q + \ldots = \sum_{n=-1}^{\infty} c(n)q^n \). Related to this function, there is a whole subject known as monster moonshine. Monster refers to the largest simple finite group of order \( \sim 10^{35} \) which this turns out to be the symmetry group of a VOA \( \mathcal{V} \) called the (monster) moonshine module. Notice the monster group as a finite group has finitely many (194) finite dimensional irreps. The moonshine is roughly the following statement:

There is a matrix of size \( \infty \times 194 \) which multiplied by the vector \( 194 \times 1 \) (dimension of monster irreps) is equal to the dimensions of the graded VOA \( \mathcal{V} = \oplus_{n=0}^{\infty} V_n \). Further, there is a unique solution (the matrix) to this equation (notice it is highly over-determined).
Now $j$ is a holomorphic function on upper half and meromorphic elsewhere. Consider the expansion of $j$ around $\rho$ and $i$. Now the second exercise is that if you expand at those points, and you normalize correctly (using $q_2, q_3$ for $i, \rho$, respectively), you would still have a moonshine.

Going back to the course, we would like to show that $M_\rho$ has a nonzero element. To do so, we will review and define a number of functions. Now consider $\Gamma/\Gamma_\infty$ where $\Gamma_\infty = \langle T \rangle$ and take $C$ as a complete coset representative of this quotient (i.e. take a unique $\gamma$ from each coset).

Now, given admissible $\rho$, we define this function of $\tau$:

$$
\tilde{P}(\rho, b, k)(\tau) = \frac{1}{2} \sum_{[\gamma]\in C} e^{2\pi i \Lambda_{bb} \gamma(\tau)} (c\tau + d)^k \rho(\gamma)^t e_b
$$

where $e_b$ is the standard basis in $\mathbb{C}^r$, for $1 \leq b \leq r$, and $\Lambda_{bb}$ is the $b$-th element on the diagonal of $\Lambda$ associated to $\rho$. The claim is:

**Claim 6.1.** If $k = 12m > 2 + \alpha$ then $P(\rho, b, k) = P/\Delta^m \in M_\rho$, where $\alpha = m_1 \log(r)$ ($m_1$ introduced in first exercise) and $\Delta$ is the discriminant $g_2^3 - 27g_3^2$.

Notice the division is in order to get zero weight. It turns out that $P$ is indeed well-defined and absolutely convergent, and in $M_\rho$, i.e. covariant wrt $\rho$.

Here, we will only show why $P$ is well-defined. Changing $\gamma \to T^n(\gamma)$, yields a term $e^{2\pi i \Lambda_{bb} n}$ in the exponent, while $\rho(T^n(\gamma)) e_b$ giving $\rho(\gamma)^t t^{-n} e_b = e^{-2\pi i \Lambda n} e_b = e^{-2\pi i \Lambda_{bb} n} e_b$.

Let us now prove the Theorem 2.1:

**Theorem.** $M_\rho$ is a free $\mathbb{C}[j]$-module of rank $r$.

**Proof.** Let’s assume that $\Lambda$ is a bijective exponent, i.e. $P_\Lambda$ is isomorphic as an infinite dimensional vector space. Now recall the standard basis $e_{n,a}$ and its pullback:

$$X^{a,n} = P_\Lambda^{-1}(e_{n,a}) = q^\Lambda (q^{-n} e_a + \sum_{m=0}^{\infty} X^{a,n}_{[m]} q^m),$$

where $e_a$ is the standard basis of $\mathbb{C}^r$. Now we have to make $M_\rho$ a module over $\mathbb{C}[j]$. The idea is to use recursive relations by multiplication by $J$. We will complete the proof in the next section. □

7. APRIL 19TH

Continuing proof of the last section:

**Proof.** Recall $J = q^{-1} + \sum_{n=1}^{\infty} c(n) q^n$ being the $J$-function. We have

$$X^{a,n} = q^\Lambda (q^{-n} e_a + \sum_{m=0}^{\infty} X^{a,n}_{[m]} q^m) \iff P_\Lambda X^{a,n} = q^{-n} e_a$$
where $X_{[m]}^{a,n} = (X_{[m],b})^r_{b=1}$ is a vector. We compute:

$$JX^{a,n} = q^\Lambda (q^{-1} + \sum_{l} c(l)q^{l})(q^{-n}e_a + \ldots)$$

where we will only keep track of the singular part (as $P_\Lambda$ is an isomorphism). So that means

$$q^\Lambda (q^{-n-1}e_a + \sum_{b=1}^{r} X_{[0],b}^{a,n}q^{-1}e_b + \sum_{l=1}^{n-1} c(l)q^{-n+l}e_a + \ldots)$$

Hence, by definition of the basis $X^{a,n}$, and the fact that $P_\Lambda$ is an isomorphism:

$$JX^{a,n} = X^{a,n+1} + \sum_{b=1}^{r} X_{[0],b}^{a,n}X^{(b,1)} + \sum_{l=1}^{n-1} c(l)X^{a,n-l}$$

which is equivalent to

$$X^{a,n+1} = JX^{a,n} - \sum_{b=1}^{r} X_{[0],b}^{a,n}X^{(b,1)} - \sum_{l=1}^{n-1} c(l)X^{a,n-l}$$

So by reverse induction, $n$ goes down and eventually $\{X^{a,1}\}$ generates $M_\rho$ as a $C[J]$ module. Hence, there are polynomials of $J$ such that

$$X^{a,n} = \sum_{b} P_b(J)X^{b,1}.$$

To show that the rank is $r$, we define the fundamental matrix $\Xi(\tau)$ and we need to show that $\det(E(\tau))$ is nonzero when $|q|$ is small. By definition of $X^n$ it can be checked that $q^{-\Lambda}\Xi$ is of the form:

$$q^{-\Lambda}\Xi(\tau) = \begin{pmatrix} q^{-1} + \ldots & q^{-1} + \ldots & \ldots \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \implies \Xi(\tau) = q^\Lambda Id_{r\times r} + O(q)$$

which shows what we desired. \[\square\]

But how do we find $\Xi(\tau)$. First, we need the characteristic matrix $\chi = (X^{a,1}_{[0],b})_{a,b}$, which means we take constant terms of the canonical module generators. Let us do an example.

**Example:** For semion, we know $s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and we had two choices for $\Lambda$ and their corresponding basis for $M_\rho$ was given in [here](#). Recall

$$\Lambda_1 = \frac{1}{24} \text{diag}(17, 11) & \Lambda_2 = \frac{1}{24} \text{diag}(23, 5)$$

For $\Lambda_1$, the fundamental matrix is:

$$q^\Lambda \begin{pmatrix} q^{-1} + 133 + 1673q + \ldots & 1248 + 49504q + \ldots \\ 56 + 968q + \ldots & q^{-1} - 377 + \ldots \end{pmatrix}$$

and for $\Lambda_2$ it is:

$$q^\Lambda \begin{pmatrix} q^{-1} + 3 + 4q + 7q^2 + \ldots & 26752 + 17340167q + \ldots \\ 2 + 2q + 6q^2 + \ldots & q^{-1} - 247 + \ldots \end{pmatrix}$$
Notice the first column should have only nonnegative integer coefficients as they are the characters of vacuum sector and semion (this is probably always true as they are supposed to be candidates for characters of a VOA and its irreps). But Zhenghan does not know about the second column and it can have negative coefficients.

How all these relate to CFT? CFT is the quantum mechanics on the moduli space of elliptic curves (or string theory in $\mathbb{C}P^2$). What does this mean?

We should not think of $\tau$ as a complex number but as an elliptic curve $C/((C^1 \oplus C^\tau)$ which can be mapped into $\mathbb{C}P^2$ by $[1, \wp, \wp'/2]$ where $\wp$ is the Weierstrass function associated to the lattice $L = C^1 \oplus C\tau$. The function $f$ we are interested in is in fact a vector-valued wavefunction on the moduli space of elliptic curves. So it has a position and momentum, which are $J(\tau)$ and $d/d\tau$. We want to define $d/d\tau$ (ultimately, you should be able to do Heisenberg uncertainty using these). But instead of $d/d\tau$, let us start with $d/d\gamma$.

Define $\nabla(\gamma) = k(\gamma) \frac{d}{2\pi i} d\gamma$, where

$$k(\gamma) = \frac{E_5(\gamma)}{\Delta} = q^{-1} - 240 - 141444q + \ldots = \sum l k(l) q^l$$

Notice the weight is $10 - 12 = -2$ and as $d/d\gamma$ changes weight by two (as we will see), $\nabla$ does not change the weight. We know $J(\gamma, \tau) = J(\tau)$. Therefore, by the chain rule:

$$\frac{d}{d\tau} J(\gamma, \tau) = (c\tau + d)^{-2} J'(\tau)$$

So $\nabla$ is weight zero and sends $M_\rho \to M_\rho$. Now let us just assume that everything works and outline what works from here:

A computation we will later is to show that:

$$\nabla X_{a,n} = (\lambda_{aa} - n) \sum_{l=-1}^{n-1} k(l) X_{a,n-l} + \sum_{b=1}^r \lambda_{bb} X_{0,b} X_{a,1}$$

Using $\nabla$ as a differential operator on the fundamental matrix gives:

$$\nabla \Xi(\tau) = \Xi(\tau)((J - 240)(\Lambda - Id) + \chi + [\Lambda, \chi])$$

Hence, if we can find $\Lambda$ and $\chi$, then we have the differential equation for $\Xi$ which we can solve to get $\Xi$. But $\chi$ is a part of $\Xi$, so how can we find it independently? We can translate the above differential equation to a Fuchsian differential equation which $\Xi$ needs to satisfy. Let us use the notation:

$$Z = (984 - J)/1728$$

using that, we can translate the differential equation above to (see here, Theorem 3.3):

$$\frac{d\Xi(Z)}{dZ} = \Xi(Z) \left( A/(2Z) + B/(3(Z - 1)) \right)$$

where $A, B$ are constant matrices such that

$$B/3 + A/2 = 1 - \Lambda,$$

$$284B - 492A = \chi + [\Lambda, \chi]$$

The residue at $0, 1, \infty$ of the differential equation above corresponds to $S, U, T$ and the monodromy is given by $\rho$. Now notice $E(\tau)$ is holomorphic which implies that

- $A(A - 1) = 0$, so $A$ has eigenvalues 0 or 1
- $B(B - 1)(B - 2) = 0$, so $B$ has eigenvalues 0 or 1 or 2.
Also notice that $A$ with $\Lambda$ give $B$. So we can substitute that into the equation for $B$ and solve for $A$. Once we have found $A$ (there might be some ambiguities), then we can have $\chi$ and finally $\Xi$.

But assuming we have $\Xi$, how do we get to a CFT? Zhenghan thinks that we can redo the theory using 3 dim braid rep on the punctured 4 sphere. We should then be able to compute the correlation functions (in those computations, the information in $\Xi$ is used) and from there it should be doable using the Osterwalder-Schrader theory to get to VOA.

8. April 22nd

So far we have described the most important objects associated to a representation $\rho$, namely $M_\rho, \Lambda, X^{a,n}, \chi, \Xi$. We have illustrated before some examples of $M_\rho, \Lambda$, now we want to do some of the same examples, and derive $X^{a,n}, \chi, \Xi$.

Take $\rho$ to be a one dimensional representation. We know already that there are 6 cases for $\rho$ as outlined in the table, let us deal with $s = 1$, which means $t^3 = s^{-1} = 1$ and so $t = 1, \omega, \overline{\omega}$.

Now take $t = 1$ which implies $\rho \equiv 1$. For $f \in M_\rho$, we have $f(\gamma, \tau) = f(\tau)$. Obviously constant function is a solution and we know by previous Theorem that $M_\rho = C[J], 1$. Next, we want $\Lambda$, and since $t = e^{i2\pi\Lambda} = 1$ so $\Lambda$ is an integer. For $\Lambda$ to be bijective, i.e. $P_\Lambda$ being an isomorphism, let us directly compute $P_\Lambda$.

Recall that $(q^{-\Lambda} f)$ is a periodic function which has the expansion $\sum_{n=-\infty}^{\infty} f[n]q^n$ where the singular part $P_\Lambda(f)$ is a polynomial (notice this finiteness is always necessary for being able to have a well-defined expansion for the product of $f$ for $f, g \in M_\rho$).

Now notice that since $M_\rho$ has only constant function and polynomials of $J$ which have at least a single pole, therefore for $P_\Lambda$ to hit $q^{-1}$ (the generator of the image in the range $C[q^{-1}]$), a choice is to have $P_\Lambda(1) = q^{-1}$. That means $q^{-\Lambda} . 1 = q^{-1}$, so $\Lambda = 1$. From here we will be able to get $X^{a,n}$ (or differential equation for $\Xi$) very easily so we move on to the more complicated example.

Let us do the more interesting case: $s = 1, t = \omega$. For covariance we need $f(T, \tau) = \Xi f(\tau)$ which is not satisfied by constant function. The answer turns out to be $j^{1/3}$ so $M_\rho = C[J], j^{1/3}$.

Notice we can take the third root of $j_\gamma$ to be able to take $n$-th root of a holomorphic $f$ it turns that Cauchy integral $\frac{1}{2\pi i} \int_C f \frac{\partial}{\partial \tau} \tau$ over any closed path $C$ needs to be in $n\mathbb{Z}$, which means the winding number has to be a multiple of $n$.

It turns out that $j^{1/3}$ has expansion

$$q^{-1/3}(1 + 248q + \ldots)$$

and we can check that $j^{1/3}$ satisfies the covariance relation:

$$j^{1/3}(\tau + 1) = e^{2\pi i (\tau + 1)(-1/3)}(1 + \ldots) = \omega j^{1/3} = \omega^2 j^{1/3}$$

where we note that $\ldots$ is the same as $j^{1/3}$ holomorphic part as $e^{2\pi i (\tau + 1)n} = e^{2\pi i \Lambda n} = q^n$.

To find $\Lambda$, we want $t = \omega = e^{2\pi i \Lambda}$. Therefore $\Lambda = 2/3$ is obviously a choice and we can check that it satisfies the trace relation for bijectivity given in Theorem 5.1. We can also see that $q^{-\Lambda} j^{1/3} = q^{-2/3} q^{-1/3}(1 + \ldots)$ so $P_\Lambda$ is surjective as we get $q^{-1}$ in the image.

Once we have $\Lambda$, we learned in the previous lecture how to get the other interesting objects: the canonical basis $X^{a,n}$. Define the standard basis $e_{n,a}$ in $C[r, q^{-1}]$ where $r = 1$, and pull it back using $P_\Lambda$, getting $X^{a,n}$.

Recall the definition of the fundamental matrix $\Xi(\tau)$. We know that it satisfies the differential equation:

$$\nabla \Xi = \Xi \times ([J - 240](\Lambda - 1) + \chi + [\Lambda, \chi])$$
We note that for the previous case where we had $s = 1, t = 1, \Lambda = 1$, and $\chi = 0$ everything is trivial.

But we would like to obtain the Fuchsian differential equation that $\Xi(J)$ satisfies in $\mathbb{C}$. To do that, we need to find $A, B$. Using the equations that $A, B$ need to satisfy outlined after (12) gives us

$$B/3 + A/2 = 1 - \Lambda, \quad 248B - 492A = \Xi + [\Lambda, \Xi]$$

Recall $\Lambda = 2/3$ so:

$$B/3 + A/2 = 1/3$$

but $A$ can only be 0 or 1 and $B$ can only be 0, 1, 2 (check equations after (12), and it turns out the only solution to this is $A = 0, B = 1$.

Then we find $\chi = 248$. So we have the right hand side completely of the Fuchsian differential equation of $\Xi$ and the $\nabla$ differential equation of $\Xi$

$$\nabla \Xi = \Xi((J - 240)(-1/3) + 248).$$

Instead of solving it, let us check that $j^{1/3}$ is a solution (notice it has a unique solution by elementary differential equation theorems). We have

$$j^{1/3} = q^{-1/3} + 248q^{2/3} + \ldots$$

Applying $\nabla$ on it, we get

$$\nabla j^{1/3} = k(\tau) \frac{d}{d\tau} j^{1/3}$$

To compute above, first notice that $\frac{d}{d\tau} q^c = 2\pi i c q^c$. So we get

$$\frac{d}{d\tau} j^{1/3} = -\frac{2\pi i}{3} q^{-1/3} + 248(2\pi) \frac{2}{3} q^{2/3} + \ldots$$

so the left side is $\nabla j^{1/3} = k(\tau)(-\frac{1}{3} q^{-1/3} + 248 \frac{2}{3} q^{2/3} + \ldots) = k(\tau)q^{-1/3}(-\frac{1}{3} + 248 \frac{2}{3} q + \ldots)$

$$= (q^{-1} - 240 - 141444q + \ldots)q^{-1/3}(-\frac{1}{3} + 248 \frac{2}{3} q + \ldots)$$

and the right hand side is

$$j^{-1/3}(-\frac{1}{3} J + 80 + 248)$$

which after expanding becomes

$$q^{-1/3} (1 + 248q + \ldots)(-\frac{1}{3} q^{-1} + 0 + 328 + \ldots)$$

Now since $P_\Lambda$ is an isomorphism, we compare only the constant and negative powers of $q$. negative power term is $1 \times -1/3 = -1/3$ for $q^{-1}$. As for the constant terms, we have

$$-240/3 + 248 \times 2/3 = 328 - 1/3 \times 248$$

which are indeed equal.

In fact coefficients of $j^{1/3}$ gives the $\Theta(E_8)$ (theta function) of $E_8$ lattice while $\Lambda = 1/3$ gives the other solution $j^{2/3}$ which is corresponding to $E_8 \oplus E_8$. 
9. APRIL 24TH

Project/Exercise:
Define a Hermitian inner product on $M_\rho$ to make it into a Hilbert space and $J$ and $\nabla$ as position and momentum operators. Find then every eigenstate $\langle n |$ such that

$$a^\dagger | n \rangle = \sqrt{n + 1} | n + 1 \rangle$$

$$a | n \rangle = \sqrt{n} | n - 1 \rangle$$

where $H = mv^2 + \frac{1}{2} k x^2 = \frac{p^2}{2m} + m \omega^2 x^2$ and $\omega = \sqrt{E/m}$. Then $a = \sqrt{\frac{m \omega^2}{2\hbar}} (\hat{x} + i \frac{m \omega}{m \omega} \hat{p})$ and $a^\dagger = \sqrt{\frac{m \omega^2}{2\hbar}} (\hat{x} - i \frac{m \omega}{m \omega} \hat{p})$.

Notice modular forms have an inner product called the Petersson inner product but it may or may not work.

Going back to the course:
Let us revise our conjecture on the construction of CFT starting from an MTC $B$:

A chiral CFT (a VOA) is the same as the MTC $B$ and central charge $c$ and the conformal weights $h_a$ and $r_{a,m}$ which we explain below.

Recall for a MTC we have labels $L = \{a, b, c, \ldots\}$ and quantum dimensions $d_a, d_b, \ldots$ and twists $\theta_a, \ldots$. We also have the total quantum dimension $D^2 = \sum_{a \in L} d_a^2$ and we denote $p_+ = \sum \theta_a d_a^2$. Then the central charge is obtained up to an integer by $p_+ / D = e^{\pi ic/4}$ where $c \in \mathbb{Q}$. We also know by Vafa’s theorem that $\theta_a = e^{2\pi i h_a}$ which means $h_a$ are rational numbers.

Now let us define $r_{a,m}$: recall from MTC we get a representation $\rho$ on $B_3$. The braiding $R_{-m}$ gives us a scalar $e^{2\pi i r_{a,m}}$ when the theory is unitary.

(Insert Picture here: In total four anyons; Three anyons $a$, forming a fusion tree with intermediate fusion $m$ and (fourth anyon) total charge $t$)

We call this representation $\rho_{a,4}$, from which we get $M_{\rho_{a,4}}$ which we claim are correlation functions on the 4-punctured sphere. Notice all we are doing here is to make a choice on the integer part of $c, h_a$ and $r_{a,m}$. So CFT is conjectured to be MTC plus some choices for the integer parts!

There is a conceptual reason for why this may be true. It comes from Grothendieck reconstruction principle (see here for an explanation). The principle basically says that for a natural theory on surfaces, due to gluing and other operations we have at our disposal, we need to know the theory only on 1-holed torus and 4-holed sphere, and 2-holed torus and 5-holed sphere.

If we have an MTC $B$ or TQFT, the group $\Gamma$ appears there twice:

From the mapping class group of torus, which is $\text{MCG}(T^2) = \Gamma$ and (indirectly) from the spherical four strand braid group $SB_4$ given by the quotient

$$B_4/\langle \langle \sigma_1 \sigma_2 \sigma_3^2 \sigma_2 \sigma_1 \rangle \rangle.$$ 

This group has the same picture as the braid group, with the exception that we imagine the braids to on the sphere. So choosing one point and going around all other three points is actually a trivial braid. This is exactly the quotient above and it is not trivial but that is the only relation.

To get from here to $\Gamma$, there is final map $\sigma_1 = \sigma_3$ which gives you the $\Gamma$. Notice $\sigma_1 = \sigma_3 \implies \sigma_1 \sigma_2 \sigma_3^2 \sigma_2 \sigma_1 = (\sigma_1 \sigma_2)^3$ which is the center of $B_3$ and we know that $B_3$ modulo that center gives us $\Gamma$.

Going back to the fundamental matrix $\Xi$, in the rest of this lecture, we want to derive that its $\nabla$ differential equation 11 and show it is equivalent to the Fuchsian differential equation 12.
Recall $J = q^{-1} + \sum c(l)q^l$ and $k(\tau) = q^{-1} - 240 - \ldots$. Proving the equations are equivalent will be a long series of calculations. Recall

$$\nabla = \frac{k(\tau)}{2\pi i} \frac{d}{d\tau}.$$

Also

$$\mathcal{X}^{a,n} = q^A(q^{-n}e_a + \sum_{m=0}^{\infty} \mathcal{X}^{a,n}_{m!} q^m)$$

and $\frac{d}{d\tau} q^c = 2\pi icq^c$. So when we differentiate $\mathcal{X}^{a,n}$, we get

$$q^A(2\pi i(\Lambda - n)q^{-n}e_a + \sum_{m=0}^{\infty} (\Lambda + m)\mathcal{X}^{a,n}_{m!} q^m)$$

Applying the rest of $\nabla$, i.e. $\frac{k(\tau)}{2\pi i}$,

$$\nabla \mathcal{X}^{a,n} = (q^{-1} - 240 + \sum_{l=1}^{\infty} k(l)q^l).q^A((\Lambda - n)q^{-n}e_a + \sum_{m=0}^{\infty} (\Lambda + m)\mathcal{X}^{a,n}_{m!} q^m)$$

as we need to keep track only of the singular part

$$q^A((\Lambda - n)q^{-n-1}e_a - 240(\Lambda - n)q^{-n}e_a + (\Lambda - n)\sum_{l=1}^{n-1} k(l)q^{-n+l}e_a + \Lambda \mathcal{X}^{a,n}_{[0]} q^{-1} + \ldots)$$

Notice that $\Lambda \mathcal{X}^{a,n}_{[0]} q^{-1} = (a \text{ matrix } \Lambda \text{ multiplied by the column vector } \mathcal{X}^{a,n}_{[0]} = \sum b \mathcal{X}^{a,n}_{[0],b} q^{-1} e_b$. Now we just need to collect the terms and write them in terms of the canonical basis, to get

$$\nabla \mathcal{X}^{a,n} = (\Lambda - n)(\mathcal{X}^{a,n+1} - 240\mathcal{X}^{a,n} + \sum_{l=1}^{n-1} k(l)\mathcal{X}^{a,n-l} + \sum_{b=1}^{r} \Lambda_{bb}\mathcal{X}^{a,n}_{[0],b} q^{-1} e_b).$$

In fact we can simplify it a little bit by absorbing the first two terms into $l = -1, 0$:

$$(\Lambda - n)(\sum_{l=-1}^{n-1} k(l)\mathcal{X}^{a,n-l}) + \sum_{b=1}^{r} \Lambda_{bb}\mathcal{X}^{a,n}_{[0],b} q^{-1} e_b.$$

We are interested only when $n = 1$ as they are the actual basis as a $\mathbb{C}[J]$-module. We get

$$\nabla \Xi = (\nabla \mathcal{X}^{a,1})^r_{a=1} = (\Lambda_{aa} - 1)(\mathcal{X}^{a,2} - 240\mathcal{X}^{a,1}) + \sum_{b=1}^{r} \Lambda_{bb}\mathcal{X}_{ab}\mathcal{X}^{h,1})^r_{a=1}$$

Computing

$$J\mathcal{X}^{a,1} = \mathcal{X}^{a,2} + \sum_{b=1}^{r} \mathcal{X}_{ab}\mathcal{X}^{h,1}$$

we can use it to substitute $\mathcal{X}^{a,2}$ in $\nabla$ equation for $\mathcal{X}^{a,1}$. Finally, after reorganizing, we obtain

$$\mathcal{X}^{a,1})^r_{a=1} \left((J - 240)(\Lambda - 1) + \chi + [\Lambda, \chi]\right)$$

which is the $\nabla$ equation we wanted for $\Xi$.

Let us now translate this to the Fuchsian differential equation. Let $Z = (984 - J)/1728$ where $1728 = 744 + 984$ so $Z - 1 = -j/1728$. What is $\nabla J$? It is still a modular function as it preserves weight so it must be polynomial of $J$. So it has to be a quadratic polynomial $aJ^2 + bJ + c$ as $J$
starts with \( q^{-1} \) and the derivative in \( \nabla \) will make it \( q^{-2} \). By computing only the singular part, this will turn out to be:

\[
\nabla J = -(J - 984)(J + 744)
\]

Next we want to compute \( \frac{d}{dZ} \). By chain rule it should be:

\[
\frac{d}{d\tau} \frac{1}{dZ} \Xi = \frac{1}{dZ} \frac{d}{d\tau} \Xi = \frac{2\pi i}{k(\tau)} k(\tau) \frac{1}{d\tau} \frac{d}{d\tau} \Xi = \frac{2\pi i}{k(\tau)} \nabla \Xi
\]

and we get to \( \frac{1}{dZ} \nabla \Xi \) which after some reorganizing done in the next section gives the Fuchsian differential equation:

\[
\frac{d\Xi}{dZ} = \Xi \left( A/(2Z) + B/(3(Z - 1)) \right)
\]

10. APRIL 26TH

Exam: Let \( \sigma_1 = e^{-\pi i/8} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \sigma_2 = e^{-\pi i/8} H \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} H \) where \( H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \).

1- Verify the braid identity \( \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \), so there is a rep \( \rho : B_3 \to U(2) \).

2- Verify \( (\sigma_1 \sigma_2)^3 = e^{-\pi i/4}.Id. \)

3- Let \( t = \tilde{\sigma}_1, s = (\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1)^{-1} \) and \( u = (\tilde{\sigma}_1 \tilde{\sigma}_2)^{-2} \) where \( \tilde{\sigma}_i = \xi^{-1} \sigma_i \) where \( \xi^6 = e^{-\pi i/4} \).

Verify \( \{t, s, u\} \) give rise to a rep of \( \rho : \Gamma \to U(2) \).

4- Find the fundamental matrix \( \Xi \) for this \( \rho \).

5- *Prove that the kernel of \( \rho \) inside \( \Gamma \) is a congruence subgroup.

We will continue where we left off in the last lecture. Recall

\[
\frac{d\Xi}{dZ} = \frac{1}{d\tau} \frac{d}{d\tau} \Xi = \frac{1}{d\tau} \frac{d}{d\tau} \Xi = \frac{2\pi i}{k(\tau)} \frac{1}{d\tau} \frac{d}{d\tau} \Xi = \frac{2\pi i}{k(\tau)} \nabla \Xi = \frac{\nabla \Xi}{\nabla Z}
\]

and we know that \( \nabla J = -(J - 984)(J + 744) \). Therefore \( \nabla Z = -\frac{(J - 984)(J + 744)}{1728} \). Substituting \( \nabla \Xi \) by \( \Xi \) and \( \nabla Z \), and using the equations defining \( A, B \) after \( \Xi \) as

\[
B/3 + A/2 = 1 - \Lambda
\]

\[
248B - 492A = \chi + [\Lambda, \chi]
\]

We obtain:

\[
\frac{d\Xi}{dZ} = \Xi \left( (J - 240)(-A/2 - B/3) + (248B - 492A) \right) \times \frac{1728}{(J - 984)(J + 744)}
\]

notice \( Z - 1 = -(J + 744)/1728 \), so by collecting the \( A/2 \) coefficients we get \( -(J - 240 - 984) = -(J - 744) \) and for coefficient of \( B/3 \), it is \( -(J + 240 + 3 \times 248) = -(J + 984) \).

Rewriting everything, we obtain the Fuchsian differential equation:

\[
\frac{d\Xi}{dZ} = \frac{\Xi}{(J - 984)(J + 744)} \left( \frac{-(J + 744) A}{1728} - \frac{(J - 984) B}{1728} \right) \times \frac{\Xi}{Z(Z - 1)} \frac{(Z - 1) A}{2} + Z B/3.
\]

Is there any reason that we should expect that we can get a Fuchsian differential equation? Notice this differential equation has three singularities 0, 1, and \( \infty \). If you think about its residue it would be \( A/2, B/3 \) and \( \Lambda - 1 \) and its monodromy would be \( S, U, T \) (which is basically \( \rho \)). More precisely, notice the holomorphic property says summing up residues should give zero so
\( A/2 + B/3 + \Lambda - 1 = 0 \) which is why we have the equation relating \( A, B, \Lambda \). Also monodromy tells us that 
\[
e^{2\pi i A/2} = S, \ e^{2\pi i B/3} = U, \ e^{2\pi i (\Lambda - 1)} = T
\]
So basically we are reverse engineering; If we know the residue and monodrmoy and solve the Riemann-Hilbert problem given that we know the monodromy group, we should get this Fuchsian differential equation.

We now want to prove a theorem we previously mentioned in [5,1]. Recall if \( \Lambda \) is bijective, then 
\[
\text{Tr}(\Lambda) = \frac{5r}{12} + \frac{\text{Tr}(S)}{4} + 2\frac{\sqrt{3}}{9}\text{Re}(e^{-\pi i/6}\text{Tr}(U)) \quad (*)
\]
Since \( \Lambda \) is bijective then \( A/2 + B/3 = 1 - \Lambda \). Notice \( A \) has eigenvalues 0, 1 and for their multiplicity assume it is \( \alpha_0, \alpha_1 \). Similarly for \( B \), assume the multiplicities are given by \( \beta_0, \beta_1, \beta_2 \). All sum up to \( r \), which is dimension of our matrices. So we get \( \text{Tr}(A/2 + B/3) = r - \text{Tr}(\Lambda) \), in other words 
\[
\text{Tr}(\Lambda) = r - \frac{\alpha_1}{2} - (\beta_1 + 2\frac{\beta_2}{3})
\]
We need to show that RHS\((*)\) is equal to above. We know that \( e^{2\pi i A/2} = S, e^{2\pi i B/3} = U \). So 
\[
\text{Tr}(S) = \alpha_0 - \alpha_1, \ \text{Tr}(U) = \beta_0 + \beta_1 \omega + \beta_2 \overline{\omega}
\]
Now multiply the last equality by \( e^{-\pi i/6} \) and compute the real part which gives 
\[
\frac{\sqrt{3}}{2}\beta_0 + 0 - \frac{\sqrt{3}}{2}\beta_2
\]
and as a result 
\[
d(\rho) := RHS(*) := \frac{5r}{12} + \frac{1}{4}(\alpha_0 - \alpha_1) + 2\frac{\sqrt{3}}{9}.\frac{\sqrt{3}}{2}(\beta_0 - \beta_2)
\]
But since \( \beta_0 = r - \beta_1 - \beta_2 \) and \( \alpha_0 = r - \alpha_1 \), after rewriting \( d(\rho) \) it gives us the equality \((*)\), i.e. 
\[
r - \frac{\alpha_1}{2} - (\beta_1 + 2\frac{\beta_2}{3})
\]

The reasons behind all this is the application of Serre duality in the Riemann Roch theorem, which makes the problem become a computation of the Euler Characteristic of a sheaf (see more here). Given any exponent \( \Lambda \), we will make assertions which we will explain next week. One is that \( \dim \ker P_\Lambda \) and \( \dim \coker P_\Lambda \) are both finite. Now, as index theory tells us, we should cook up an index 
\[
\text{Index } P_\Lambda = \dim \ker P_\Lambda - \dim \coker P_\Lambda
\]
This is a version of Riemann-Roch for nonabelian variety. Now the actual general theorem (Theorem 3.2 in Gannon’s paper) is that 
\[
\text{Index } P_\Lambda = d(\rho) - \text{Tr}(\Lambda)
\]
Serre duality comes in as the above is basically Euler characteristic of some \( H^0 - H^1 \). So if it is bijective, then \( \dim \ker P_\Lambda = \dim \coker P_\Lambda = 0 \) and so \( d(\rho) = \text{Tr}(\Lambda) \).

The other assertion is the following theorem:

**Theorem 10.1.** For any admissible \( \rho \), there exists a bijective exponent \( \Lambda \) such that \( P_\Lambda \) is an isomorphism as a \( \mathbb{C} \)-vector space.
There are two proofs: one is in Theorem 3.2 of Gannon where it uses a lot of bif theorems of the literature.

There could be a second proof: There is a paper by G. Mason and M. Knopp. Zhenghan believes that using their paper one can come up with another proof.

First, we take $f \in M_\rho$ and we know $q^{-\Lambda} f$ has an expansion and suppose $f$ is in the kernel so $q^{-\Lambda} f = \sum_{n \geq 0} f[n] q^n$ (which is called a $\Lambda$-holomorphic function). Now inside their paper, in the first part, they show that $\dim(\Lambda\text{-hol of } M_\rho) < r$. The second part is that: if $0 < \Lambda_{bb} \leq 1$ for all $b$'s, then $P_\Lambda$ is surjective.

Actually, if we take poincare series we introduced in 6.1 and take their fourier expansion, then if we look in there we can find something that has singluarity $q^{-1}$, so that means we ave surjectivity like above.

So suppose we have a surjective $P_\Lambda$. To have it bijective, let us look at two exponents $\Lambda$ and $\Lambda'$ and then their difference

$$\Lambda' - \Lambda = \text{diag}(n_1, \ldots, n_r), \text{ where } n_b \in \mathbb{Z}, \forall b$$

Now substituting that for $P_{\Lambda'}$ we get

$$P_{\Lambda'}(f) = \text{principal part of } (q^{-\Lambda - \text{diag}(n_1, \ldots, n_r)} f)$$

So if the kernel is finite, it is generated by finitely many $f_i$s. Then we can shift our exponent to the right direction (for large enough $n_b > 0$), so that we can kill the kernel (by killing the positive powers coefficients in all $f_i$s). Notice in this operation, surjectivity is also preserved as you can use the $j$ to compensate those added powers.

11. ADDITIONAL MATERIAL

April 1st

On Tannaka duality and its proof: [Reference: T. Wasserman’s PhD Thesis]: “The relationship between symmetric fusion categories and finite groups is captured by Tannaka duality. The form of Tannaka duality we will be using in this thesis is due to Deligne. His Theorem roughly says that every symmetric fusion category is braided monoidally equivalent to the representation category of a finite (super-)group. The notion of finite super-group used here is that of a finite group equipped with a choice of central element of order two. Part of the content of this Theorem is that every symmetric fusion category admits a braided monoidal functor to the category of super-vector spaces, called the fibre functor. If the essential image of this functor is contained in the subcategory of vector spaces, the symmetric fusion category is called Tannakian, otherwise we will call it super-Tannakian. The finite (super-)group is found by computing the (super-)group of tensor automorphisms of the fibre functor. In the super-Tannakian case, the grading involution of super-vector spaces will always be an automorphism of the fibre functor and gives rise to the central order two element of the super-group”.

The Moonshine Story The coefficients of the $j$–function are actually related to the dimensions of the irreducible representations of the Monster group. More precisely, they are the dimension of a graded vector space (a VOA) which is preserved by an action of the Monster group. I highly recommend to read the fascinating story behind all of this and VOA from Chapter 1.1. Motivation, in Introduction to Vertex Operator Algebras and Their Representations, by James Lepowsky and Haisheng Li (if you are using UCSB wifi, you should be able to access the Springer Link in here to download the book).

April 3rd
Operator Product Expansion (OPE in CFT) This is a complicated topic which is hard to address rigorously without first knowing the definition of VOA. I refer to Remark 3.3.13 of [here]. Roughly speaking, if you consider inserting a field (particle) \( v \) at \( z_2 \) and then another \( u \) at \( z_1 \), i.e. \( Y(u, z_1)Y(v, z_2) \), this operator product can be expanded as a Taylor series of field insertions \( Y(w_n, z_2) \) only at \( z_2 \) weighted by \( (z_1 - z_2)^{-n-1} \). The singular part of this expansion, which is finite (i.e. \( 0 \leq n \leq N \) for some \( N \)), is referred to as the OPE. [Go back to lecture]